BAREKENG: Journal of Mathematics and Its Applications
December 2025 Volume 19 Issue 4 Page 2891–2906
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol19iss4pp2891-2906

DYNAMICS OF A PREY-PREDATOR MODEL WITH ALLEE EFFECTS AND HOLLING TYPE IV FUNCTIONAL RESPONSE: LOCAL STABILITY AND NUMERICAL EXPLORATION OF BIFURCATIONS

Resmawan Resmawan[1*], **Agus Suryanto**[2], **Isnani Darti**[3],
Hasan S. Panigoro[4]

[1,2,3]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Brawijaya
Jln. Veteran No 10-11, Ketawanggede, Lowokwaru, Malang, Jawa Timur, 65145, Indonesia.

[1,4]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Negeri Gorontalo
Jln. Prof. Dr. Ing. B. J. Habibie, Moutong, Tilongkabila, Bone Bolango, Gorontalo, 96554, Indonesia

*Corresponding author's e-mail: * **resmawan@student.ub.ac.id***

Article History:

Received: 9th March 2025
Revised: 2nd May 2025
Accepted: 17th June 2025
Available online: 1st September 2025

Keywords:

Allee Effect;
Forward bifurcation;
Hopf Bifurcation;
Prey group defence;
Prey-predator model.
.

ABSTRACT

This study presents a prey-predator model incorporating the Allee effect and Holling Type IV Functional Response. The model identifies three equilibrium points: the zero-equilibrium, the predator extinction equilibrium, and the positive equilibrium. Under specific conditions, all these points exhibit local asymptotic stability. The Allee effect is an important factor in determining the stability of the equilibrium point. A weak Allee effect can destabilize the zero-equilibrium point, while a strong Allee effect ensures its local asymptotic stability, potentially leading to the extinction of both species. Additionally, forward and Hopf bifurcation under weak Allee conditions occur at the predator extinction equilibrium point. In contrast, a strong Allee effect may cause bistability between the zero-equilibrium and predator extinction equilibrium points. This evidence suggests that prey can survive without predators; however, a strong Allee effect might result in prey extinction if the population decreases significantly. The Holling Type IV functional response illustrates the impact of prey group defense, which diminishes predation pressure as prey density increases, thereby facilitating the development of limit cycles and establishing a positive equilibrium under specific parameter conditions. This mechanism is crucial for managing predator-prey cohabitation and influencing the system's bifurcation structure. The final section of the study includes numerical simulations to support the analytical findings. The interplay between the Allee effect and the Holling Type IV functional response yields complex dynamics, encompassing bistability, oscillation behavior, and sensitivity to initial conditions. Their collaborative interaction amplifies the system's nonlinearity, enabling the creation of various dynamic behaviors that are extremely sensitive to fluctuations in parameter values.

How to cite this article:

R. Resmawan, A. Suryanto, I. Darti and H. S. Panigoro., "DYNAMICS OF A PREY-PREDATOR MODEL WITH ALLEE EFFECTS AND HOLLING TYPE IV FUNCTIONAL RESPONSE: LOCAL STABILITY AND NUMERICAL EXPLORATION OF BIFURCATIONS," *BAREKENG: J. Math. & App.*, vol. 19, iss. 4, pp. 2891-2906, December, 2025.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*

Research Article · Open Access

1. INTRODUCTION

One of the most significant environmental issues in the world remains the extinction of species. Habitat loss, climate change, and human activity contribute to species' rapid extinction [1]. Several factors, such as the Allee effect, make small groups more susceptible to extinction. This is because the effect makes it harder for individuals to reproduce and survive in low population densities [2]. Besides that, alterations in behavior, such as collective defense among prey species, can influence predator-prey interactions and environmental stability [3]. Mathematical models can be crucial in advancing our understanding of complex ecological interactions and informing effective conservation strategies. Expanding on Malthus's population growth model [4], Lotka [5] and Volterra [6] developed models to examine the interactions between two species. Their work, now called the Lotka-Volterra model, has established a basis for following significant models, including those created by Leslie-Gower [7] and Rosenzweig-MacArthur [8]. The examination of prey-predator models is essential for comprehending various biological processes. Current initiatives seek to develop more accurate models that closely correspond with observable biological events. These developments have facilitated integrating various biological processes into prey-predator models, thus representing the complexity of actual ecological systems more realistically.

Various biological processes in prey and predators have been extensively studied through model development. Specific phenomena have been incorporated into certain models, such as changes in prey age structure [9][10], anti-predator behavior [11], fear effects on prey [12], competition within species [13][14], and disease occurrence among species [12]. Prey-predator models continue to undergo rigorous examination, integrating additional ecological factors relevant to particular species.

The Allee effect plays a crucial role in ecology, particularly for species at risk of extinction, as it outlines the reproductive challenges that heighten the likelihood of species loss. Numerous studies have explored prey-predator models that incorporate the Allee effect, assessing its impact on established models such as the Leslie-Gower model [15], Lotka-Volterra model [16], and Rosenzweig-MacArthur model [17]. Scholars have also investigated the Allee effect using various functional responses, including Holling type I functional response [18], Holling type III functional response [19], Beddington-DeAngelis [20], and Michaelis-Menten [21][22]. Anggriani et al. [23] examined the Allee effect in conjunction with intraspecific competition among predators, while its implications for eco-epidemiological models were explored by Rahmi et al. [24] and Sidik et al. [25]. Studies incorporating the Allee effect on predators can be found in the works of [15] [26][27], whereas the Allee effect on prey is analyzed in models by [28][29][30][31].

Beyond the Allee effect, researchers have observed that prey groups often engage in collective defense against predator attacks [32]. This behavior has been integrated into various mathematical models. For example, Zhang et al. [32] examine Hopf bifurcation in predator-prey models incorporating prey group defense and time delays, representing this defense mechanism through an exponential function. In a different approach, Jiao et al. [33] construct a Leslie-Gower model that includes prey group defense with a threshold value, employing a type IV functional response. Patra et al. [34] also explore prey group defense, utilizing a modified Holling type IV functional response in their study. The collective defensive actions of prey can impact predator population density and improve the survival chances of potential victims.

The ongoing enhancement of models incorporating biological phenomena in predator and prey species aims to create more realistic representations. Our study introduces a prey-predator model that combines the Allee effect and prey group defense. We draw upon the model by [28], which integrates the Allee effect, and the model by [33], which accounts for prey group defense. While [28] utilized a Holling type I functional response in their prey-predator model with the Allee effect, we adapt this by implementing Holling's type IV functional response. This response function offers greater ecological relevance and reflects the defensive mechanisms of prey groups. It suggests that larger prey populations experience reduced predation rates, as they can collectively defend against predators, making it more challenging for predators to capture them. Consequently, the predator attack rate decreases, demonstrating the protective effect of group behavior in prey. Although Jiao et al [33] employed the Holling type IV functional response to illustrate prey group defense, they did not account for the Allee effect. Our research develops a model incorporating the Holling Type IV Functional Response for prey group defense, as presented by [33], and the Allee effect on prey, as examined by [28]. We analyze the resulting prey-predator model for local stability and further explore bifurcations and population dynamics through numerical simulations.

2. RESEARCH METHODS

The research employs a systematic approach to evaluate the suggested mathematical model, as depicted in the flowchart in **Figure 1**.

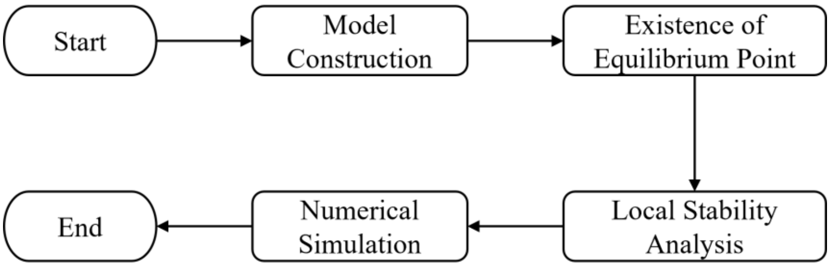


Figure 1. **Flowchart of the Research Stages**

The study starts with the model construction, which is then followed by the existence of equilibrium points and their stability. Numerical simulations are performed to confirm the theoretical results, culminating in the final conclusions and insights. The detailed phases of the research are outlined below.

2.1 Model Construction

This research was conducted by constructing a mathematical model that describes the interaction between two populations: prey and predator. The model was constructed by considering the ecological phenomena of prey and predator in the real world and referring to previously developed models. Following the model's formulation, analytical dynamic analysis was performed, accompanied by numerical simulations to support the analytical results. The following section describes how the model was developed in this research.

The developed model incorporates two key variables: the density of the prey population (X) and the density of the predator population (Y) at a given time t. This model is known as the prey-predator model, which takes into account both the Allee effect and Holling Type IV Functional Response, as shown in system of **Equation (1)**.

$$\begin{aligned} \frac{dX}{dt} &= \rho X\left(1-\frac{X}{\kappa}-\frac{\varphi}{\zeta+X}\right)-\frac{\alpha XY}{\beta+X^2} \\ \frac{dY}{dt} &= \frac{\sigma XY}{\beta+X^2}-\delta Y \end{aligned} \tag{1}$$

with $\rho, \alpha, \beta, \sigma, \varphi, \zeta, \delta$ and κ as a positive parameter. ρ and α represent the intrinsic growth rate of the prey and the predation rate on the prey by the predator, respectively. β represents the environmental protection rate, and σ represent the biomass conversion coefficients, i.e., the conversion rate of prey predation to predator birth. δ and κ represent the natural mortality rate of predators and the environmental carrying capacity of prey, respectively. Meanwhile $\varphi, \zeta > 0$ describes the level of the Allee effect, with φ is the severity of Allee, and ζ is the degree of Allee effect. In particular, if $\varphi < \zeta$ or $\zeta < \varphi$, then **Equation (1)** exhibits a weak or strong Allee effect, respectively [29].

Specifically, **Equation (1)** describes the dynamics of prey population density, denoted by $\frac{dX}{dt}$, and predator population density, denoted by $\frac{dY}{dt}$. The prey population grows logistically, represented by $\rho X\left(1-\frac{X}{\kappa}\right)$, but its growth is further limited by the Allee effect, represented by $\frac{\varphi}{\zeta+X}$. Additionally, the prey population decreases due to predation, characterized by the Holling type IV functional response, represented by $\frac{\alpha XY}{\beta+X^2}$. Meanwhile, the predator population increases through predation on prey, represented by $\frac{\sigma XY}{\beta+X^2}$, and decreases due to natural mortality, represented by δY.

2.2 Existence of Equilibrium Point

To determine the equilibrium points of Equation (1), one must concurrently solve the equations $\frac{dX}{dt} = 0$ and $\frac{dY}{dt} = 0$. This process involves:

$$\rho X\left(1 - \frac{X}{\kappa} - \frac{\varphi}{\zeta + X}\right) - \frac{\alpha XY}{\beta + X^2} = 0, \tag{2}$$

$$\frac{\sigma XY}{\beta + X^2} - \delta Y = 0.$$

2.3 Local Stability Analysis

The stability of the equilibrium points is analyzed using the Jacobian matrix evaluation. If we suppose $f_1 = \frac{dX}{dt}$ and $f_2 = \frac{dY}{dt}$, then the Jacobian matrix from Equation (1) can be defined as:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} \end{bmatrix}.$$

The stability of the equilibrium point can be determined by calculating the eigenvalues of the Jacobian matrix. The equilibrium point is considered locally asymptotically stable if all eigenvalues have negative values.

2.4 Numerical Simulation

Numerical simulations were carried out to support the analytical findings by selecting parameter values that were in agreement with the analytical results. Several variations of parameter values are given to observe the sensitivity of changes in parameter values to the system dynamics.

3. RESULTS AND DISCUSSION

3.1 The Existence of Equilibrium Points

When resolving Equation (2), three distinct categories of equilibrium points emerge: zero-equilibrium, predator-free, and positive.

3.1.1 Zero-equilibrium Point

The state of extinction for all populations is represented by the zero-equilibrium point. This point, symbolized by $E_0(0,0)$, is always exists in $R_+^2 \cup (0,0)$.

3.1.2 Predator-free Equilibrium Point

The predator-free equilibrium points are symbolized by $E_x(X_x, 0),\ x = 1,2,3$. These points are derived from Equation (3):

$$X^2 - (\kappa - \zeta)X + \kappa(\varphi - \zeta) = 0. \tag{3}$$

If we consider X_1 and X_2 as the two solutions to Equation (3), we can conclude that:

$$X_1 = \frac{(\kappa - \zeta) + \sqrt{(\kappa - \zeta)^2 - 4\kappa(\varphi - \zeta)}}{2} \tag{4}$$

$$X_2 = \frac{(\kappa - \zeta) - \sqrt{(\kappa - \zeta)^2 - 4\kappa(\varphi - \zeta)}}{2} \tag{5}$$

The existence of X_x can be determined by examining the Allee effect condition $(\varphi - \zeta)$, the value of $(\kappa - \zeta)$, and the discriminant value from Equation (3), namely:

$$\Pi_1 = (\kappa - \zeta)^2 - 4\kappa(\varphi - \zeta). \tag{6}$$

1. Weak Allee Effect Case

   The weak Allee effect in Equation (1) occurs if $\varphi < \zeta$. If $\varphi < \zeta$, then $\Pi_1 > 0$, which leads to the existence of predator-free equilibrium points depending on the value of $\kappa - \zeta$:

   In Equation (1), the weak Allee effect occurs when $\varphi < \zeta$. When $\varphi < \zeta$, it follows that $\Pi_1 > 0$, resulting in the presence of predator-free equilibrium points that are contingent on the value of $\kappa - \zeta$:

   a. When $\kappa > \zeta$, $\sqrt{(\kappa - \zeta)^2 - 4\kappa(\varphi - \zeta)} > (\kappa - \zeta)$, resulting $X_1 > 0$ and $X_2 < 0$.

   b. When $\kappa < \zeta$, $\sqrt{(\zeta - \kappa)^2 - 4\kappa(\varphi - \zeta)} > (\zeta - \kappa)$, resulting $X_1 > 0$ and $X_2 < 0$.

   Therefore, in the presence of a weak Allee effect within Equation (1), a single predator-free equilibrium point exists, specifically $E_1(X_1, 0)$.

2. Strong Allee Effect Case

   In Equation (1), the strong Allee effect occurs when $\varphi > \zeta$. The existence of predator-free equilibrium points is contingent on the discriminant (Π_1) in Equation (6) and the $(\kappa - \zeta)$ value, provided:

   a. $\Pi_1 < 0$ case

      If $\Pi_1 < 0$ then the predator-free equilibrium point $E_x(X_x, 0)$ do not exist.

   b. $\Pi_1 > 0$ case

      If $\Pi_1 > 0$, then $\varphi < \frac{(\kappa+\zeta)^2}{4K}$. Additionally, the $\kappa - \zeta$ value determines whether the predator-free equilibrium point exists:

      i. When $\kappa > \zeta$, $\sqrt{(\kappa - \zeta)^2 - 4\kappa(\varphi - \zeta)} < (\kappa - \zeta)$, resulting $X_1 > 0$ and $X_2 > 0$. For this scenario, two predator-free equilibrium points exist: specifically, $E_1(X_1, 0)$ and $E_2(X_2, 0)$.

      ii. When $\kappa < \zeta$, $\sqrt{(\kappa - \zeta)^2 - 4\kappa(\varphi - \zeta)} < (\kappa - \zeta)$, resulting $X_1 < 0$ and $X_2 < 0$. In this scenario, $E_1(X_1, 0)$ and $E_2(X_2, 0)$ do not exist.

   c. $\Pi_1 = 0$ case

      When the condition is met, a single predator-free equilibrium point exists, specifically $E_3(X_3, 0)$, with $X_3 = \frac{\kappa-\zeta}{2}$. E_3 exists if $\kappa > \zeta$ and do not exist if $\kappa < \zeta$.

Consequently, there exist three equilibrium points without predators, specifically $E_1(X_1, 0), E_2(X_2, 0)$, and $E_3(X_3, 0)$, whose presence is contingent upon the conditions of the Allee effect. Theorem 1 establishes the conditions under which the predator-free equilibrium point exists under a weak Allee effect. In contrast, Theorem 2 establishes the conditions under which the predator-free equilibrium point exists under a strong Allee effect.

Theorem 1. *If the Equation (1) exhibits a weak Allee effect* $(\varphi < \zeta)$, *then the equilibrium point* $E_1(X_1, 0)$ *exists and is unique*.

Theorem 2. *Let* $\kappa > w$ *and the Equation (1) has a strong Allee effect* $(\varphi > \zeta)$:

1. *If* $\varphi > \frac{(\kappa+\zeta)^2}{4\kappa}$, *then there are no predator-free equilibrium points.*
2. *If* $\varphi = \frac{(\kappa+\zeta)^2}{4\kappa}$, *then there exists exactly one predator-free equilibrium point, namely,* E_3.
3. *If* $\varphi < \frac{(\kappa+\zeta)^2}{4\kappa}$, *then there are two predator-free equilibrium point, namely,* E_1 *and* E_2.

3.1.3 Positive Equilibrium Point

The positive equilibrium points are denoted by $E_i(X_i, Y_i), i = 4,5,6$, which represent the condition where all populations exist, with $X_i$and Y_i are derived from **Equation (7)** and **Equation (8)**.

$$\frac{\sigma X_i}{\beta + X_i^2} - \delta = 0 \tag{7}$$

$$\rho\left(1 - \frac{X_i}{\kappa} - \frac{\varphi}{\zeta + X_i}\right) - \frac{\alpha Y_i}{\beta + X_i^2} = 0 \tag{8}$$

From **Equation (7)**, we obtain

$$\Pi_2 = \sigma^2 - 4\beta\delta^2,\ X_4 = \frac{\sigma + \sqrt{\Pi_2}}{2\delta} \quad \text{and} \quad X_5 = \frac{\sigma - \sqrt{\Pi_2}}{2\delta}. \tag{9}$$

$N_{4,5}$ exists if $\Pi_2 \geq 0$ or $\beta \leq \left(\frac{\sigma}{2\delta}\right)^2$.

From **Equation (8)**, we obtain

$$Y_i = \frac{\rho(\beta + X_i^2)[(\kappa - \zeta)X_i - \kappa(\varphi - \zeta) - X_i^2]}{\kappa\alpha(\zeta + X_i)},\quad i = 4,5,6.$$

Theorem 3 establishes the conditions under which the existence of positive equilibrium point depends on b.

Theorem 3. *Define* $\Pi_2 = \sigma^2 - 4\beta\delta^2$ *and let* $X_4 = \frac{\sigma+\sqrt{\Pi_2}}{2\delta}, X_5 = \frac{\sigma-\sqrt{\Pi_2}}{2\delta}, X_6 = \frac{\sigma}{2\delta}$. *Furthermore, define* $Y_i = \frac{\rho(\beta+X_i^2)[(\kappa-\zeta)X_i-\kappa(\varphi-\zeta)-X_i^2]}{\kappa\alpha(\zeta+X_i)}, i = 4,5,6,$ *with* $(\kappa - \zeta)X_i - \kappa(\varphi - \zeta) - X_i^2 > 0$.

1. *If* $\beta > \left(\frac{\sigma}{2\delta}\right)^2$, *then there are no positive equilibrium points.*
2. *If* $\beta = \left(\frac{\sigma}{2\delta}\right)^2$, *then there exists exactly one positive equilibrium point, namely* $E_6(X_6, Y_6)$.
3. *If* $\beta < \left(\frac{\sigma}{2\delta}\right)^2$, *then there are two positive equilibrium points, namely* $E_4(X_4, Y_4)$ *and* $E_5(X_5, Y_5)$.

3.2 Local Stability

Equation (1) is linearized around its equilibrium point to analyze local stability. The linear component of the linearized model is referred to as the Jacobian matrix. Furthermore, the eigenvalues of the Jacobian matrix are used to assess the stability of the equilibrium points in the prey-predator model [35]. This stability indicates whether the predator and prey populations will remain balanced, fluctuate, or even become extinct, depending on the eigenvalues. This analysis is crucial for understanding how biological interactions and external factors affect the balance of the prey-predator ecosystem. Linearization around the equilibrium point is carried out so that the Jacobian matrix is obtained as

$$J = \begin{bmatrix} \rho - \frac{2\rho}{\kappa}X - \frac{\rho\varphi\zeta}{(\zeta + X)^2} - \frac{\alpha Y(\beta - X^2)}{(\beta + X^2)^2} & -\frac{\alpha X}{\beta + X^2} \\ \frac{\sigma Y}{\beta + X^2} - \frac{2\sigma X^2 Y}{(\beta + X^2)^2} & \frac{\sigma X}{\beta + X^2} - \delta \end{bmatrix}. \tag{10}$$

The stability of the equilibrium points of **Equation (1)** is determined by the Jacobian matrix **Equation (10)**, eigenvalues, and the result is presented in the following theorem. **Theorem 4** provides a condition that determines how the strength of the Allee effect influences the local stability of the zero-equilibrium point.

Theorem 4. *If the Allee effect is strong* $(\varphi > \zeta)$, *then the zero-equilibrium point* $E_0(0,0)$ *is locally asymptotically stable. Conversely, if the Allee effect is weak* $(\varphi < \zeta)$, *this equilibrium point is unstable.*

Proof. By substituting $E_0(0,0)$ to the Jacobian matrix **Equation (10)**, gives

$$J_{E_0} = \begin{bmatrix} \dfrac{\rho(\zeta-\varphi)}{\zeta} & 0 \\ 0 & -\delta \end{bmatrix},$$

and we get two eigen values $\lambda_1 = \frac{\rho(\zeta-\varphi)}{\zeta}$ and $\lambda_2 = -\delta < 0$. Hence, E_0 is locally asymptotically stable if $\varphi > \zeta$ and unstable if $\varphi < \zeta$. ■

Theorem 5 establishes a condition that explains how the local stability of the predator-free equilibrium point, under the weak Allee effect, is influenced by the parameter c.

Theorem 5. *Suppose that the* **Equation (1)** *exhibits a weak Allee effect. The predator-free equilibrium point,* $E_1(X_1, 0)$ *is locally asymptotically stable when* $\sigma < \frac{\delta(\beta+X_1^2)}{X_1}$ *and unstable when* $\sigma > \frac{\delta(\beta+X_1^2)}{X_1}$.

Proof. By substituting $E_1(X_1, 0)$ to the Jacobian matrix **Equation (10)**, we obtain

$$J_{E_1} = \begin{bmatrix} \rho X_1\left(\dfrac{\varphi}{(\zeta+X_1)^2} - \dfrac{1}{\kappa}\right) & -\dfrac{\alpha X_1}{\beta+X_1^2} \\ 0 & \dfrac{\sigma X_1 - \delta(\beta+X_1^2)}{\beta+{X_1}^2} \end{bmatrix}.$$

And we get eigen values $\lambda_1 = \rho X_1\left(\frac{\varphi}{(\zeta+X_1)^2} - \frac{1}{\kappa}\right)$ and $\lambda_2 = \frac{\sigma X_1 - \delta(\beta+X_1^2)}{\beta+{X_1}^2}$. If the Allee effect is weak, then $\varphi < \frac{(K+\zeta)^2}{4\kappa}$, and it can be shown that $\lambda_1 < 0$. Furthermore, it can be shown that the value of λ_2 depends on σ. If $\sigma < \frac{\delta(\beta+X_1^2)}{X_1}$, then $\lambda_2 < 0$ making the predator-free equilibrium point, E_1, is locally asymptotically stable and if $\sigma > \frac{\delta(\beta+X_1^2)}{X_1}$ then $\lambda_2 > 0$ making the predator-free equilibrium point is unstable. ■

Theorem 6 establishes a condition that explains how the local stability of the predator-free equilibrium point, under the strong Allee effect, is influenced by the parameter h.

Theorem 6. *Suppose* $E_x = (X_x, 0), x = 1,2,3,\ \kappa > \zeta,\ \sigma < \frac{\delta(\beta+X^2)}{X}$ *and the* **Equation (1)** *has a strong Allee effect:*

1. *If* $\varphi = \frac{(\kappa+\zeta)^2}{4\kappa}$, *then the predator-free equilibrium point,* E_3 *is non-hyperbolic,*
2. *If* $\varphi < \frac{(\kappa+\zeta)^2}{4\kappa}$, *then the predator-free equilibrium point,* E_1 *is locally asymptotically stable and the predator-free equilibrium point,* E_2 *is unstable (saddle node).*

Proof. By substituting $E_x(X_x, 0)$ to the Jacobian matrix **Equation (10)**, gives

$$J_{E_x} = \begin{bmatrix} \rho X_x\left(\dfrac{\varphi}{(\zeta+X_x)^2} - \dfrac{1}{\kappa}\right) & -\dfrac{\alpha X_x}{\beta+X_x^2} \\ 0 & \dfrac{\sigma X - \delta(\beta+X_x^2)}{\beta+X_x^2} \end{bmatrix},$$

where its eigenvalues are,

$$\lambda_1 = \rho X_x\left(\frac{\varphi}{(\zeta+X_x)^2} - \frac{1}{\kappa}\right) \quad \text{and} \quad \lambda_2 = \frac{\sigma X - \delta(\beta+X_x^2)}{\beta+X_x^2}.$$

We can show that $\lambda_2 < 0$ if $\sigma < \frac{\delta(\beta+X_x^2)}{X_x}$. Furthermore, λ_1 is depend on the Allee effect case. For the strong Allee effect ($\varphi > \zeta$), we have the following case:

1. If $\varphi = \frac{(\kappa+\zeta)^2}{4\kappa}$, then $\lambda_1 = 0$. Since $\lambda_1 = 0$, the predator-free equilibrium point $E_3\left(\frac{\kappa-\zeta}{2}, 0\right)$ is non-hyperbolic.

2. If $\varphi < \frac{(\kappa+\zeta)^2}{4\kappa}$, then

$$\lambda_1 = \rho X_x\left(\frac{\varphi}{(\zeta+X_x)^2} - \frac{1}{\kappa}\right) < \frac{\rho X_{1,2}}{\kappa}\left(\frac{(\kappa+\zeta)^2 - \left((\kappa+\zeta) \pm \sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi}\right)^2}{\left((\kappa+\zeta) \pm \sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi}\right)^2}\right).$$

   a. For E_1,

$$\lambda_1 < \frac{\rho X_1}{\kappa}\left(\frac{(\kappa+\zeta)^2 - \left((\kappa+\zeta) + \sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi}\right)^2}{\left((\kappa+\zeta) + \sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi}\right)^2}\right).$$

   Furthermore, it is shown that

$$\begin{aligned}(\kappa+\zeta)^2 - \left((\kappa+\zeta) + \sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi}\right)^2 &= -2(\kappa+\zeta)\sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi} - (\kappa+\zeta)^2 + 4\kappa\varphi \\ &< -2(\kappa+\zeta)\sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi} - (\kappa+\zeta)^2 + (\kappa+\zeta)^2 \\ &= -2(\kappa+\zeta)\sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi} < 0.\end{aligned}$$

   Since $\lambda_1 < 0$ and $\lambda_2 < 0$, then E_1 is locally asymptotically stable.

   b. For E_2,

$$\lambda_1 < \frac{\rho X_2}{\kappa}\left(\frac{(\kappa+\zeta)^2 - \left((\kappa+\zeta) - \sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi}\right)^2}{\left((\kappa+\zeta) - \sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi}\right)^2}\right)$$

   Moreover, it has been demonstrated that

$$\begin{aligned}(\kappa+\zeta)^2 - \left((\kappa+\zeta) - \sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi}\right)^2 &= 2(\kappa+\zeta)\sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi} - (\kappa+\zeta)^2 + 4\kappa\varphi \\ &< 2(\kappa+\zeta)\sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi} - (\kappa+\zeta)^2 + (\kappa+\zeta)^2 \\ &= 2(\kappa+\zeta)\sqrt{(\kappa+\zeta)^2 - 4\kappa\varphi} > 0\end{aligned}$$

   Since $\lambda_1 > 0$ and $\lambda_2 < 0$, then E_2 is unstable (saddle-node). ■

Theorem 7 provides a condition that elucidates how the local stability of the positive equilibrium point is affected by the parameter β.

Theorem 7. *Suppose* $E_i = (X_i, Y_i)$, $i = 4,5,6$, $\Pi_2 = \sigma^2 - 4\beta\delta^2$, $X_4 = \frac{\sigma+\sqrt{\Pi_2}}{2\delta}$, $X_5 = \frac{\sigma-\sqrt{\Pi_2}}{2\delta}$, $X_6 = \frac{\sigma}{2\delta}$, *and* $Y_i = \frac{\rho(\beta+X_i^2)[(\kappa-\zeta)X_i-\kappa(\varphi-\zeta)-X_i^2]}{\kappa\alpha(\zeta+X_i)}$. *Also suppose* $Y_4 > 0$, $Y_5 > 0$, *and* $Y_6 > 0$.

1. *If* $\beta = \left(\frac{\sigma}{2\delta}\right)^2$, *then the positive equilibrium point* E_6 *is non-hyperbolic.*
2. *If* $\beta < \left(\frac{\sigma}{2\delta}\right)^2$, *then the positive equilibrium point* E_4 *is unstable. Moreover, if* $tr(J_{E_i}) < 0$, *then* E_4 *is saddle-node and* E_5 *is locally asymptotically stable.*

Proof. By substituting $E_i(X_i, 0)$ to the Jacobian matrix **Equation (10)**, we obtain

$$J_{E_i} = \begin{bmatrix} \dfrac{\rho\varphi X_i}{(\zeta + X_i)^2} + \dfrac{2\alpha X_i^2 Y_i}{\left(\beta + X_i^2\right)^2} - \dfrac{\rho X_i}{\kappa} & -\dfrac{\delta}{\sigma} \\ \dfrac{(\sigma - 2\delta X_i)Y_i}{\beta + X_i^2} & 0 \end{bmatrix}, \tag{11}$$

where

$$X_4 = \frac{\sigma + \sqrt{\Pi_2}}{2\delta}, X_5 = \frac{\sigma - \sqrt{\Pi_2}}{2\delta}, X_6 = \frac{\sigma}{2\delta}, Y_i = \frac{\rho\left(\beta + X_i^2\right)\left[(\kappa - \zeta)X_i - \kappa(\varphi - \zeta) - X_i^2\right]}{\kappa\alpha(\zeta + X_i)}, i = 4,5,6.$$

From **Equation (11)**, the determinant and trace of the Jacobian matrix are obtained as follows:

$$\det\left(J_{E_i}\right) = -\left(-\frac{\delta}{\sigma}\right)\left(\frac{(\sigma - 2\delta X_i)Y_i}{\beta + X_i^2}\right) = \frac{\delta(\sigma - 2\delta X_i)Y_i}{\sigma\left(\beta + X_i^2\right)},$$

$$\operatorname{tr}\left(J_{E_i}\right) = \frac{\rho\varphi X_i}{(\zeta + X_i)^2} + \frac{2\alpha X_i^2 Y_i}{\left(\beta + X_i^2\right)^2} - \frac{\rho X_i}{\kappa}.$$

Furthermore, the stability of E_i can be determined by examining the determinant and trace of J_{E_i}:

1. For $E_6(X_6, Y_6)$, we obtain $\det\left(J_{E_6}\right) = 0$, so that E_6 is non-hyperbolic.
2. For $E_4(X_4, Y_4)$, we can show that

   $$\sigma - 2\delta X_4 = \sigma - 2\delta\left(\frac{\sigma + \sqrt{\sigma^2 - 4\delta^2\beta}}{2\delta}\right) = -\sqrt{\sigma^2 - 4\delta^2\beta} < 0.$$

   Since $\sigma - 2\delta X_4 < 0$, then $\det\left(J_{E_4}\right) < 0$. Hence, E_4 is unstable. Moreover, if $\operatorname{tr}\left(J_{E_4}\right) < 0$, then E_4 is saddle-node.
3. For $E_5(X_5, Y_5)$, we can show that

   $$\sigma - 2\delta X_5 = \sigma - 2\delta\left(\frac{\sigma - \sqrt{\sigma^2 - 4\delta^2\beta}}{2\delta}\right) = \sqrt{\sigma^2 - 4\delta^2\beta} > 0.$$

   Since $\sigma - 2\delta X_5 > 0$, then $\det\left(J_{E_5}\right) > 0$. Hence, E_5 is locally asymtotically stable if $\operatorname{tr}\left(J_{E_5}\right) < 0$. ∎

3.3 Numerical Simulation

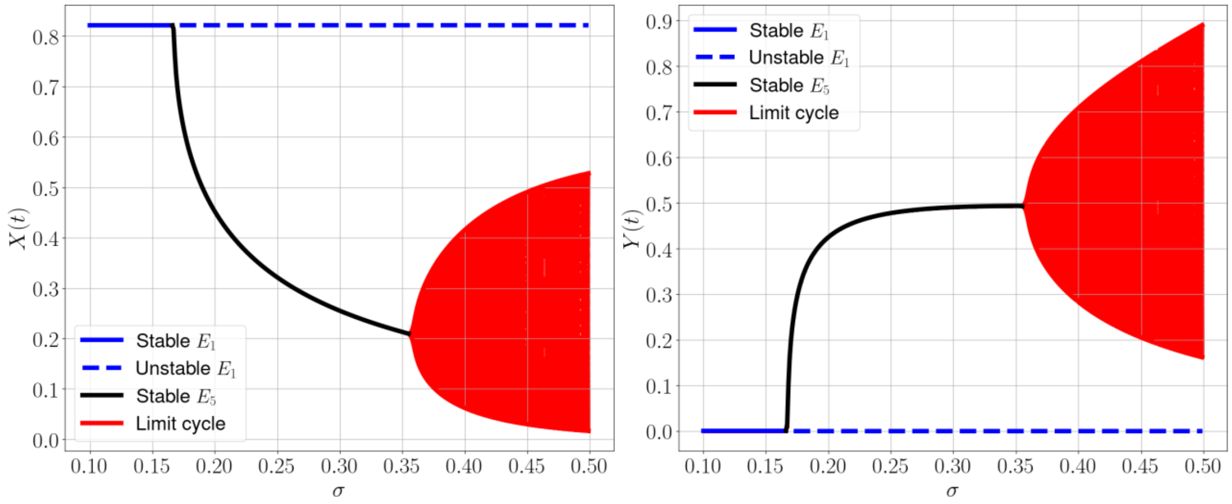
This work does not particularly investigate real-world ecological scenarios. Instead, it concentrates on the dynamics of prey-predator interactions, incorporating ecological elements such as the Allee effect and prey group defense. Consequently, the parameter values employed in the numerical simulations are hypothetically selected to satisfy the mathematical conditions and constraints established in the preceding analytical results. The analysis indicates that the predation conversion rate (σ) and the environmental protection rate (β) affect the model's stability; therefore, numerical simulations are performed by altering the parameters σ and β. The parameter values listed in **Table 1** were chosen to facilitate the simulation.

Table 1. Hypothetical Parameter Values

Parameter	ρ	κ	ζ	α	δ
Value	1.00	1.00	0.30	0.60	0.10

3.3.1 The Influence of Predation Conversion Rate

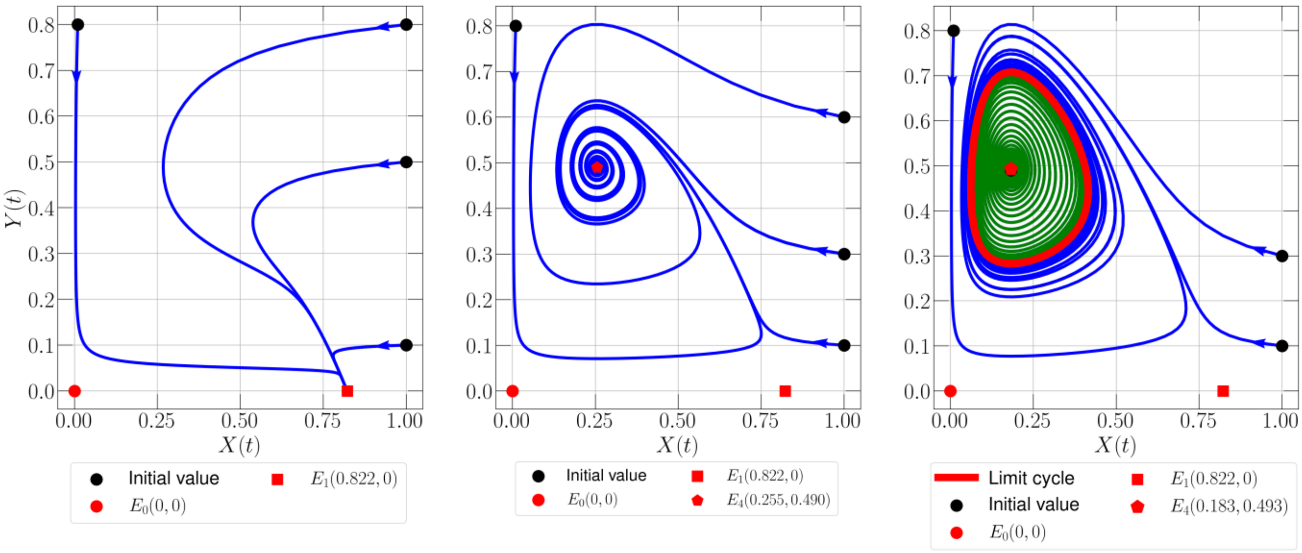
The simulations presented in this section employ the parameter values listed in **Table 1**, $\beta = 0.70$ and utilize the predation conversion rate $\sigma \in [0.10,0.50]$. **Figure 2** displays bifurcation diagrams that demonstrate how elevating the predation conversion rate affects the convergence of the system's solution under weak Allee effect conditions.



(a) $\sigma - X(t)$ (b) $\sigma - Y(t)$

Figure 2. Bifurcation Diagrams of the Equation (1) With a Weak Allee Effect ($\varphi = 0.2$) and Parameter Values as Given in Table 1
(a) Influence of σ on Prey (X) and (b) Influence of σ on Predator (Y)

The bifurcation diagram depicted in Figure 2 highlights two critical bifurcation points, $\sigma_1^* \approx 0.167$ and $\sigma_2^* \approx 0.359$, which are connected to fluctuations in the predation conversion rate. A forward bifurcation causes the equilibrium point E_1 to transition from a locally asymptotically stable state to an unstable one. Specifically, E_1 maintains local asymptotic stability when $\sigma < \sigma_1^*$, but loses this stability when $\sigma > \sigma_1^*$. For $\sigma > \sigma_1^*$, the positive point E_5 achieves local asymptotic stability, provided that E_1 is unstable and $\sigma < \sigma_2^*$. The occurrence of a Hopf bifurcation is signaled by the formation of a limit cycle surrounding E_5 at and $\sigma = \sigma_2^*$. Figure 3 provides a visual representation of these stability alterations within the phase portrait.

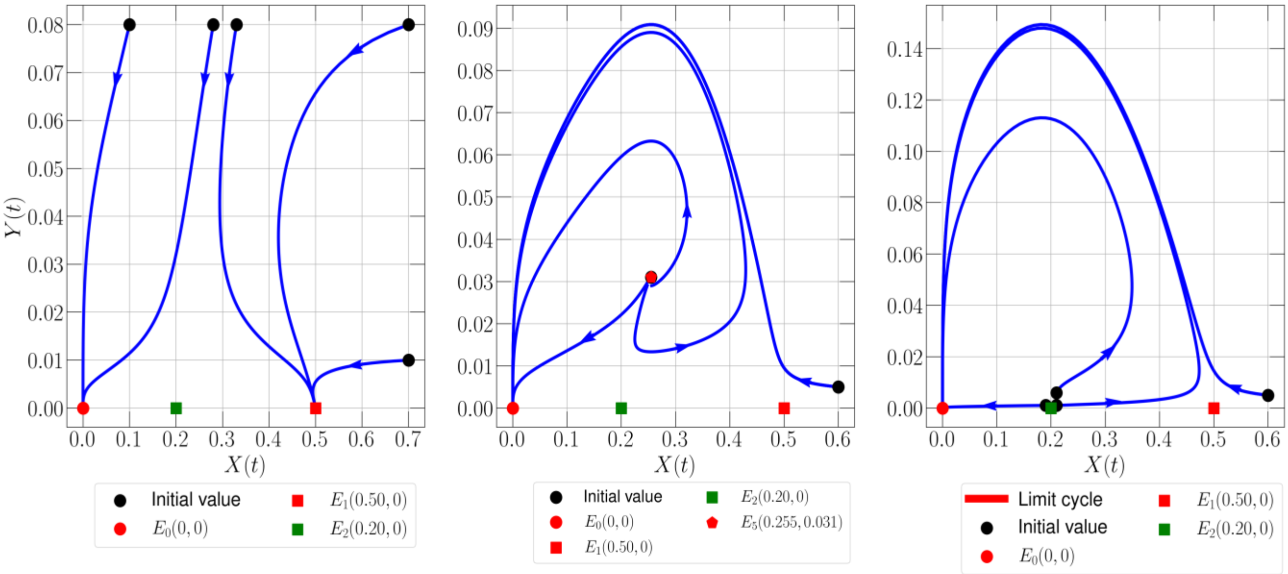


(a) $\sigma = 0.10$ (b) $\sigma = 0.30$ (c) $\sigma = 0.40$

Figure 3. The Phase Portraits of the Equation (1) With a Weak Allee Effect ($\varphi = 0.2$), Demonstrate the Influence of Increasing Predation Conversion Rates on the Stability of Equilibrium Points, Highlighting the Transition from Stable Predator-Free Equilibrium to Oscillatory Behavior
(a) Small Predation Conversion Rate, (b) Medium Predation Conversion Rate, (c) High Predation Conversion Rate

Figure 3 displays the population dynamics of Equation (1) under a weak Allee effect through phase portraits. When the predation conversion rate is $\sigma = 0.10$ (Figure 3 (a)), the system exhibits two equilibrium points: the zero-equilibrium point $E_0(0,0)$ and the predator-free equilibrium point $E_1(0.822,0)$. The system converges to E_1, signifying predator extinction and prey survival. As the predation rate increases to $\sigma = 0.30$ (Figure 3 (b)), three equilibrium points emerge, with convergence to the positive equilibrium point $E_5(0.255,0.490)$, indicating stable coexistence of both populations. At $\sigma = 0.40$ (Figure 3 (c)), the system maintains three equilibrium points but converges to a limit cycle around E_5, suggesting long-term stable oscillations between prey and predator populations. These simulations imply that a slight increase in predation efficiency could potentially disrupt prey-predator coexistence, leading to population oscillations in real-world scenarios.

Further simulations explore the impact of the predation conversion rate on population dynamics characterized by a strong Allee effect in **Equation (1)**. **Figure 4** presents phase portraits for various σ values, with corresponding parameter values listed in **Table 1**.



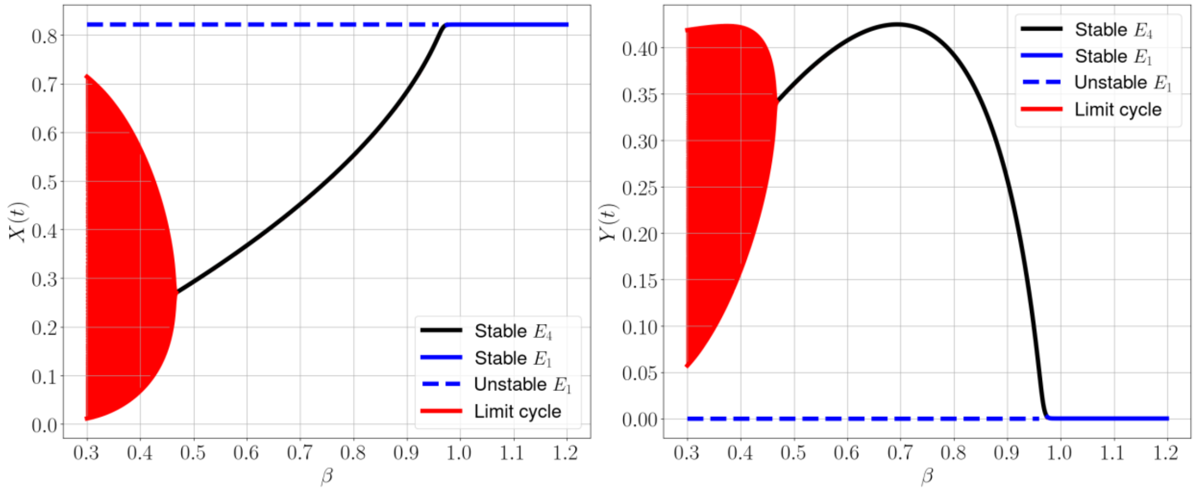
(a) $\sigma = 0.10$ (b) $\sigma = 0.30$ (c) $\sigma = 0.40$

Figure 4. The Phase Portraits of the Equation (1) With a Strong Allee Effect ($\varphi = 0.4$), Demonstrate the Influence of Increasing Predation Conversion Rates on the Stability of Equilibrium Points, Highlighting the Potential Extinction of Both Populations (a) Small Predation Conversion Rate, (b) Medium Predation Conversion Rate, (c) High Predation Conversion Rate

Figure 4 displays the population dynamics of **Equation (1)** with a strong Allee effect through phase portraits. When the predation conversion rate is $\sigma = 0.10$ (**Figure 4** (a)), the system exhibits three equilibrium points: the zero-equilibrium point $E_0(0,0)$ and two predator-free equilibrium points $E_1(0.50,0)$ and $E_2(0.20,0)$. The system's convergence to either E_0 or E_1 indicates bistability, suggesting that the initial population size determines prey survival. At $\sigma = 0.30$ **Figure 4** (b), four equilibrium points emerge: the zero-equilibrium point $E_0(0,0)$, predator-free equilibrium points $E_1(0.50,0)$ and $E_2(0.20,0)$, and the positive equilibrium point $E_5(0.255,0.031)$. The system's convergence to E_0 signifies the extinction of both populations. In **Figure 4** (c), where $\sigma = 0.40$, three equilibrium points exist: the zero-equilibrium point $E_0(0,0)$ and predator-free equilibrium points $E_1(0.50,0)$ and $E_2(0.20,0)$. The positive equilibrium point does not exist, and the system converges to E_0, implying that increased predation conversion rates lead to the extinction of both populations. These simulations indicate that in real-world scenarios, a minor increase in predation efficiency, coupled with strong Allee effects, may result in population extinction.

3.3.2 The Influence of Environmental Protection Rate

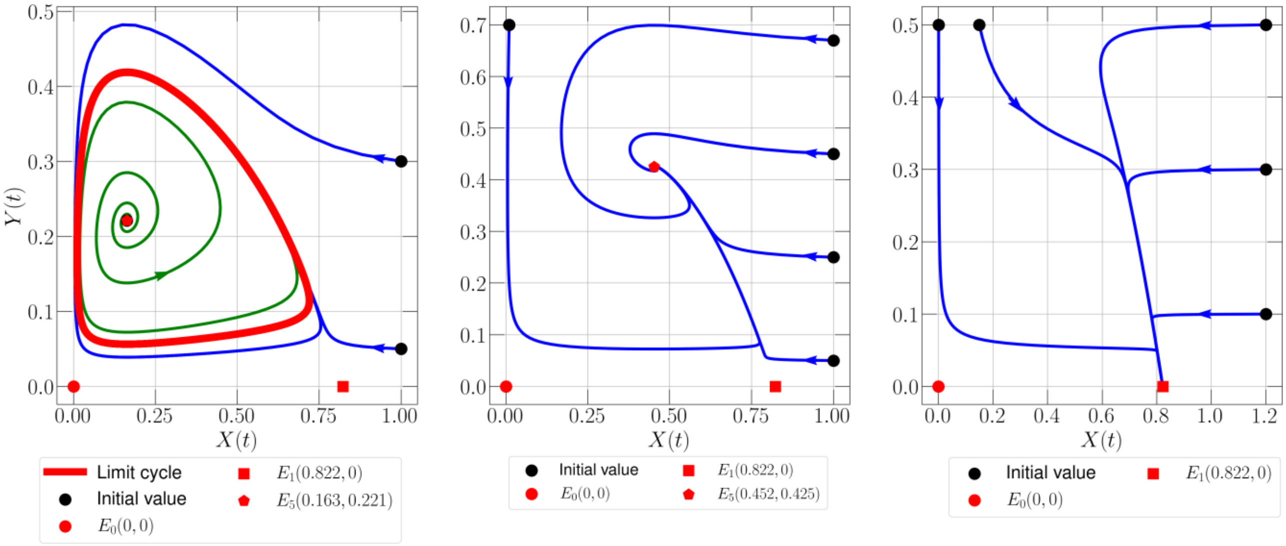
The parameter values listed in **Table 1**, $\sigma = 0.20$ and the environmental protection rate $\beta \in [0.1, 1.2]$ are utilized in the simulations for this section. **Figure 5** presents bifurcation diagrams that demonstrate how elevating the environmental protection rate affects the convergence of the system's solution under weak Allee effect conditions.



(a) $\beta - X(t)$ (a) $\beta - X(t)$

Figure 5. Bifurcation Diagrams of the Equation (1) With a Weak Allee Effect ($\varphi = 0.2$) and Parameter Values as Given in Table 1
(a) Influence of β on Prey (X) and (b) Influence of β on Predator (Y)

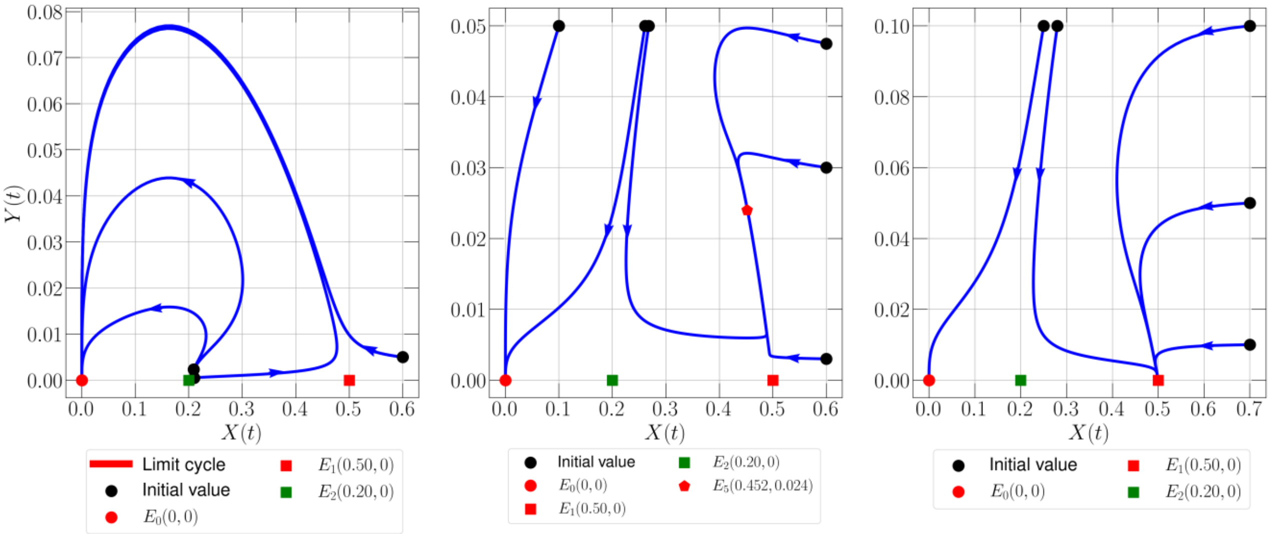
Figure 5 displays a bifurcation diagram revealing two critical points for environmental protection rate changes: $\beta_1^* \approx 0.465$ and $\beta_2^* \approx 0.972$. The first bifurcation point indicates a Hopf bifurcation, transitioning from a limit cycle to a stable positive point, E_5. When $\beta < \beta_1^*$, a limit cycle surrounds E_5. As β reaches $\beta_1^* < \beta < \beta_2^*$, the point E_5 becomes locally asymptotically stable, while E_1 loses stability. Once β exceeds β_2^*, a forward bifurcation occurs, causing E_5 to become unstable and E_1 to gain locally asymptotically stable. These stability shifts are visually represented in the phase portrait shown in Figure 6.



(a) $\beta = 0.30$ (b) $\beta = 0.70$ (c) $\beta = 1.10$

Figure 6. The Phase Portraits of the Equation (1) With a Weak Allee Effect ($\varphi = 0.2$), Demonstrate the Influence of Increasing Environmental Protection Rates on the Stability of Equilibrium Points, Highlighting the Transition from Oscillatory Behavior to Stable Predator-Free Equilibrium
(a) Small Environmental Protection Rate, (b) Medium Environmental Protection Rate, (c) High Environmental Protection Rate

Figure 6 displays the population dynamics of Equation (1) under a weak Allee effect through phase portraits. When $\beta = 0.30$ (Figure 6 (a)), the system exhibits three equilibrium points: the zero-equilibrium point $E_0(0,0)$, the predator-free equilibrium point $E_1(0.822,0)$, and the positive equilibrium point $E_5(0.163,0.221)$. The solution oscillates stably around E_5, forming a limit cycle. For $\beta = 0.70$ (Figure 6 (b)), three equilibrium points persist, with the solution converging to $E_5(0.452,0.425)$. This indicates that E_5 is locally asymptotically stable, while E_0 and E_1 are unstable, suggesting stable positive conditions. At $\beta = 1.10$ (Figure 6 (c)), only E_0 and E_1 remain, and the solution converges to E_1, signifying predator extinction and prey survival. Furthermore, simulations demonstrate how the environmental protection rate influences population dynamics in **Equation (1)** with a strong Allee effect. **Figure 7** presents phase portraits using parameter values from **Table 1** and varying environmental protection rates (β).



(a) $\beta = 0.30$ (b) $\beta = 0.70$ (c) $\beta = 1.10$

Figure 7. The Phase Portraits of the System (1) With a Strong Allee Effect ($\varphi = 0.4$), Demonstrate the Influence of Increasing Environmental Protection Rates on the Stability of Equilibrium Points, Highlighting the Transition from Potential Extinction of Both Populations to Potential Survival (a) Small Environmental Protection Rate, (b) Medium Environmental Protection Rate, (c) High Environmental Protection Rate

Figure 7 displays phase portraits depicting population dynamics in **Equation (1)** under a strong Allee effect. When the environmental protection rate is $\beta = 0.30$ (**Figure 7** (a)), three equilibrium points exist: the zero-equilibrium point $E_0(0,0)$ and two predator-free equilibrium points $E_1(0.5,0)$ and $E_2(0.2,0)$. As E_0 is stable and the others are unstable, solutions converge to E_0, suggesting possible extinction. As the protection rate increases to $\beta = 0.70$ (**Figure 7** (b)), a positive point $E_5(0.452,0.024)$ emerges, and solutions converge to either E_0 or E_5. This demonstrates bistability, where survival is contingent on the initial population size. At $\beta = 1.10$ (**Figure 7** (c)), only E_0 and E_1 remain, with solutions converging to these points. This indicates potential extinction unless the initial prey population surpasses the Allee threshold.

This study has analytically proven local stability and the occurrence of bifurcations through numerical simulation. Future research may focus on broadening existing findings to understand the system's dynamics better. Future research could be done to derive global stability criteria through mathematical analysis. Moreover, analytically demonstrating the existence of bifurcations before numerical validation would enhance the theoretical framework of the model. Alongside the study of stability and bifurcations, forthcoming enhancements to the model could integrate additional significant ecological variables to augment its realism and applicability for practical conservation initiatives.

4. CONCLUSION

This research examines the dynamic behavior of a prey-predator model that incorporates both the Allee effect and Holling Type IV functional response, with several conclusions following:

1. The model reveals three equilibrium points: zero, predator extinction, and positive, each demonstrating local asymptotic stability under certain conditions. When a weak Allee effect is present, the zero-equilibrium point becomes unstable; however, with a strong Allee effect, it exhibits local asymptotic stability. A significant Allee effect may result in both populations becoming extinct. In the case of a weak Allee effect, forward and Hopf bifurcations occur at the predator extinction equilibrium point. Conversely, a strong Allee effect indicates bistability at zero and predator extinction equilibrium points. This suggests that prey can survive without predators, but a strong Allee effect might lead to prey extinction if the population falls below a critical threshold.

2. The study concludes with numerous numerical simulations that support these findings. The model can be applied to study the population dynamics of endangered species in their interactions with predators, particularly species that exhibit social behaviors such as group defense and potential extinction due to ecological factors. These factors may include difficulties finding mates, challenges in regeneration, monogamous living, or living in small groups.
3. In ecosystem management, this model can assist in designing more effective predator control or prey conservation strategies by accounting for how group protection and population density influence prey-predator interactions. Additionally, the model can be utilized to predict the impact of human interventions, such as habitat modifications or species protection measures, on ecosystem balance.

AUTHOR CONTRIBUTIONS

Resmawan: Conceptualization, Formal Analysis and Investigation, Writing – Original Draft Preparation, Resources, Visualization. Agus Suryanto: Conceptualization, Writing - Review and Editing, Supervision, Validation. Isnani Darti: Conceptualization, Methodology, Writing - Review and Editing, Supervision. Hasan S. Panigoro: Conceptualization, Software, Writing - Review and Editing, Supervision. All authors discussed the results and contributed to the final manuscript.

FUNDING STATEMENT

This work was supported by the Ministry of Higher Education, Science, and Technology of the Republic of Indonesia through the Center for Higher Education Funding and Assessment (PPAPT) and Indonesia Endowment Funds for Education (LPDP) for providing the Indonesian Education Scholarship (BPI-Beasiswa Pendidikan Indonesia).

ACKNOWLEDGMENT

The authors would like to express their sincere gratitude to R.R. Musafir for her invaluable assistance during the research process. Her support and contributions were instrumental in the successful completion of this study. The authors also extend their appreciation to the anonymous reviewers and the editor for their constructive comments and suggestions, which have significantly improved the quality of the manuscript. We extend our gratitude to the International Symposium on Biomathematics (Symomath) 2024 organizers for providing a platform to present and discuss this research.

CONFLICT OF INTEREST

The authors declare that no conflicts of interest exist in this study.

REFERENCES

[1] D. Leclère *et al.*, “BENDING THE CURVE OF TERRESTRIAL BIODIVERSITY NEEDS AN INTEGRATED STRATEGY,” *Nature*, vol. 585, no. 7826, pp. 551–556, Sep. 2020, doi: https://doi.org/10.1038/s41586-020-2705-y.

[2] L. Berec, E. Angulo, and F. Courchamp, “MULTIPLE ALLEE EFFECTS AND POPULATION MANAGEMENT,” *Trends Ecol Evol*, vol. 22, no. 4, pp. 185–191, Apr. 2007, doi: https://doi.org/10.1016/j.tree.2006.12.002.

[3] G. Beauchamp, *SOCIAL PREDATION: HOW GROUP LIVING BENEFITS PREDATORS AND PREY*. New York: Elsevier, 2014. doi: https://doi.org/10.1016/C2012-0-03532-6.

[4] T. R. Malthus, *An Essay on the Principle of Population*. London: Reeves and Turner, 1872.

[5] A. J. Lotka, *Elements of Physical Biology*. Baltimore: Williams and Wikkins, 1925.

[6] V. Volterra, *VARIAZIONI E FLUTTUAZIONI DEL NUMERO D'INDIVIDUI IN SPECIE ANIMALI CONVIVENTI*. Societá anonima tipografica" Leonardo da Vinci", 1927.

[7] P. H. Leslie and J. C. Gower, "THE PROPERTIES OF A STOCHASTIC MODEL FOR THE PREDATOR-PREY TYPE OF INTERACTION BETWEEN TWO SPECIES," *Biometrika*, vol. 47, no. 3/4, p. 219, 1960, doi: https://doi.org/10.2307/2333294.

[8] M. L. Rosenzweig and R. H. MacArthur, "GRAPHICAL REPRESENTATION AND STABILITY CONDITIONS OF PREDATOR-PREY INTERACTIONS," *Am Nat*, vol. 97, no. 895, pp. 209–223, 1963, doi: https://doi.org/10.1086/282272.

[9] L. K. Beay, A. Suryanto, I. Darti, and Trisilowati, "HOPF BIFURCATION AND STABILITY ANALYSIS OF THE ROSENZWEIG-MACARTHUR PREDATOR-PREY MODEL WITH STAGE-STRUCTURE IN PREY," *Mathematical Biosciences and Engineering*, vol. 17, no. 4, pp. 4080–4097, 2020, doi: https://doi.org/10.3934/mbe.2020226.

[10] S. Zhang, S. Yuan, and T. Zhang, "A PREDATOR-PREY MODEL WITH DIFFERENT RESPONSE FUNCTIONS TO JUVENILE AND ADULT PREY IN DETERMINISTIC AND STOCHASTIC ENVIRONMENTS," *Appl Math Comput*, vol. 413, p. 126598, 2022, doi: https://doi.org/10.1016/j.amc.2021.126598.

[11] S. Sirisubtawee, N. Khansai, and A. Charoenloedmongkhon, "INVESTIGATION ON DYNAMICS OF AN IMPULSIVE PREDATOR–PREY SYSTEM WITH GENERALIZED HOLLING TYPE IV FUNCTIONAL RESPONSE AND ANTI-PREDATOR BEHAVIOR," *Adv Differ Equ*, vol. 2021, no. 1, p. 160, Mar. 2021, doi: https://doi.org/10.1186/s13662-021-03324-w.

[12] A. S. Purnomo, I. Darti, A. Suryanto, and W. M. Kusumawinahyu, "FEAR EFFECT ON A MODIFIED LESLIE-GOWER PREDATOR-PREY MODEL WITH DISEASE TRANSMISSION IN PREY POPULATION.," *Engineering Letters*, vol. 31, no. 2, pp. 764–773, 2023.

[13] Y. Long, L. Wang, and J. Li, "UNIFORM PERSISTENCE AND MULTISTABILITY IN A TWO-PREDATOR–ONE-PREY SYSTEM WITH INTER-SPECIFIC AND INTRA-SPECIFIC COMPETITION," *J Appl Math Comput*, vol. 68, no. 2, pp. 767–794, 2022, doi https://doi.org/10.1007/s12190-021-01551-8.

[14] H. S. Panigoro, E. Rahmi, and R. Resmawan, "BIFURCATION ANALYSIS OF A PREDATOR–PREY MODEL INVOLVING AGE STRUCTURE, INTRASPECIFIC COMPETITION, MICHAELIS–MENTEN TYPE HARVESTING, AND MEMORY EFFECT," *Front Appl Math Stat*, vol. 8, 2023, doi: https://doi.org/10.3389/fams.2022.1077831.

[15] E. Rahmi, I. Darti, A. Suryanto, Trisilowati, and H. S. Panigoro, "STABILITY ANALYSIS OF A FRACTIONAL-ORDER LESLIE-GOWER MODEL WITH ALLEE EFFECT IN PREDATOR," *J Phys Conf Ser*, vol. 1821, no. 1, p. 12051, 2021, doi: https://doi.org/10.1088/1742-6596/1821/1/012051.

[16] H. S. Panigoro and E. Rahmi, "COMPUTATIONAL DYNAMICS OF A LOTKA-VOLTERRA MODEL WITH ADDITIVE ALLEE EFFECT BASED ON ATANGANA-BALEANU FRACTIONAL DERIVATIVE," *Jambura Journal of Biomathematics (JJBM)*, vol. 2, no. 2, pp. 96–103, 2021, doi: https://doi.org/10.34312/jjbm.v2i2.11886.

[17] H. S. Panigoro and E. Rahmi, "IMPACT OF FEAR AND STRONG ALLEE EFFECTS ON THE DYNAMICS OF A FRACTIONAL-ORDER ROSENZWEIG-MACARTHUR MODEL," *Nonlinear Dynamics and Applications*, pp. 611–619, 2022, doi: https://doi.org/10.1007/978-3-030-99792-2_50.

[18] Y. Ye, H. Liu, Y. Wei, K. Zhang, M. Ma, and J. Ye, "DYNAMIC STUDY OF A PREDATOR-PREY MODEL WITH ALLEE EFFECT AND HOLLING TYPE-I FUNCTIONAL RESPONSE," *Adv Differ Equ*, vol. 2019, no. 1, p. 369, Dec. 2019, doi: https://doi.org/10.1186/s13662-019-2311-1.

[19] K. Baisad and S. Moonchai, "ANALYSIS OF STABILITY AND HOPF BIFURCATION IN A FRACTIONAL GAUSS-TYPE PREDATOR–PREY MODEL WITH ALLEE EFFECT AND HOLLING TYPE-III FUNCTIONAL RESPONSE," *Adv Differ Equ*, vol. 2018, no. 1, p. 82, Dec. 2018, doi: https://doi.org/10.1186/s13662-018-1535-9.

[20] E. Rahmi, I. Darti, A. Suryanto, and Trisilowati, "A MODIFIED LESLIE–GOWER MODEL INCORPORATING BEDDINGTON–DEANGELIS FUNCTIONAL RESPONSE, DOUBLE ALLEE EFFECT AND MEMORY EFFECT," *Fractal and Fractional*, vol. 5, no. 3, p. 84, 2021, doi: https://doi.org/10.3390/fractalfract5030084.

[21] H. S. Panigoro, E. Rahmi, A. Suryanto, and I. Darti, "A FRACTIONAL ORDER PREDATOR–PREY MODEL WITH STRONG ALLEE EFFECT AND MICHAELIS–MENTEN TYPE OF PREDATOR HARVESTING," in *The 8th Symposium on Biomathematics (SYMOMATH) 2021: Bridging Mathematics and Covid-19 Through Multidisciplinary Collaboration*, AIP Conference Proceedings , 2022, p. 20018. doi: https://doi.org/10.1063/5.0082684.

[22] E. Rahmi, I. Darti, A. Suryanto, and Trisilowati, "GLOBAL DYNAMICS OF A FRACTIONAL-ORDER LESLIE-GOWER MODEL WITH ALLEE EFFECT," in *The 8th Symposium on Biomathematics (SYMOMATH) 2021: Bridging Mathematics and Covid-19 Through Multidisciplinary Collaboration*, AIP Conference Proceedings, 2022, p. 20014. doi: https://doi.org/10.1063/5.0082943.

[23] N. Anggriani, H. S. Panigoro, E. Rahmi, O. J. Peter, and S. A. Jose, "A PREDATOR–PREY MODEL WITH ADDITIVE ALLEE EFFECT AND INTRASPECIFIC COMPETITION ON PREDATOR INVOLVING ATANGANA–BALEANU–CAPUTO DERIVATIVE," *Results Phys*, vol. 49, p. 106489, 2023, doi: https://doi.org/10.1016/j.rinp.2023.106489.

[24] E. Rahmi, I. Darti, A. Suryanto, and T. Trisilowati, "A FRACTIONAL-ORDER ECO-EPIDEMIOLOGICAL LESLIE–GOWER MODEL WITH DOUBLE ALLEE EFFECT AND DISEASE IN PREDATOR," *International Journal of Differential Equations*, vol. 2023, pp. 1–24, 2023, doi: https://doi.org/10.1155/2023/5030729.

[25] A. T. R. Sidik, H. S. Panigoro, R. Resmawan, and E. Rahmi, "THE EXISTENCE OF NEIMARK-SACKER BIFURCATION ON A DISCRETE-TIME SIS-EPIDEMIC MODEL INCORPORATING LOGISTIC GROWTH AND ALLEE EFFECT," *Jambura Journal of Biomathematics (JJBM)*, vol. 3, no. 2, pp. 58–62, 2022, doi: https://doi.org/10.34312/jjbm.v3i2.17515.

[26] N. Anggriani, H. S. Panigoro, E. Rahmi, O. J. Peter, and S. A. Jose, "A PREDATOR–PREY MODEL WITH ADDITIVE ALLEE EFFECT AND INTRASPECIFIC COMPETITION ON PREDATOR INVOLVING ATANGANA–BALEANU–CAPUTO DERIVATIVE," *Results Phys*, vol. 49, p. 106489, 2023, doi: https://doi.org/10.1016/j.rinp.2023.106489.

[27] H. S. Panigoro, E. Rahmi, N. Achmad, and S. L. Mahmud, "THE INFLUENCE OF ADDITIVE ALLEE EFFECT AND PERIODIC HARVESTING TO THE DYNAMICS OF LESLIE-GOWER PREDATOR- PREY MODEL," *Jambura Journal of Mathematics*, vol. 2, no. 2, pp. 87–96, 2020.doi: https://doi.org/10.34312/jjom.v2i2.4566

[28] D. Bai and X. Zhang, "DYNAMICS OF A PREDATOR–PREY MODEL WITH THE ADDITIVE PREDATION IN PREY," *Mathematics*, vol. 10, no. 4, 2022, doi: https://doi.org/10.3390/math10040655.

[29] C. C. García and J. V. Cuenca, "ADDITIVE ALLEE EFFECT ON PREY IN THE DYNAMICS OF A GAUSE PREDATOR–PREY MODEL WITH CONSTANT OR PROPORTIONAL REFUGE ON PREY AT LOW OR HIGH

DENSITIES,” *Commun Nonlinear Sci Numer Simul*, vol. 126, p. 107427, 2023, doi: https://doi.org/10.1016/j.cnsns.2023.107427.

[30] B. Xie and Z. Zhang, “IMPACT OF ALLEE AND FEAR EFFECTS IN A FRACTIONAL ORDER PREY–PREDATOR SYSTEM INCORPORATING PREY REFUGE,” *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 33, no. 1, p. 13131, Feb. 2023, doi: https://doi.org/10.1063/5.0130809.

[31] Y. Ye and Y. Zhao, “BIFURCATION ANALYSIS OF A DELAY-INDUCED PREDATOR–PREY MODEL WITH ALLEE EFFECT AND PREY GROUP DEFENSE,” *International Journal of Bifurcation and Chaos*, vol. 31, no. 10, p. 2150158, 2021, doi: https://doi.org/10.1142/S0218127421501583.

[32] C. Zhang, R. Wu, and M. Chen, “HOPF BIFURCATION IN A DELAYED PREDATOR-PREY SYSTEM WITH GENERAL GROUP DEFENCE FOR PREY,” *Journal of Applied Analysis & Computation*, vol. 11, no. 2, pp. 810–840, 2021, doi: https://doi.org/10.11948/20200011.

[33] X. Jiao, X. Li, and Y. Yang, “DYNAMICS AND BIFURCATIONS OF A FILIPPOV LESLIE-GOWER PREDATOR-PREY MODEL WITH GROUP DEFENSE AND TIME DELAY,” *Chaos Solitons Fractals*, vol. 162, p. 112436, 2022, doi: https://doi.org/10.1016/j.chaos.2022.112436.

[34] R. R. Patra, S. Kundu, and S. Maitra, “EFFECT OF DELAY AND CONTROL ON A PREDATOR-PREY ECOSYSTEM WITH GENERALIST PREDATOR AND GROUP DEFENCE IN THE PREY SPECIES,” *The European Physical Journal Plus*, vol. 137, no. 1, p. 28, 2022, doi: https://doi.org/10.1140/epjp/s13360-021-02225-x.

[35] S. Wiggins, *INTRODUCTION TO APPLIED NONLINEAR DYNAMICAL SYSTEMS AND CHAOS*, vol. 2. in Texts in Applied Mathematics, vol. 2. New York: Springer-Verlag, 2003. doi: https://doi.org/10.1007/b97481.