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THE LINEARITY OF THE EXPECTED VALUE OF A FUZZY VARIABLE

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ABSTRACT

In this research, we introduce a novel credibility measure defined as a non-empty set satisfying the axioms of normality, monotonicity, self-duality, and maximality. Based on this credibility measure, a credibility space is constructed, upon which a fuzzy variable can be defined. Similar to fuzzy numbers, fuzzy variables are characterized by membership functions. The membership function of this fuzzy variable is directly derived from the credibility measure. Subsequently, by integrating the credibility measure, the expected value of the fuzzy variable is obtained. The linearity property of fuzzy expected value on fuzzy variables will be proven. This linearity property is highly useful in solving various problems involving fuzzy variables. Therefore, the proposed credibility measure provides a new framework in fuzzy variable theory. This credibility measure not only offers a more formal approach to measuring uncertainty but also opens up possibilities for the development of more complex and applicable fuzzy models.

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1. INTRODUCTION

The fuzzy set theory was first introduced by Zadeh in 1965 through the concept of membership functions to address uncertainty in systems [1]. In 1978, Zadeh proposed the possibility measure to evaluate events in a fuzzy manner, which later became the foundation for the development of fuzzy measure theory [2]. Since then, the possibility theory has attracted the attention of many researchers, including Nahmias [3], Zimmermann [4], and Liu [5]. However, the possibility measure has a limitation in that it lacks self-duality, a property that is essential both theoretically and practically. To address this limitation, Liu introduced the credibility measure as an alternative. The credibility measure has more specific axioms compared to ordinary measures [6]. Subsequently, provide sufficient and necessary conditions for the credibility measure [7].

Credibility theory was formulated by Liu as a branch of mathematics aimed at studying fuzzy phenomena [8]. This theory is based on five axioms that define the credibility measure and serve as the foundation for key concepts, such as fuzzy variables, membership functions, credibility distributions, expected values, variances, critical values, entropy, distance, and fundamental theorems [9]. Some of the main theorems in this theory include the subadditivity theorem of credibility, the credibility extension theorem, the semicontinuity law of credibility, the credibility multiplication theorem, and the credibility inversion theorem. In Liu's study [10], the focus was on introducing credibility theory using an axiomatic approach to study fuzzy phenomena. The study also covered credibility measures and credibility spaces, including the subadditivity theorem, the extension theorem, the semicontinuity law, and the multiplication theorem. Furthermore, fuzzy variables were defined as functions within credibility spaces, with discussions on membership functions, sufficient and necessary conditions, the credibility inversion theorem, and the primary concepts and theorems in credibility theory. As an extension of credibility theory, Liu introduced fuzzy random theory and random fuzzy theory. In general, a fuzzy random variable is a function from a probability space to the set of fuzzy variables, while a random fuzzy variable is a function from a credibility space to the set of random variables. Both are special cases of hybrid variables, defined as functions from a probability space to the set of real numbers. The expected value of fuzzy variables was first described by Liu [11] in 2002 using possibility and necessity measures. In 2005, Liu [12] further developed the concept of expected value, particularly for fuzzy random variables within credibility spaces. Subsequently, the theory of fuzzy variables in the credibility space was further developed by Xue Feng in 2008. In that article, the focus was on the continuous membership functions of fuzzy variables [13]. Liu's 2009 research revealed that the linearity of fuzzy expected values simplifies the analysis of models and optimization problems, as well as the property of linearity for the expected value of random fuzzy variables [14].

In fuzzy problems, fuzzy numbers are commonly applied by utilizing their membership functions [15], [16], or utilizing their alpha cut values on fuzzy integral to solve inventory problems [17]. In addition, fuzzy problems can be solved with fuzzy cognitive maps [18] and Choquet integral to solve decision problems or defuzzified [19], such as [20]. However, Li Xiang's study demonstrated that credibility measures can also be applied to optimization problems [21]. Based on this, this article focuses on the analytical solution of the expected value of fuzzy variables defined in credibility spaces with membership functions derived from credibility measures. Fuzzy variables can be analyzed using the expected value approach, where the expected value is obtained through the integration of credibility measures. To simplify the solution process, the novelty of this article lies in demonstrating that the linearity property of fuzzy random variables also holds for fuzzy variables solved through the integration of their credibility measures. Consequently, a theorem on the linearity of fuzzy variables will be proven.

2. RESEARCH METHOD

2.1 Credibility Measure and Credibility Space

Given Θ a nonempty set, $P(\Theta)$ is the power set of Θ with each element on $P(\Theta)$ is called an event. To define credibility, for each $A \in P(\Theta)$ a measure $Cr\{A\}$ is expressed as the credibility of the event A will occur. To ensure that the measure of $Cr\{A\}$ has properties in accordance with credibility, then the following axioms are given:

1. (Normality) $Cr\{\Theta\} = 1$.
2. (Monotonicity) $Cr\{A\} \leq Cr\{B\}$ for $A \subset B$, with $A, B \in P(\Theta)$.

3. (Self-Duality) $Cr\{A\} + Cr\{A^c\} = 1$ for every event A.
4. (Maximality)

$$Cr\{\cup_i A_i\} \wedge 0.5 = \begin{cases} \sup_i Cr\{A_i\} & , \sup Cr\{A_i\} < 0.5, \\ (\sup_i Cr\{A_i\}) \wedge 0.5 & , \sup Cr\{A_i\} \geq 0.5. \end{cases}$$

There are definitions of credibility measures.

Definition 1. *The set function Cr is said to be a credibility measure if it satisfies the axioms of normality, monotonicity, self-duality, and maximality.*

Definition 2. *Given f a nonnegative function on $\Theta = \mathbb{R}$ such that $sup_{\{x \in \mathbb{R}\}} f(x) = 1$.*

The function set is defined

$$Cr\{A\} = \frac{1}{2}\left(sup_{\{x \in A\}} f(x) + 1 - sup_{\{x \in A^c\}} f(x)\right) \tag{1}$$

is a credibility measure on Θ.

Based on the axioms of credibility measure, we can conclude that:

1. Considering **Equation (1)**, we find $Cr\{\Theta\} = 1$ then it satisfies normality.
2. For each set $A, B \in P(\Theta)$ with $A \subset B$ then $Cr\{A\} \leq Cr\{B\}$ in other words, Monotonicity is satisfied.
3. For each $A \in P(\Theta)$, $Cr\{A\} + Cr\{A^c\} = 1$ then Cr satisfies self-duality.
4. For each $\{A_i\} \in P(\Theta)$, the maximality

$$Cr\{\cup_i A_i\} \wedge 0.5 = \begin{cases} sup_i\, Cr\{A_i\}, & sup\, Cr\{A_i\} < 0.5 \\ (sup_i\, Cr\{A_i\}) \wedge 0.5, & sup\, Cr\{A_i\} \geq 0.5. \end{cases}$$

Remark: If the supremum of a function $f(x)$ is not equal to 1. The function can be normalized using the following equation

$$f_{normal}(x) = \left(\frac{1}{\sup f(x)}\right) f(x).$$

Example 1. For example, given a sigmoid function $f(x)$ on $\mathbb{R}$

$$f(x) = \begin{cases} \dfrac{1}{1 + e^{-x}}, & x \in \mathbb{R} \\ 0, & others. \end{cases} \tag{2}$$

such that $sup_{\{x \in \mathbb{R}\}} f(x) = 1$. The graph of the function f is shown in **Figure 1**.

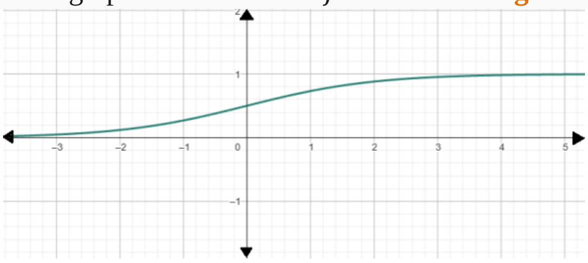

Figure 1. Graph of the Sigmoid Function $f(x)$ on $\mathbb{R}$

Credibility of $A \subset \mathbb{R}$ can be defined using **Equation (1)**.

The following is a definition of credibility space.

Definition 3. *Given a nonempty set Θ, P power set of Θ, and Cr is a credibility measure then (Θ, P, Cr) is defined as a credibility space.*

Theorem 1. *Given the sigmoid function in **Equation (2)**, the credibility space can be formed as $(\mathbb{R}, P(\mathbb{R}), Cr)$ with $\mathbb{R}$ the real set numbers, $P(\mathbb{R})$ power set of $\mathbb{R}$, and Cr the credibility function defined on $\mathbb{R}$ with*

$$Cr\{A\} = \frac{1}{2}\left(sup_{\{x\in A\}}f(x) + 1 \; - sup_{\{x\in A^c\}}f(x)\right), \forall\, A \in P(\mathbb{R}).$$

Proof. It will be shown that the function Cr is a credibility measure on $\mathbb{R}$.

1. Considering **Equation (1)**, it will be shown that $Cr\{\mathbb{R}\} = 1$ such that,

$$Cr\{\mathbb{R}\} = \frac{1}{2}\,(sup_{\{x\in\mathbb{R}\}}f(x) + 1 - sup_{\{x\notin\mathbb{R}\}}f(x)) = \frac{1}{2}(1 + 1 - 0) = \;1.$$

We find $Cr\{\mathbb{R}\} = 1$ then it satisfies normality.

2. For each set $A, B \in P(\mathbb{R})$ with $A \subset B$. It will be shown $Cr\{A\} \le Cr\{B\}$.

Note that $A \subset B$ then $sup_{\{x\in\mathbb{R}\}}f(x) \le sup_{\{x\in\mathbb{R}^c\}}f(x)$. If $A \subset B$ then $B^c \subset A^c$ so,

$$sup_{\{x\in A^c\}}f(x) \ge sup_{\{x\in B^c\}}f(x) \;\; \Rightarrow \; -sup_{\{x\in A^c\}}f(x) \le -sup_{\{x\in B^c\}}f(x).$$

Based on **Equation (1)**, then

$$\begin{aligned} Cr\{A\} &= \frac{1}{2}\left(sup_{\{x\in A\}}f(x) + 1 - sup_{\{x\in A^c\}}f(x)\right) \\ &\le \frac{1}{2}\left(sup_{\{x\in B\}}f(x) + 1 - sup_{\{x\in B^c\}}f(x)\right) \\ &\le Cr\{B\}. \end{aligned}$$

For $A \subset B$ then $Cr\{A\} \le Cr\{B\}$ in other words, monotonicity is satisfied.

3. For each $A \in P(R)$. It will be shown $Cr\{A\} + Cr\{A^c\} = 1$.

Based on **Equation (1)**, we obtain:

$$Cr\{A\} = \frac{1}{2}(sup_{\{x\in A\}}f(x) + 1 - sup_{\{x\in A^c\}}f(x))$$

and

$$\begin{aligned} Cr\{A^c\} &= \frac{1}{2}(sup_{\{x\in A^c\}}f(x) + 1 - sup_{\{x\in (A^c)^c\}}f(x)) \\ &= \frac{1}{2}(sup_{\{x\in A^c\}}f(x) + 1 - sup_{\{x\in A\}}f(x)) \end{aligned}$$

such that $Cr\{A\} + Cr\{A^c\}$ is

$$\begin{aligned} &= \frac{1}{2}(sup_{\{x\in A\}}\mu(x) + 1 - sup_{\{x\in A^c\}}f(x)) + \frac{1}{2}(sup_{\{x\in A^c\}}f(x) + 1 - sup_{\{x\in A\}}f(x)) \\ &= \frac{1}{2}\left\{\left(sup_{\{x\in A\}}f(x) - sup_{\{x\in A\}}f(x)\right) + (1+1) + \left(sup_{\{x\in A^c\}}f(x) + \left(-sup_{\{x\in A^c\}}f(x)\right)\right)\right\} \\ &= \frac{1}{2}(2) \\ &= 1. \end{aligned}$$

Since $Cr\{A\} + Cr\{A^c\} = 1$ then Cr satisfies self-duality.

4. It will be shown that the maximality condition is satisfied.

$$Cr\{\cup_i A_i\} \wedge 0.5 = \begin{cases} sup_i\; Cr\{A_i\}, sup\; Cr\{A_i\} < 0.5 \\ (sup_i\; Cr\{A_i\}) \wedge 0.5\,, sup\; Cr\{A_i\} \ge 0.5. \end{cases}$$

For each $\{A_i\} \in P(\mathbb{R})$.

a. First case: for $Cr\{\cup_i A_i\} < \; 0.5$.

Based on **Equation (1)**, we obtain

$$\begin{aligned} Cr\{\cup_i A_i\} \wedge 0.5 &= Cr\{\cup_i A_i\} \\ &= \frac{1}{2}(sup_{\{x\in \cup_i A_i\}}f(x) + 1 - sup_{\{x\in(\cup_i A_i)^c\}}f(x)) \\ &= sup_i\left(\frac{1}{2}\left(sup_{\{x\in A_i\}}f(x) + 1 - sup_{\{x\in A_i^c\}}f(x)\right)\right) \end{aligned}$$

$$= sup_i\ Cr\{A_i\}.$$

b. Second Case: for $Cr\{\cup_i A_i\} \geq 0.5$.

There is $Cr\{\cup_i A_i\} = 0.5$ such that

$$sup_i\ Cr\{A_i\} = sup_i\ (\frac{1}{2}(sup_{\{x\in A_i\}} f(x) + 1 - sup_{\{x\in A_i^c\}} f(x)))$$
$$= \frac{1}{2}(sup_{\{x\in \cup_i A_i\}} f(x) + 1 - sup_{\{x\in (\cup_i A_i)^c\}} f(x))$$
$$= Cr\{\cup_i A_i\}$$
$$= 0.5$$
$$= Cr\{\cup_i A_i\} \wedge 0.5.$$

Based on the evidence of the first and second cases, it is proven that the maximality of the credibility measure is

$$Cr\{\cup_i A_i\} \wedge 0.5 = \begin{cases} sup_i\ Cr\{A_i\}\ , sup\ Cr\{A_i\} < 0.5 \\ (sup_i\ Cr\{A_i\}) \wedge 0.5\ , sup\ Cr\{A_i\} \geq 0.5. \end{cases}$$

Based on the four pieces of evidence, Cr is a measure of credibility in $\mathbb{R}$. ■

In this section, we present the credibility measure of the union and intersection of intervals that are subsets of the power set. Note that, $e^{-x} > 0 \Rightarrow 1 + e^{-x} > 1 \Rightarrow \frac{1}{1+e^{-x}} < 1$. Therefore, the supremum of $f(x)$ on $\mathbb{R}$ is 1. However, there is no $x \in \mathbb{R}$ for which $f(x) = 1$.

Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Based on **Equation (1)**, the credibility measures of their union and intersection are as follows:

Credibility Measure of the Union of A and B.

Case 1: when $a_1 < b_1 < b_2 < a_2$, then $a_2 \in A \cup B$ such that: $\sup_{\{x\in A\cup B\}} f(x) = f(a_2) = \frac{1}{1 + e^{-a_2}}$.

Thus, the credibility measure is

$$Cr\{A \cup B\} = \frac{1}{2(1\ +\ e^{-a_2})}.$$

Case 2: when $a_1 < b_1 < a_2 < b_2$, then $b_2 \in A \cup B$ such that: $\sup_{\{x\in A\cup B\}} f(x) = f(b_2) = \frac{1}{1 + e^{-b_2}}$.

Thus, the credibility measure is

$$Cr\{A \cup B\} = \frac{1}{2(1\ +\ e^{-b_2})}.$$

Case 3: when $a_1 < a_2 < b_1 < b_2$, then $b_2 \in A \cup B$ such that: $\sup_{\{x\in A\cup B\}} f(x) = f(b_2) = \frac{1}{1 + e^{-b_2}}$.

Thus, the credibility measure is

$$Cr\{A \cup B\} = \frac{1}{2(1\ +\ e^{-b_2})}.$$

2. Credibility Measure of the Intersection of A and B.

Case 1: when $a_1 < b_1 < b_2 < a_2$, then $b_2 \in A \cap B$ such that: $\sup_{\{x\in A\cup B\}} f(x) = f(b_2) = \frac{1}{1 + e^{-b_2}}$.

Thus, the credibility measure is

$$Cr\{A \cup B\} = \frac{1}{2(1\ +\ e^{-b_2})}.$$

Case 2: when $a_1 < b_1 < a_2 < b_2$, then $a_2 \in A \cap B$ such that: $\sup_{\{x\in A\cup B\}} f(x) = f(a_2) = \frac{1}{1 + e^{-a_2}}$.

Thus, the credibility measure is

$$Cr\{A \cup B\} = \frac{1}{2(1\ +\ e^{-a_2})}.$$

Case 3: when $a_1 < a_2 < b_1 < b_2$, since $\emptyset \in A \cap B$ such that : $\sup_{\{x \in A \cup B\}} f(x) = 0$.

Thus, the credibility measure is

$$Cr\{A \cup B\} = 0.$$

Note that the function $f(x)$ maps $\mathbb{R}$ to the interval $(0, 1)$, and

$$f(x) = \frac{1}{1 + e^{-x}} < 1.$$

Therefore, for all $A \in P(\mathbb{R}), Cr\{A\} < 0.5$.

Now, consider a sequence of closed intervals $\{A_i\} \subset \mathbb{R}$, where $i = \{1,2, \dots\}$. The credibility measures of their union and intersection are as follows:

1. Credibility Measure of $\bigcup_i A_i$. Based on **Equation (1)**,

$$Cr(\cup A_i) \wedge 0.5 = Cr(\cup A_i) = sup_i \left(sup_{\{x \in A_i\}} \frac{1}{2(1 + e^{-x})} \right).$$

2. Credibility Measure of $\bigcap_i A_i$. There are two cases:
   a. Case 1: If $\cap A_i \neq \emptyset$, then

$$Cr(\cap A_i) = inf_i \left(sup_{\{x \in A_i\}} \frac{1}{2(1 + e^{-x})} \right).$$

   b. Case 2: If $\cap A_i = \emptyset$, then $f(x) = 0$, then

$$Cr(\cap A_i) = 0.$$

Therefore, the maximum credibility measure of the union is:

$$Cr(\cup A_i) = sup_i Cr(A_i) = sup_i \left(sup_{\{x \in A_i\}} \frac{1}{2(1 + e^{-x})} \right), Cr(A_i) < 0.5$$

and the credibility measure of the intersection is:

$$Cr(\cap A_i) = \begin{cases} inf_i \left(sup_{\{x \in A_i\}} \frac{1}{2(1 + e^{-x})} \right), & \cap A_i \neq \emptyset \\ 0, & \cap A_i = \emptyset. \end{cases}$$

2.2 Membership Function of Fuzzy Variable

Fuzzy variables are functions defined in the credibility space and are characterized by membership functions. The following is the definition of fuzzy variables and their membership functions.

Definition 4. *A fuzzy variable is defined as a (measurable) function from the credibility space* $(\Theta, P(\Theta), Cr)$ *to the set of real numbers.*

Definition 5. *Given* $\tilde{\xi}$ *fuzzy variables defined on the credibility space* $(\Theta, P(\Theta), Cr)$. *The membership function of the fuzzy variables is derived from the credibility space as follows*

$$\mu(x) = (2Cr\{\mu^{-1}(\alpha) = x\}) \wedge 1, \qquad x \in \mathbb{R}.$$

The following example provides an illustration of how to determine the membership function of a fuzzy variable based on credibility space.

Example 2. Given credibility space $(\mathbb{R}, P(\mathbb{R}), Cr)$.

$$Cr\{A\} = \frac{1}{2}\left(sup_{\{x \in A\}} f(x) + 1 - sup_{\{x \in A^c\}} f(x) \right), \ \forall \ A \in P(\mathbb{R}),$$

with $f(x)$ is a sigmoid function.

Furthermore, for singleton $A = \{x\}$ then

$$Cr\{A\} = \frac{1}{2}(f(x) + 1 - sup_{x \neq y}(y\, f(y)) = \frac{1}{2}(\frac{1}{1+e^{-x}} + 1 - sup_{\{x \neq y\}} f(x) = \frac{1}{2(1+e^{-x})}.$$

The fuzzy variable $\tilde{\xi}$ is defined in the credibility space $(\mathbb{R}, P(\mathbb{R}), Cr)$ to $\mathbb{R}$. The membership function of $\tilde{\xi}$ will be sought. Based on **Definition 5**, the membership function is obtained.

$$\begin{aligned}\mu(x) &= (2\, Cr\{\mu^{-1}\,(\alpha) = x\}) \wedge 1 \\ &= (2\, Cr\{A\}) \wedge 1 \\ &= (2\, (\frac{1}{2}(1+e^{-x}))) \wedge 1 \\ &= min\{\frac{1}{1+e^{-x}}\},1\} \\ &= \frac{1}{1+e^{-x}}.\end{aligned}$$

3. RESULTS AND DISCUSSION

3.1 Expected Value of Fuzzy Variable

The fuzzy variables are defined in credibility space and have membership functions. As with regular random variables, the expected value can be defined. The expected value has a role in optimization with random parameters. The following is the definition of the expected value of fuzzy variables.

Definition 6. *Given $\tilde{\xi}$ a fuzzy variable, then the expected value of $\tilde{\xi}$ is defined as*

$$E[\tilde{\xi}] = \int_0^{+\infty} Cr\{\mu^{-1}(\alpha) \geq x\}dx - \int_{-\infty}^{0} Cr\{\, \mu^{-1}(\alpha) \leq x\}\, dx$$

with one or two of these integrals are finite.

Example 3. Given function $f(x)$ on $\mathbb{R}$,

$$f(x) = \frac{1}{1+e^{-x}}.$$

Define variable fuzzy $\tilde{\xi}$ on credibility space $(\mathbb{R}, P(\mathbb{R}), Cr)$ to $\mathbb{R}$ and the membership function of $\tilde{\xi}$ as

$$\mu(x) = \frac{1}{1+e^{-x}}.$$

Next, we will look for the expected value of the fuzzy variable $\tilde{\xi}$.

1. For $x < 0$ then

$$Cr\{\, \mu^{-1}(\alpha) \leq x\} = \frac{1}{2}\left(sup_{y\leq x}\, f(y) + 1 - sup_{y>x}\, f(y)\right) = \frac{1}{2}(0+1-1) = 0.$$

2. For $x > 0$ then

$$Cr\{\mu^{-1}(\alpha) \geq x\} = \frac{1}{2}\left(sup_{y\geq x}\, f(y) + 1 - sup_{y<x}\, f(y)\right) = \frac{1}{2}(1+1-0) = 1.$$

Based on **Definition 6**, the expected value of $\tilde{\xi}$ is

$$E[\tilde{\xi}] = \int_0^{+\infty} Cr\{\, \mu^{-1}(\alpha) \geq x\}dx - \int_{-\infty}^{0} Cr\{\, \mu^{-1}\,(\alpha) \leq x\,\}dx = \int_0^{+\infty} 1dx - \int_{-\infty}^{0} 0dx = +\infty.$$

Example 4. Given function $f(x)$ on $\mathbb{R}$ with $a < b < c$,

$$f(x) = \begin{cases} \dfrac{x-a}{b-a}, a \leq x < b \\ \dfrac{c-x}{c-b}, b \leq x \leq c \\ 0\,, x \text{ others.} \end{cases}$$

such that $sup_{x\in\mathbb{R}}\, f(x) = 1$. Define a triangular fuzzy variable $\tilde{\xi}$ on credibility space $(\mathbb{R}, P(\mathbb{R}), Cr)$ to $\mathbb{R}$, and the membership function of $\tilde{\xi}$ as

$$\mu(x)=\begin{cases}\frac{x-a}{b-a}, a\le x<b\\ \frac{c-x}{c-b}, b\le x\le c\\ 0, x \text{ others.}\end{cases}$$

Next, we will look for the expected value of the fuzzy variable $\tilde{\xi}$. If $a<0<b<c$ then obtain

1. For $x<a$ then
$$Cr\{\mu^{-1}(\alpha)\le x\}=\frac{1}{2}\left(sup_{y\le x} f(y)+1-sup_{y>x} f(y)\right)=\frac{1}{2}(0+1-1)=0.$$
2. For $a<x<0$ then
$$Cr\{\mu^{-1}(\alpha)\le x\}=\frac{1}{2}\left(sup_{y\le x} f(y)+1-sup_{y>x}\ f(y)\right)=\frac{1}{2}\left(\frac{x-a}{b-a}+1-1\right)=\frac{x-a}{2(b-a)}.$$
3. For $0<x<b$ then
$$Cr\{\mu^{-1}(\alpha)\ge x\}=\frac{1}{2}\left(sup_{y\ge x} f(y)+1-sup_{y<x} f(y)\right)$$
$$=\frac{1}{2}\left(\frac{x-a}{b-a}+1-\frac{-a}{b-a}\right)=\frac{x}{2(b-a)}+\frac{1}{2}.$$
4. For $b<x<c$ then
$$Cr\{\mu^{-1}(\alpha)\ge x\}=\frac{1}{2}\left(sup_{y\ge x} f(y)+1-sup_{y<x} f(y)\right)=\frac{1}{2}\left(\frac{c-x}{c-b}+1-1\right)=\frac{c-x}{2(c-b)}.$$
5. For $x>c$ then
$$Cr\{\mu^{-1}(\alpha)\ge x\}=\frac{1}{2}\left(sup_{y\ge x} f(y)+1-sup_{y<x} f(y)\right)=\frac{1}{2}(0+1-1)=0.$$

Based on **Definition 6**, the expected value of $\tilde{\xi}$ is

$$\begin{aligned}E[\tilde{\xi}]&=\int_0^{+\infty} Cr\{\mu^{-1}(\alpha)\ge x\}dx-\int_{-\infty}^{0} Cr\{\mu^{-1}(\alpha)\le x\}dx\\
&=\int_0^b\left(\frac{x}{2(b-a)}+\frac{1}{2}\right)dx+\int_b^c\frac{c-x}{2(c-b)}\,dx-\int_a^0\frac{x-a}{2(b-a)}\,dx\\
&=\frac{b^2}{4(b-a)}+\frac{b}{2}+\frac{c}{2}-\frac{c+b}{4}+\frac{a^2}{4(b-a)}-\frac{2a^2}{4(b-a)}\\
&=\frac{b^2-a^2}{4(b-a)}+\frac{b}{4}+\frac{c}{4}\\
&=\frac{b+a}{4}+\frac{b}{4}+\frac{c}{4}\\
&=\frac{a+2b+c}{4}.\end{aligned}$$

For example, given $\tilde{\xi}=(0,60,100)$ then the expected value is

$$E[\tilde{\xi}]=\frac{a+2b+c}{4}=\frac{0+2(60)+100}{4}=\frac{220}{4}=55.$$

3.2 The Linearity of Fuzzy Expected Value

The expected value of a fuzzy variable satisfies the linearity condition. In this section, we will provide a proof of the theorem of the linearity of fuzzy expected values.

Theorem 2. *A Fuzzy variable $\tilde{\xi}$ with a finite expected value, then for any number a and b, we have*
$$E[a\tilde{\xi}+b]=aE[\tilde{\xi}]+b.$$

Proof. For each $b\in\mathbb{R}$, it will be shown that $E[a\tilde{\xi}+b]=aE[\tilde{\xi}]+b$.

1. In the first step, it will be shown that $E[\tilde{\xi}+b]=E[\tilde{\xi}]+b$.

a. Case 1: for $b \geq 0$, then

$$E[\tilde{\xi} + b] = \int_0^\infty Cr\{\mu^{-1}(\alpha) + b \geq x\}dx - \int_{-\infty}^0 Cr\{\mu^{-1}(\alpha) + b \leq x\}\,dx$$
$$= \int_0^\infty Cr\{\mu^{-1}(\alpha) \geq x - b\}\,dx - \int_{-\infty}^0 Cr\{\mu^{-1}(\alpha) \leq x - b\}\,dx$$
$$= \int_0^b Cr\{\mu^{-1}(\alpha) \geq x - b\}dx + \int_b^\infty Cr\{\mu^{-1}(\alpha) \geq x - b\}dx$$
$$- \int_b^0 Cr\{\mu^{-1}(\alpha) \leq x - b\}dx - \int_{-\infty}^b Cr\{\mu^{-1}(\alpha) \leq x - b\}\,dx$$
$$= \int_b^\infty Cr\{\mu^{-1}(\alpha) \geq x - b\}dx - \int_{-\infty}^b Cr\{\mu^{-1}(\alpha) \leq x - b\}dx$$
$$+ \int_0^b Cr\{\mu^{-1}(\alpha) \geq x - b\}dx - \int_b^0 Cr\{\mu^{-1}(\alpha) \leq x - b\}dx$$
$$= E[\tilde{\xi}] + \int_0^b Cr\{\mu^{-1}(\alpha) \geq x - b\}dx + \int_0^b Cr\{\mu^{-1}(\alpha) < x - b\}\,dx$$
$$= E[\tilde{\xi}] + \int_0^b (Cr\{\mu^{-1}(\alpha) \geq x - b\} + Cr\{\mu^{-1}(\alpha) < x - b\})\,dx$$
$$= E[\tilde{\xi}] + b.$$

b. Case 2: for $b < 0$, then

$$E[\tilde{\xi} + b] = \int_0^\infty Cr\{\mu^{-1}(\alpha) + b \geq x\}\,dx - \int_{-\infty}^0 Cr\{\mu^{-1}(\alpha) + b \leq x\}dx$$
$$= \int_0^\infty Cr\{\tilde{\xi} \geq \mu^{-1}(\alpha) - b\}dx - \int_{-\infty}^0 Cr\{\mu^{-1}(\alpha) \leq x - b\}dx$$
$$= \int_0^b Cr\{\mu^{-1}(\alpha) \geq x - b\}dx + \int_b^\infty Cr\{\mu^{-1}(\alpha) \geq x - b\}dx$$
$$- \int_b^0 Cr\{\mu^{-1}(\alpha) \leq x - b\}\,dx - \int_{-\infty}^b Cr\{\mu^{-1}(\alpha) \leq x - b\}dx$$
$$= \int_b^\infty Cr\{\mu^{-1}(\alpha) \geq x - b\}dx - \int_{-\infty}^b Cr\{\mu^{-1}(\alpha) \leq x - b\}dx$$
$$+ \int_0^b Cr\{\mu^{-1}(\alpha) \geq x - b\}dx - \int_b^0 Cr\{\mu^{-1}(\alpha) \leq x - b\}dx$$
$$= E[\tilde{\xi}] - \int_b^0 Cr\{\mu^{-1}(\alpha) \geq x - b\}\,dx - \int_b^0 Cr\{\mu^{-1}(\alpha) < x - b\}dx$$
$$= E[\tilde{\xi}] - \int_b^0 (Cr\{\mu^{-1}(\alpha) \geq x - b\} + Cr\{\mu^{-1}(\alpha) < x - b\})\,dx$$
$$= E[\tilde{\xi}] + b.$$

2. In the second step, it will be shown that $E[a\tilde{\xi}] = aE[\tilde{\xi}]$.

a. Case 1: for $a > 0$, then

$$E[a\tilde{\xi}] = \int_0^\infty Cr\{a\,\mu^{-1}(\alpha) \geq x\}\,dx - \int_{-\infty}^0 Cr\{a\,\mu^{-1}(\alpha) \leq x\}\,dx$$
$$= \int_0^\infty Cr\{\mu^{-1}(\alpha) \geq \frac{x}{a}\}\,dx - \int_{-\infty}^0 Cr\{\mu^{-1}(\alpha) \leq \frac{x}{a}\}\,dx$$
$$= \int_0^\infty Cr\{\mu^{-1}(\alpha) \geq \frac{x}{a}\}\,d(\frac{ax}{a}) - \int_{-\infty}^0 Cr\{\mu^{-1}(\alpha) \leq \frac{x}{a}\}\,d(\frac{ax}{a})$$
$$= a\int_0^\infty Cr\{\mu^{-1}(\alpha) \geq \frac{x}{a}\}\,d(\frac{x}{a}) - a\int_{-\infty}^0 Cr\{\mu^{-1}(\alpha) \leq \frac{x}{a}\}\,d(\frac{x}{a})$$

$$= aE[\tilde{\xi}].$$

b. Case 2: for $a < 0$, then

$$\begin{aligned}
E[a\tilde{\xi}] &= \int_0^{\infty} Cr\{a\,\mu^{-1}(\alpha) \geq x\}\,dx - \int_{-\infty}^{0} Cr\{a\,\mu^{-1}(\alpha) \leq x\}\,dx \\
&= \int_0^{\infty} Cr\{\mu^{-1}\,(\alpha) \leq \frac{x}{a}\}\,dx \;-\int_{-\infty}^{0} Cr\{\mu^{-1}(\alpha) \geq \frac{x}{a}\}\,dx \\
&= \int_0^{\infty} Cr\{\mu^{-1}(\alpha) \leq \frac{x}{a}\}\,d(\frac{ax}{a}) \;-\int_{-\infty}^{0} Cr\{\mu^{-1}(\alpha) \geq \frac{x}{a}\}\,d(\frac{ax}{a}) \\
&= a\int_0^{\infty} Cr\{\mu^{-1}(\alpha) \geq \frac{x}{a}\}\,d(\frac{x}{a}) \;-\; a\int_{-\infty}^{0} Cr\{\mu^{-1}(\alpha) \leq \frac{x}{a}\}\,d(\frac{x}{a}) \\
&= aE[\tilde{\xi}]
\end{aligned}$$

Based on the evidence in the first and second steps, for $E[a\tilde{\xi} + b]$ occur $E[a\tilde{\xi} + b] = aE[\tilde{\xi}] + b$, such that **Theorem 2** is proven. ■

The linearity of fuzzy expected values is important in fuzzy optimization. This will simplify solving optimization problems with fuzzy variables using the expected value approach.

Based on **Theorem 2**, the following example is given.

Example 5. Given a triangular fuzzy variable $\tilde{\xi} = (0,60,100)$. From Example 4, $E[\tilde{\xi}] = 55$. Then

$$\begin{aligned}
E[3\tilde{\xi} + 2] &= 3E[\tilde{\xi}] + 2 \\
&= 3(55) + 2 \\
&= 167.
\end{aligned}$$

4. CONCLUSION

The membership function of a fuzzy variable can be formed from its credibility measure. However, the definition of its membership function is not unique. The expected value of a fuzzy variable can be obtained by integrating its credibility measure. The nature of the fuzzy expected value is to meet the linearity condition, where this condition can facilitate the solution of problems related to fuzzy variables. Suggestions for further research are other methods to determine the expected value other than its credibility measure, for example, with its possibility or necessity.

AUTHOR CONTRIBUTIONS

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CONFLICT OF INTEREST

The authors declare no conflicts of interest regarding the publication of this paper.

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