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PERSYMMETRIC MATRIX AND ITS APPLICATION IN CODING THEORY

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ABSTRACT

A persymmetric matrix is a square matrix that is symmetric concerning its antidiagonal. This article discusses some characteristics of a persymmetric matrix and its application in coding theory. A persymmetric matrix is used to form a generator matrix of binary reversible self-dual codes. A binary reversible self-dual code is a self-dual code whose reverse element is contained in the code. The methodology involves the implementation of flip transpose and column reversal to ensure the generator matrix satisfies both self-duality and reversibility. We begin with small-sized persymmetric matrices (e.g., 2×2 and 3×3) to extend the construction of the larger matrices. Combining a self-dual code and a reversible self-dual code of shorter length, and embedding persymmetric blocks, we construct new binary reversible self-dual codes of longer length. The novelty of this research lies in developing a new construction method for binary reversible self-dual codes derived directly from self-dual codes in the standard form, which has not been explicitly addressed in previous studies.

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1. INTRODUCTION

A persymmetric matrix is a special class of matrices with a unique structure. This type of matrix was first introduced by Arthur Cayley in 1858. A persymmetric matrix has the property that its elements are symmetric with respect to its lower-left to upper-right diagonal [1]. The concept of persymmetric matrices is almost similar to that of a centrosymmetric matrix [2] and a Leslie matrix [3].

Some research on persymmetric matrices has advanced over time. The application of persymmetric matrices can be used in various fields, such as radar detection, graph, fractional analysis, mathematical physics, and coding theory. In 2008, Pailloux et al. introduced the persymmetric adaptive matched filter for radar detection by utilizing the properties of a persymmetric matrix [4]. Subsequently, in 2012, Nouri and Mahdi used persymmetric matrices to determine the adjacency matrix and Laplacian matrices in graphs [5]. Aleorov et al. (2015) explained that a class of persymmetric matrices generated by boundary value problems of fractional differential equations has simple and real eigenvalues that play an essential role in fractional analysis [6].

In 2015, Julio and Soto introduced a new concept called flip transpose, which is an operation on matrices involving reversing the order of elements along their diagonal. They found a strong correlation between flip transpose and persymmetric matrices [7]. Building on this conceptual development, Soto et al. in 2015 discussed nonnegative persymmetric matrices with predetermined elementary divisors, exploring their properties and construction methods [8].

Next, in 2017, Genest et al. explored isospectral deformations and orthogonal polynomials related to these matrices, contributing to spectral theory and mathematical physics [9]. In 2020, Vaia discussed a persymmetric Jacobi matrix with specific eigenvalues and their relevance to the dynamics of mass-spring chains [10]. Later in 2020, Li and Wang discussed and verified the real eigenvalues of a persymmetric matrix with validated error bounds [11].

In coding theory, Kim et al. explored the relationship between a persymmetric matrix and the construction of binary reversible self-dual codes [12]. They discovered that the unique structure of persymmetric matrices could be utilized to construct binary codes with self-dual property codes that are identical to their dual but in a reversible form. A new binary reversible self-dual code is constructed from smaller binary reversible self-dual code. Typically, the construction of reversible self-dual codes relies on ad hoc algebraic structures or circulant matrices. However, these methods may fail to provide sufficient flexibility when generalizing to different code lengths or maintaining both self-duality and reversibility simultaneously. Persymmetric matrices offer a compelling alternative due to their natural antidiagonal symmetry, which aligns well with the requirements of reversible and self-dual codes. Their well-defined structure facilitates the enforcement of orthogonality in generator matrices. It enables matrix operations such as flip, transpose, and column reversal to play a constructive role in code formation.

Reversible self-dual codes have essential applications in coding theory, especially in the context of error-correcting codes and cryptography to improve encoding/decoding performance and reduce redundancy [13], [14], [15], and [16]. Additionally, these codes have significant applications in DNA computing, where the primary goal is to efficiently store and transfer genetic information while minimizing errors caused by mutations or distortions. These codes play a crucial role in DNA-based data storage and error correction, especially due to their inherent structure of self-duality and reversibility [17] and [18]. Moreover, reversible self-dual codes contribute to secure communication systems by enhancing resistance against attacks such as hardware Trojans, side-channel attacks, and data tampering, owing to their properties, which facilitate secure and fault-tolerant circuit design [13] and [15].

This paper investigates the structural properties of persymmetric matrices and proposes a novel construction of binary reversible self-dual codes based on these properties. Unlike previous works that rely primarily on circulant or double circulant matrices, our method begins with a self-dual code in the standard form. It generates a new binary reversible self-dual code by embedding a carefully selected persymmetric matrix. This construction uses flip transpose and column reversal operations to ensure self-duality and reversibility are preserved. The main contribution of this work is a systematic algebraic framework for constructing new binary reversible self-dual codes using embedded persymmetric matrices. This offers a new direction for code design in both theoretical and practical applications.

2. RESEARCH METHODS

The methodology used in this research article involves conducting a comprehensive literature review of various relevant articles and books. The method used in this study consists of the following five main steps:

1. Definition setup: Establish fundamental matrix operations, including flip transpose and column reversal, which are crucial for recognizing persymmetric properties.
2. Characterization of persymmetric matrices: Analyze and classify persymmetric matrices of small sizes (2×2, 3×3), and systematically extend the construction to larger sizes.
3. Linking matrices to codes: Utilize the properties of persymmetric matrices to construct generator matrices for binary self-dual codes, ensuring the orthogonality conditions.
4. Extension techniques: Develop new codes by introducing eigenvector-based perturbations, maintaining the self-duality and reversibility through carefully designed matrix additions.
5. Verification: Validate the constructions by mathematically checking the self-duality and reversibility properties and providing explicit examples.

To support these steps, we present the basic concepts and operations related to persymmetric matrices and demonstrate how they can be used to construct binary reversible self-dual codes.

Definition 1. [12] *Let A be a matrix of size $m \times n$ denoted by $(a_{i,j})_{m \times n}$. The column reversed matrix of A is $A^r = (a_{m,n-j+1})_{m\times n}$.*

We denote the column reversed matrix of the identity matrix of size $n \times n$ by R_n. Therefore, the column reversed matrix of a square matrix A of size $n \times n$ can be expressed as follows:

$$A^r = AR_n.$$

Example 1. Consider the matrix A as follows:

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}.$$

Subsequently, we get

$$A^r = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix},$$

where

$$A^r = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = AR_2.$$

Definition 2. [7] *Let A be a matrix of size $m \times n$ denoted by $(a_{i,j})_{m \times n}$. The flip transpose of matrix A is $A^F = (a_{n-j+1,m-i+1})_{n \times m}$.*

The flip transpose matrix of a square matrix A of size $n \times n$ can also be expressed as follows:

$$A^F = R_n A^T R_n,$$

where A^T is the transpose of matrix A. Also, the flip transpose matrix of a column vector $\mathbf{x}^T$ can be expressed as follows:

$$(\mathbf{x}^T)^F = R_n\,\mathbf{x}.$$

Example 2. Given the matrix A as follows:

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}.$$

Then,

$$A^F = R_2 A^T R_2 = \begin{pmatrix} 5 & 2 \\ 4 & 1 \end{pmatrix}.$$

Next, we discuss the main terminology in coding theory, taken from [19]. A binary linear code C of length n is a subspace of F_2^n. A dual code of C is defined by

$$C^{\perp} = \{ \mathbf{x} \in F_2^n \mid \mathbf{x} \cdot \mathbf{c} = 0 \ \forall \mathbf{c} \in C \}.$$

The code C is said to be a self-dual if $C = C^{\perp}$. A generator matrix of linear code C is a matrix whose rows form a basis of C. A binary linear code C of length n and dimension k, with generator G, can be stated as follows:

$$C = \{\mathbf{w}G \mid \mathbf{w} \in F_2^k\}.$$

The standard generator matrix of code C is defined by,

$$G = (I_k \ \ A),$$

where G is a matrix of size $k \times n$, I_k is the identity matrix of size $k \times k$, and A is a matrix of size $k \times (n - k)$. The following theorem describes the sufficient and necessary conditions of the generator matrix of a binary self-dual code.

Theorem 1. [19] *Let C be a binary linear code of length $2n$ with generator matrix*

$$G = (I_n \ \ A).$$

Then C is self-dual if and only if $AA^T = I_n$.

In other words, **Theorem 1** states that the matrix A is orthogonal.

Definition 3. [20] *Let C be a binary linear code with length n digit. C is said to be a binary reversible code if for all $\mathbf{c} = (c_1, c_2, \dots, c_{n-1}, c_n) \in C$, the codeword $\mathbf{c}^r = (c_n, c_{n-1}, \dots, c_2, c_1) \in C$.*

3. RESULTS AND DISCUSSION

This section discusses further relations between persymmetric matrices and the construction of a binary reversible self-dual code.

3.1 Properties of Persymmetric Matrix

We study the definition of a persymmetric matrix and the general form of such a matrix.

Definition 4. [7] *Let A be a square matrix of size $n \times n$. A is said to be a persymmetric matrix if $A = A^F$.*

Example 3. Given a square matrix A as follow:

$$A = \begin{pmatrix} 9 & 2 & 5 \\ 4 & -3 & 2 \\ 5 & 4 & 9 \end{pmatrix}.$$

Then, A is a persymmetric matrix because

$$\begin{aligned} A^F &= R_3 A^T R_3 \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 9 & 4 & 5 \\ 2 & -3 & 4 \\ 5 & 2 & 9 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 2 & 5 \\ 4 & -3 & 2 \\ 5 & 4 & 9 \end{pmatrix} \\ &= A. \end{aligned}$$

Next, we present the general forms of the 2×2 and 3×3 persymmetric matrices, which will be constructed in the following lemma.

Lemma 1. *Let A and B be square matrices of sizes 2×2 and 3×3, respectively. If A and B are in the form*

$$A = \begin{pmatrix} a & c \\ b & a \end{pmatrix},$$

$$B = \begin{pmatrix} a & b & c \\ d & e & b \\ c & d & a \end{pmatrix},$$

then the matrices A and B are persymmetric.

Proof. Consider that,

$$\begin{aligned} A^F &= R_2 A^T R_2 \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & a \\ a & c \end{pmatrix} \\ &= \begin{pmatrix} a & c \\ b & a \end{pmatrix} \\ &= A. \end{aligned}$$

Thus, by **Definition 4**, the matrix A is persymmetric. Similarly, it can be shown that B is also persymmetric. ∎

Example 4. Given the matrices A and B as follows,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 3 & 3 \\ 4 & 5 & 3 \\ 3 & 4 & 2 \end{pmatrix}.$$

Here, the matrices A and B are persymmetric according to **Lemma 1**.

Subsequently, the form of persymmetric matrices of sizes 2×2 and 3×3 can be extended to larger sizes in the following lemma.

Lemma 2. *Let A, B and C be square matrices of size $n \times n$. If B and C are persymmetric, then*

$$E = \begin{pmatrix} A & C \\ B & A^F \end{pmatrix}$$

is persymmetric matrix of size $2n \times 2n$.

Proof. We obtain the flip transpose of matrix E,

$$\begin{aligned} E^F &= R_{2n} \begin{pmatrix} A & C \\ B & A^F \end{pmatrix}^T R_{2n} \\ &= \begin{pmatrix} O_n & R_n \\ R_n & O_n \end{pmatrix} \begin{pmatrix} A^T & B^T \\ C^T & (A^T)^F \end{pmatrix} \begin{pmatrix} O_n & R_n \\ R_n & O_n \end{pmatrix} \\ &= \begin{pmatrix} R_n (A^F)^T R_n & R_n C^T R_n \\ R_n B^T R_n & R_n A^T R_n \end{pmatrix} \\ &= \begin{pmatrix} A & C \\ B & A^F \end{pmatrix} \\ &= E. \end{aligned}$$

Since $E = E^F$, by **Definition 4**, E is persymmetric. ∎

Corollary 1. *Let A be a square matrix of size $n \times$ n. If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of A and A^F, respectively, then $\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}$ are eigenvectors of $\begin{pmatrix} A & O \\ O & A^F \end{pmatrix}$, where O is the zero matrix of size $n \times n$.*

Proof. Assume that $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of A and A^F, respectively. Then,

$$A\mathbf{x} = \lambda_1 \mathbf{x},$$

where λ_1 is an eigenvalue of A. Also,

$$A^F \mathbf{y} = \lambda_2 \mathbf{y},$$

where λ_2 is an eigenvalue of A^F.

Now, consider the following computation:

$$\begin{pmatrix} A & O \\ O & A^F \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} A & O \\ O & A^F \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} A\mathbf{x} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \lambda_1\mathbf{x} \\ \mathbf{0} \end{pmatrix} = \lambda_1\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}.$$

Thus, $\mathbf{x} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} A & O \\ O & A^F \end{pmatrix}$.

Similarly, for eigenvector $\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}$, we have

$$\begin{pmatrix} A & O \\ O & A^F \end{pmatrix}\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} A & O \\ O & A^F \end{pmatrix}\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ A^F\mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \lambda_2\mathbf{y} \end{pmatrix} = \lambda_2\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}.$$

Therefore, $\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} A & O \\ O & A^F \end{pmatrix}$. ■

Example 5. Consider the square matrix A given by

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

By **Lemma 2**, we can construct a persymmetric matrix of size 4×4 as follows:

$$E = \begin{pmatrix} A & O \\ O & A^F \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here, the zero matrix is persymmetrix. Now, consider the eigenvectors of the matrix A are $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ each corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$, respectively. Meanwhile, the eigenvectors of the matrix A^F are $\mathbf{y}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{y}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ each corresponding to the eigenvalues $\lambda_3 = 1$ and $\lambda_4 = 3$, respectively. By **Corollary 1**, the eigenvectors of the matrix E are $\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \mathbf{x}_2 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \mathbf{0} \\ \mathbf{y}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} \mathbf{0} \\ \mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. The eigenvectors $\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{0} \\ \mathbf{y}_1 \end{pmatrix}$ correspond to the eigenvalue $\lambda_5 = 1$. Subsequently, the eigenvectors $\begin{pmatrix} \mathbf{x}_2 \\ \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{0} \\ \mathbf{y}_2 \end{pmatrix}$ correspond to the eigenvalue $\lambda_6 = 3$. These results can be verified as follows:

$$E\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix} = \lambda_5\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix}.$$

$$E\begin{pmatrix}\mathbf{0}\\ \mathbf{y}_2\end{pmatrix} = \begin{pmatrix}1 & 2 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 3 & 2\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\3\\0\end{pmatrix} = 3\begin{pmatrix}\mathbf{0}\\ \mathbf{y}_2\end{pmatrix} = \lambda_6\begin{pmatrix}\mathbf{0}\\ \mathbf{y}_2\end{pmatrix}.$$

For the other eigenvectors of E can be verified in the same way.

We now construct a persymmetric matrix of odd order, with the result given in the following lemma.

Lemma 3. *Let A be a square matrix of size $n \times n$, B and C be the persymmetric matrices of size $n \times n$. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ and α is a constant. Then, the following matrix D is persymmetric matrix of size* $(2n+1) \times (2n+1)$.

$$D = \begin{pmatrix} A & \mathbf{x} & C\\ \mathbf{y}^T & \alpha & \mathbf{x}^F\\ B & (\mathbf{y}^F)^T & A^F\end{pmatrix}.$$

Proof. We first note that for $(\mathbf{x}^F)^T \in \mathbb{C}^n$, it can be expressed as

$$(\mathbf{x}^F)^T = R_n\mathbf{x}.$$

From this, we obtain the relation

$$\mathbf{x}^F = \mathbf{x}^T R_n \Leftrightarrow \mathbf{x}^F R_n = \mathbf{x}^T.$$

Now, we compute the flip transpose of D.

$$\begin{aligned}
D^F &= R_{2n+1}\begin{pmatrix} A & \mathbf{x} & C\\ \mathbf{y}^T & \alpha & \mathbf{x}^F\\ B & (\mathbf{y}^F)^T & A^F\end{pmatrix}^T R_{2n+1}\\
&= \begin{pmatrix} O & O & R_n\\ O & 1 & O\\ R_n & O & O\end{pmatrix}\begin{pmatrix} A^T & \mathbf{y} & B^T\\ \mathbf{x}^T & \alpha & \mathbf{y}^F\\ C^T & (\mathbf{x}^F)^T & (A^F)^T\end{pmatrix}\begin{pmatrix} O & O & R_n\\ O & 1 & O\\ R_n & O & O\end{pmatrix}\\
&= \begin{pmatrix} O & O & R_n\\ O & 1 & O\\ R_n & O & O\end{pmatrix}\begin{pmatrix} B^T R_n & \mathbf{y} & A^T R_n\\ \mathbf{y}^F R_n & \alpha & \mathbf{x}^T R_n\\ (A^F)^T R_n & (\mathbf{x}^F)^T & C^T R_n\end{pmatrix}\\
&= \begin{pmatrix} R_n (A^F)^T R_n & R_n(\mathbf{x}^F)^T & R_n C^T R_n\\ \mathbf{y}^F R_n & \alpha & \mathbf{x}^T R_n\\ R_n (B)^T R_n & R_n\mathbf{y} & R_n A^T R_n\end{pmatrix}\\
&= \begin{pmatrix} A & \mathbf{x} & C\\ \mathbf{y}^T & \alpha & \mathbf{x}^F\\ B & (\mathbf{y}^F)^T & A^F\end{pmatrix}\\
&= D.
\end{aligned}$$

Since $D^F = D$, by **Definition 4**, D is persymmetric matrix of size $(2n+1) \times (2n+1)$. ∎

Example 6. Given the square matrices A, B, and C, along with vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, and a scalar α as follows:

$$A = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\ B = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix},\ C = \begin{pmatrix}3 & 4\\ 1 & 3\end{pmatrix},$$
$$\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{y} = \begin{pmatrix}3\\4\end{pmatrix},\ \alpha = 3.$$

Here, the matrices B and C are persymmetric. By Applying **Lemma 3**, we obtain the following persymmetric matrix D.

$$D = \begin{pmatrix} A & \mathbf{x} & C\\ \mathbf{y}^T & \alpha & \mathbf{x}^F\\ B & (\mathbf{y}^F)^T & A^F\end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 & 4\\ 3 & 4 & 0 & 1 & 3\\ 3 & 4 & 3 & 0 & 1\\ 2 & 1 & 4 & 4 & 2\\ 1 & 2 & 3 & 3 & 1\end{pmatrix}.$$

3.2 The Application of the Persymmetric Matrix in Coding Theory

This section discusses constructing binary reversible self-dual codes associated with a persymmetric matrix. First, we present the construction results from previous research. Then, we introduce a new theorem.

Lemma 4. [12] *Let A be a square matrix of size $n \times n$ and A^r be the column reversed matrix of A. Then, the following two statements imply a third:*

1. *A is an orthogonal matrix.*
2. $(A^r)^2 = I_n$.
3. *A is a persymmetric matrix.*

According to **Theorem 1**, the submatrix in the standard form of the generator matrix of a binary self-dual code is orthogonal. Therefore, this lemma establishes a connection between persymmetric and orthogonal matrices.

Example 7. Consider the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Note that,

$$AA^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = I_2$$

and

$$(A^r)^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Since $AA^T = I_2$ and $(A^r)^2 = I_2$, it follows from **Lemma 4** that the matrix A is a persymmetric matrix. This can be explicitly verified as follows:

$$A^F = R_2A^TR_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A.$$

By applying **Lemma 4**, we establish the properties of a binary reversible self-dual code, as stated in the following lemma.

Lemma 5. [12] *Let C be a binary self-dual code of length $2n$ with generator matrix $G = (I_n\ A)$. The code C is reversible if and only if it satisfies one of the following conditions.*

1. $(A^r)^2 = I_n$.
2. *A is a persymmetric matrix.*

Because C is a self-dual code, then A is an orthogonal matrix. By **Lemma 4**, both conditions are satisfied if at least one of them holds. This property is crucial because it provides a practical criterion to ensure the reversibility of self-dual codes, facilitating the construction of such codes with predictable structural behavior. This motivates the construction of binary reversible self-dual codes.

Theorem 2. [12] *Let $(I_n\ A)$ be the generator matrix of a binary reversible self-dual code of length $2n$. Suppose the column vector $\mathbf{x} = (x_i)$ is an eigenvector of A^r with odd weight and $E = \mathbf{x}\mathbf{x}^F$. Then, the matrix*

$$G = \begin{pmatrix} I_n & O & \mathbf{x} & A+E \\ O & 1 & 0 & \mathbf{x}^F \end{pmatrix}$$

generates a binary reversible self-dual code of length $2n+2$.

Example 8. Given a reversible self-dual code of length 4 digits that has the following generator matrix,

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

This matrix can be written in the form $G = (I_2\ A)$. Hence,

$$A^r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If the eigenvectors of A^r with odd weight are computed, the result is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So, we get

$$E = \mathbf{x}\mathbf{x}^F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By applying **Theorem 2**, we can construct a binary reversible self-dual code of length 6, denoted as C', with the following generator matrix.

$$G' = \begin{pmatrix} I_2 & O & \mathbf{x} & A+E \\ O & 1 & 0 & \mathbf{x}^F \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Then, elements of C' are

$$\begin{array}{ll} \mathbf{c_1} = 000000 & \mathbf{c_5} = 100001 \\ \mathbf{c_2} = 001010 & \mathbf{c_6} = 101011 \\ \mathbf{c_3} = 010100 & \mathbf{c_7} = 110101 \\ \mathbf{c_4} = 011110 & \mathbf{c_8} = 111111. \end{array}$$

Furthermore, every element $\mathbf{c}^r$ of C' are in C' since,

$$\begin{array}{ll} \mathbf{c}_1^r = \mathbf{c}_1 & \mathbf{c}_5^r = \mathbf{c}_5 \\ \mathbf{c}_2^r = \mathbf{c}_3 & \mathbf{c}_6^r = \mathbf{c}_7 \\ \mathbf{c}_3^r = \mathbf{c}_2 & \mathbf{c}_7^r = \mathbf{c}_6 \\ \mathbf{c}_4^r = \mathbf{c}_4 & \mathbf{c}_8^r = \mathbf{c}_8. \end{array}$$

In 2020, Kim et al. was constructed a binary reversible self-dual code of length $2n$ from a self-dual code of length n in its general form as follows:

Proposition 1. [12] If G is a generator matrix of a self-dual code of length n, then

$$G' = \begin{pmatrix} G & O \\ O & G^r \end{pmatrix}$$

generates a reversible self-dual code of length $2n$.

Based on **Proposition 1**, the generator matrix of G' in general form, then not every matrix G' can be associated with a persymmetric matrix. However, G' can be associated with a persymmetric matrix if G' in the standard form. Therefore, we construct the generator matrix of the reversible self-dual code in the standard form as follows:

Theorem 3. *If* $(I_n \;\; A)$ *is the generator matrix of a binary self-dual code of length* $2n$*, then*

$$G = \begin{pmatrix} I_n & O & A & O \\ O & I_n & O & A^F \end{pmatrix}$$

generates a binary reversible self-dual code of length $4n$*.*

Proof. Let

$$D = \begin{pmatrix} A & O \\ O & A^F \end{pmatrix}.$$

Since $(I_n \;\; A)$ is the generator matrix of a binary self-dual code then $AA^T = I_n$. By **Definition 4**, we obtain

$$\begin{aligned} A^F(A^F)^T &= R_nA^TR_nR_nAR_n \\ &= R_nA^TAR_n \\ &= R_nI_nR_n \\ &= I_n. \end{aligned}$$

Thus, we compute

$$
\begin{aligned}
DD^T &= \begin{pmatrix} A & O \\ O & A^F \end{pmatrix} \begin{pmatrix} A & O \\ O & A^F \end{pmatrix}^T \\
&= \begin{pmatrix} A & O \\ O & A^F \end{pmatrix} \begin{pmatrix} A^T & O \\ O & (A^F)^T \end{pmatrix} \\
&= \begin{pmatrix} AA^T & O \\ O & A^F(A^F)^T \end{pmatrix} \\
&= \begin{pmatrix} I_n & O \\ O & I_n \end{pmatrix} \\
&= I_{2n}
\end{aligned}
$$

and $D^F = D$. Based on **Lemma 5**, G generates a binary reversible self-dual code of length $4n$. ■

Example 9. Given a self-dual code C of length 4 with generator matrix as follows:

$$D = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

According to **Theorem 3,** we can construct a binary reversible self-dual code of length 8, denoted as C', with the following generator matrix.

$$G = \begin{pmatrix} I_2 & O & A & O \\ O & I_2 & O & A^F \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, the elements of the code C' are as follows:

$$
\begin{array}{llll}
\mathbf{c}_1 = 00000000 & \mathbf{c}_5 = 01001000 & \mathbf{c}_9 = 10000100 & \mathbf{c}_{13} = 11001100 \\
\mathbf{c}_2 = 00010010 & \mathbf{c}_6 = 01011010 & \mathbf{c}_{10} = 10010110 & \mathbf{c}_{14} = 11011110 \\
\mathbf{c}_3 = 00100001 & \mathbf{c}_7 = 01101001 & \mathbf{c}_{11} = 10100101 & \mathbf{c}_{15} = 11101101 \\
\mathbf{c}_4 = 00110011 & \mathbf{c}_8 = 01111011 & \mathbf{c}_{12} = 10110111 & \mathbf{c}_{16} = 11111111.
\end{array}
$$

Additionally, the reverse of each codeword is always contained in C', as shown below.

$$
\begin{array}{llll}
\mathbf{c}_1^r = \mathbf{c}_1 & \mathbf{c}_5^r = \mathbf{c}_2 & \mathbf{c}_9^r = \mathbf{c}_3 & \mathbf{c}_{13}^r = \mathbf{c}_4 \\
\mathbf{c}_2^r = \mathbf{c}_5 & \mathbf{c}_6^r = \mathbf{c}_6 & \mathbf{c}_{10}^r = \mathbf{c}_7 & \mathbf{c}_{14}^r = \mathbf{c}_8 \\
\mathbf{c}_3^r = \mathbf{c}_9 & \mathbf{c}_7^r = \mathbf{c}_{10} & \mathbf{c}_{11}^r = \mathbf{c}_{11} & \mathbf{c}_{15}^r = \mathbf{c}_{12} \\
\mathbf{c}_4^r = \mathbf{c}_{13} & \mathbf{c}_8^r = \mathbf{c}_{14} & \mathbf{c}_{12}^r = \mathbf{c}_{15} & \mathbf{c}_{16}^r = \mathbf{c}_{16}.
\end{array}
$$

4. CONCLUSION

A persymmetric matrix is a square matrix equal to its flip transpose. Some large sizes of persymmetric matrices can be constructed from smaller ones. Based on these properties, we have constructed a new binary reversible self-dual code of length $4n$, by using generator matrices in the standard form. This construction from a self-dual code of length $2n$. However, this construction does not guarantee that all resulting codes are optimal or feasible for practical communication systems. Limitations such as decoding complexity, minimum distance, or code performance under real-world noise conditions need further analysis. In addition, the method has been applied only to binary fields and specific matrix configurations. As future research suggestions, this framework can be extended to other classes of structured matrices, such as centrosymmetric or Toeplitz matrices, or adapted to other finite fields. Moreover, performance evaluation of the constructed codes in practical applications, such as secure communication or DNA data storage, is a promising area to explore. Incorporating error-detection and correction codes into the construction process may also enhance the applicability of the resulting codes.

AUTHOR CONTRIBUTIONS

Ardi Nur Hidayat: Conceptualization, Formal Analysis, Investigation, Methodology, Resources, Writing - Original Draft, Writing - Review and Editing. Vira Hari Krisnawati: Supervision, Validation, Writing - Review and Editing. Abdul Rouf Alghofari: Supervision, Validation, Writing - Review and Editing. All authors discussed the results and contributed to the final manuscript.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interests

REFERENCES

[1] Arthur Cayley ,."II. A memoir on the theory of matrices," *Philos. Trans. R. Soc. Lond.*, vol. 148, pp. 17–37, Dec. 1858, doi: https://doi.org/10.1098/rstl.1858.0002.

[2] B. P. Tomasouw, "KARAKTERISTIK MATRIKS CENTRO-SIMETRIS," *BAREKENG J. Ilmu Mat. Dan Terap.*, vol. 10, no. 2, pp. 69–76, Dec. 2016, doi: https://doi.org/10.30598/barekengvol10iss2pp69-76.

[3] D. Patty, Z. A. Leleury, and J. A. Popla, "LESLIE MATRIX CONSTRUCTION AND ITS PROPERTIES IN PREDICTING THE SIZE OF THE FEMALE POPULATION," *BAREKENG J. Ilmu Mat. Dan Terap.*, vol. 18, no. 1, pp. 0637–0642, Mar. 2024, doi: https://doi.org/10.30598/barekengvol18iss1pp0637-0642.

[4] G. Pailloux, P. Forster, J. P. Ovarlez, and F. Pascal, "ON PERSYMMETRIC COVARIANCE MATRICES IN ADAPTIVE DETECTION," in *2008 IEEE International Conference on Acoustics, Speech and Signal Processing*, Las Vegas, NV: IEEE, Mar. 2008, pp. 2305–2308. doi: https://doi.org/10.1109/ICASSP.2008.4518107.

[5] M. Nouri, "BISYMMETRIC, PERSYMMETRIC MATRICES AND ITS APPLICATIONS IN EIGEN-DECOMPOSITION OF ADJACENCY AND LAPLACIAN MATRICES," *Int. J. Math. Comput. Sci.*, vol. 6, no. 7, pp. 766–769, 2012.

[6] M. Aleroev, T. Aleroev, M. Kirane, and Y.-F. Tang, "ON ONE CLASS OF PERSYMMETRIC MATRICES GENERATED BY BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER," *Appl. Math. Comput.*, vol. 268, pp. 151–163, Oct. 2015, doi: https://doi.org/10.1016/j.amc.2015.06.076.

[7] A. I. Julio and R. L. Soto, "PERSYMMETRIC AND BISYMMETRIC NONNEGATIVE INVERSE EIGENVALUE PROBLEM," *Linear Algebra Its Appl.*, vol. 469, pp. 130–152, Mar. 2015, doi: https://doi.org/10.1016/j.laa.2014.11.025.

[8] R. L. Soto, A. I. Julio, and M. Salas, "NONNEGATIVE PERSYMMETRIC MATRICES WITH PRESCRIBED ELEMENTARY DIVISORS," *Linear Algebra Its Appl.*, vol. 483, pp. 139–157, Oct. 2015, doi: https://doi.org/10.1016/j.laa.2015.05.032.

[9] V. X. Genest, S. Tsujimoto, L. Vinet, and A. Zhedanov, "PERSYMMETRIC JACOBI MATRICES, ISOSPECTRAL DEFORMATIONS AND ORTHOGONAL POLYNOMIALS," *J. Math. Anal. Appl.*, vol. 450, no. 2, pp. 915–928, Jun. 2017, doi: https://doi.org/10.1016/j.jmaa.2017.01.056.

[10] R. Vaia and L. Spadini, "PERSYMMETRIC JACOBI MATRICES WITH SQUARE-INTEGER EIGENVALUES AND DISPERSIONLESS MASS-SPRING CHAINS," *Linear Algebra Its Appl.*, vol. 585, pp. 164–177, Jan. 2020, doi: https://doi.org/10.1016/j.laa.2019.10.002.

[11] Z. Li and X. Wang, "VERIFIED ERROR BOUNDS FOR REAL EIGENVALUES OF REAL SYMMETRIC AND PERSYMMETRIC MATRICES," *Math. Probl. Eng.*, vol. 2020, pp. 1–9, Dec. 2020, doi: https://doi.org/10.1155/2020/9240852.

[12] H. J. Kim, W.-H. Choi, and Y. Lee, "CONSTRUCTION OF REVERSIBLE SELF-DUAL CODES," *Finite Fields Their Appl.*, vol. 67, p. 101714, Oct. 2020, doi: https://doi.org/10.1016/j.ffa.2020.101714.

[13] X. T. Ngo, S. Bhasin, J.-L. Danger, S. Guilley, and Z. Najm, "LINEAR COMPLEMENTARY DUAL CODE IMPROVEMENT TO STRENGTHEN ENCODED CIRCUIT AGAINST HARDWARE TROJAN HORSES," in *2015 IEEE International Symposium on Hardware Oriented Security and Trust (HOST)*, Washington, DC: IEEE, May 2015, pp. 82–87. doi: https://doi.org/10.1109/HST.2015.7140242.

[14] E. Nachmani, E. Marciano, L. Lugosch, W. J. Gross, D. Burshtein, and Y. Be'ery, "DEEP LEARNING METHODS FOR IMPROVED DECODING OF LINEAR CODES," *IEEE J. Sel. Top. Signal Process.*, vol. 12, no. 1, pp. 119–131, Feb. 2018, doi: https://doi.org/10.1109/JSTSP.2017.2788405.

[15] P. K. Kadbe and M. G. Waje, "ERROR DETECTION IN FAULT-TOLERANT REVERSIBLE CIRCUIT USING FREDKIN GATES," in *ICCCE 2021*, vol. 828, A. Kumar and S. Mozar, Eds., in Lecture Notes in Electrical Engineering, vol. 828. , Singapore: Springer Nature Singapore, 2022, pp. 579–585. doi: https://doi.org/10.1007/978-981-16-7985-8_58.

[16] A. N. Hidayat, V. H. Krisnawati, and A. R. Alghofari, "REVERSIBLE SELF-DUAL CODES OVER FINITE FIELD," *CAUCHY J. Mat. Murni Dan Apl.*, vol. 9, no. 2, pp. 287–296, Nov. 2024, doi: https://doi.org/10.18860/ca.v9i2.29116.

[17] H. J. Kim, W.-H. Choi, and Y. Lee, "DESIGNING DNA CODES FROM REVERSIBLE SELF-DUAL CODES OVER GF (4)," *Discrete Math.*, vol. 344, no. 1, p. 112159, Jan. 2021, doi: https://doi.org/10.1016/j.disc.2020.112159.

[18] V. H. Krisnawati, A. R. Alghofari, and A. N. Hidayat, "REVERSIBLE (BISYMMETRIC) SELF-DUAL CODES OVER FINITE FIELDS WITH CHARACTERISTIC TWO AND THEIR APPLICATIONS TO DNA CODES," *Commun. Math. Biol. Neurosci.*, vol. 2025, https://doi.org/10.28919/cmbn/9267.

[19] S. Ling and C. Xing, *CODING THEORY: A FIRST COURSE*, 1st ed. Cambridge University Press, 2004. doi: https://doi.org/10.1017/CBO9780511755279.

[20] F. Gursoy, E. S. Oztas, and I. Siap, "REVERSIBLE DNA CODES USING SKEW POLYNOMIAL RINGS," *Appl. Algebra Eng. Commun. Comput.*, vol. 28, no. 4, pp. 311–320, Aug. 2017, doi: https://doi.org/10.1007/s00200-017-0325-z.