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NIL DERIVATIONS AND d-IDEALS ON POLYNOMIAL RINGS

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ABSTRACT

Let R be a ring. An additive mapping $d : R \to R$ is called derivation if d satisfies Leibniz's rule, i.e., $d(ab) = d(a)b + ad(b)$, for every $a, b \in R$. In a special case, for each $x \in R$ there exists a positive integer n which depends on x such that $d^n(x) = 0$, then d is called as a nil derivation on R. The concept of d- ideal which is an ideal that remains stable under the derivation operation d. This research presents a systematic construction of nil derivations on polynomial rings and investigates their corresponding nilpotency indices. Unlike prior studies that often treat derivations in abstract terms, this work emphasizes explicit constructions, offering concrete examples and techniques for generating such derivations. A key focus is the relationship between nil derivations and general nilpotent derivations, including an analysis of their linear combinations. Furthermore, the study provides new insights into the behavior of nil derivations in the context of d-ideals, shedding light on their structural properties within ring theory. To enhance understanding, each theoretical development is supported by illustrative examples, reinforcing the applicability and significance of the results.

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1. INTRODUCTION

A ring is an algebraic structure consisting of a set equipped with two binary operations, addition and multiplication, that satisfy certain axioms. Rings are foundational tools in various disciplines, including physics, chemistry, economics, finance, and cryptography, highlighting their broad applicability. Among the core concepts in ring theory is derivation, a special type of function from a ring to itself that satisfies linearity and the Leibniz rule.

Over the past few decades, there has been a significant increase in research on derivations in ring structures and their diverse applications. A notable element in ring theory is the nil element, which plays a crucial role in several branches of mathematics, particularly in commutative algebra, algebraic geometry, deformation theory, Lie algebras, and mathematical physics. A nil derivation is a type of derivation that has a special property, namely that there is a positive integer n that depends on the ring element so that when the derivation is applied repeatedly n times on the ring element, it will result in zero. The smallest positive integer n that satisfies this condition is called the index of nilpotency. This has led to the development of the concept of nil derivations, a growing area of study with applications in algebra and functional analysis.

The study of derivations across different algebraic structures continues to attract significant scholarly attention. Numerous studies have explored derivations in various contexts, including different types of rings and modules. In the context of ring derivations, Guven [1] investigated special derivations on prime rings. This was followed by Golbasi and Koc [2], who extended the concept to Lie ideals. In 2014, Ali et al. [3] examined the commutativity of derivation maps on prime and semiprime rings, and this line of research was further expanded by Ali and Alhazmi [4] through generalizations of derivation and commutativity in prime rings. Atteya [5] continued these efforts by studying the commutativity of derivations in semiprime rings.

Belkadi et al. contributed to this area by investigating nilpotent homoderivations [6] and n-Jordan homoderivations [7] on prime rings. El-Sayiad et al. [8] also studied homoderivations in semiprimary rings. El-Deken and El-Soufi [9] explored derivation bindings and their generalizations, which were later developed into homoderivation structures [10]. The research on derivations continues to evolve. Thomas et al. [11] examined derivation aspects on various ring types, while Ezzat [12] explored the idea of higher-derivations. In 2024, Gouda and Nabiel [13] extended the concept of left derivations.

Within the framework of modules, Bracic [14] studied derivations and their representations on modules, whereas Gurjar and Patra [15] explored minimum generators of module derivations. The topic of commuting derivations has also been widely investigated. Retert [16] analyzed commuting derivations in simple rings, followed by Chen and Wang [17], who applied this study to Lie algebras. Maubach [18] proposed a conjecture on commuting derivations in rings, Pogudin [19] examined such derivations in fields, and Fitriani et al. [20] investigated commuting and centralizing in modules. Additionally, Fitriani et al. [21] conducted research on f-derivations in polynomial modules, where derivations on rings served as the foundation for constructing derivations in more complex module structures.

Given the expanding landscape of research in this area, the exploration of nil derivations on polynomial rings presents a promising avenue. This study aims to construct nil derivations on polynomial rings, investigate their connections to derivations, and examine the properties of d-ideals, ideal that remains stable against the derivation operation d, in this context.

2. RESEARCH METHODS

This research is focused on nil derivation on the polynomial ring, properties of nil derivation, d-ideal on nil derivation, and followed by nil derivation on the quotient ring. At the beginning of the research, a literature study is conducted on rings, polynomial ring, derivations on rings, nil derivation on rings, composition and linear combination of derivations, d-ideal and some definitions found in [22], [23], [24], [11], [25], [26], [27], [28], [29], [30]. After that, we construct a conjecture regarding the properties of nil derivation on the specified ring and d-ideal. In the last step, we will prove some properties that we have established. In investigating the properties of nil derivations, we will first introduce the polynomial ring, nil derivation, d-ideal, and derivation on the quotient ring.

One type of ring is the polynomial ring, an algebra structure consisting of a set of polynomials with coefficients from a ring, equipped with the addition and multiplication operations of polynomials that fulfill the properties of the ring.

Definition 1. [22] Given a ring R. The set $R[x]$ is denoted as the set of all infinite series $(a_0, a_1, a_2, \ldots)$ with $a_i \in R, i = 0,1,2, \ldots$ and there exists a nonnegative integer n such that for every $k \geq n, a_k = 0$. The elements of $R[x]$ are called polynomials over R.

In the introduction, we explained the definition of derivation. Suppose given any derivation d on the ring R, if for every $x \in R$ there exists $n \in \mathbb{N}$ such that $d^n(x) = 0$, then the derivation d is called a nil derivation as described in the following definition.

Definition 2. [23] A mapping $f : R \to R$ is said to be nil if for every $x \in R$ there exists a number n (depending on x) such that $f^n(x) = 0$. The smallest number n is called the index of nilpotency of f with respect to x, denoted by $nil(f, x)$.

In ring, we know the concept of an ideal which is a special subring of a ring. For example, the ring R with derivative d and ideal I. Ideal I is called a d-ideal if I remain stable against the derivation operation d as explained in the following definition.

Definition 3. [24] Given any ring R, an ideal I in R, and a derivation of d on R. Ideal I is called d-ideal if $d(I) \subseteq I$.

Furthermore, the quotient ring of a ring and its d-ideal can be constructed as described in the following theorem.

Theorem 1. [11] *Let a ring R with unity, a map $d : R \to R$ is a derivation on ring R, and set I is a d-ideal of R. A map $\bar{d} : R/I \to R/I$ with the definition $\bar{d}(a + I) = d(a) + I$ for all $a + I \in R/I$ is a derivation on the quotient ring R/I.*

After understanding the underlying definitions and theorems, the next step is to formulate and prove the conjecture into a theorem or proposition.

3. RESULTS AND DISCUSSION

The derivation discussed in this paper is the nil derivation on the ring, especially the nil derivation on the polynomial ring, the relationship between nil derivation and nilpotent derivation, the properties of nil derivation, nil derivation on d-ideal, and nil derivation on the quotient ring.

3.1 Nil Derivation on Polynomial Ring

To extend the concept of nil derivation of a ring as in Definition 1 to a polynomial ring, the relevant theorem is presented below. Before proving the main theorem, the following lemma will first be proved which forms the basis of the proof.

Lemma 1. *Given ring R and $d : R \to R$ a derivation on ring R. If we define $\hat{d} : R[x] \to R[x]$, with $\hat{d}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d(a_i) x^i$, then for every positive integer N holds: $\hat{d}^N\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d^N(a_i) x^i$.*

Proof. It will be shown $\hat{d}^N\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d^N(a_i) x^i$ by using mathematical induction on N. For $N = 1$ it is obtained:

$$\hat{d}^1\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d^1(a_i) x^i = \sum_{i=0}^{n} d(a_i) x^i.$$

Furthermore, suppose that $N = k$ holds:

$$\hat{d}^k\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d^k(a_i) x^i.$$

It will be shown that for $N = k + 1$ holds: $\hat{d}^{k+1}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d^{k+1}(a_i) x^i$.

From the definition of $\hat{d}$, we obtain:

$$\begin{aligned}
\hat{d}^{k+1}\left(\sum_{i=0}^{n} a_i x^i\right) &= \hat{d}\left(\hat{d}^{k}\left(\sum_{i=0}^{n} a_i x^i\right)\right)\\
&= \hat{d}\left(\sum_{i=0}^{n} d^k(a_i)x^i\right)\\
&= \hat{d}\left(d^k(a_0) + d^k(a_1)x + \cdots + d^k(a_n)x^n\right)\\
&= d\left(d^k(a_0)\right) + d\left(d^k(a_i)\right)x + \cdots + d\left(d^k(a_n)\right)x^n\\
&= \sum_{i=0}^{n} d\left(d^k(a_i)\right)x^i\\
&= \sum_{i=0}^{n} d^{k+1}(a_i)x^i.
\end{aligned}$$

Thus, the statement is proven for $N = k + 1$. Therefore, for every positive integer N holds:

$$\hat{d}^N\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d^N(a_i)x^i\,. \qquad \blacksquare$$

The result of Lemma 1 becomes the basis for the proof of the following theorem.

Theorem 2. *Given a ring R. If $d : R \to R$ is a nil derivation on the ring R, then there exists $\hat{d} : R[x] \to R[x]$ which is a nil derivation on the polynomial ring $R[x]$.*

Proof. Given an arbitrary derivation of nil $d : R \to R$ with $d(a) = a$ for every $a \in R$. We define:

$\hat{d} : R[x] \to R[x]$

$$\begin{aligned}
\sum_{i=0}^{n} a_i x^i \longmapsto \hat{d}\left(\sum_{i=0}^{n} a_i x^i\right)
&= \hat{d}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)\\
&= d(a_0) + d(a_1)x + d(a_2)x^2 + \cdots + d(a_n)x^n\\
&= \sum_{i=0}^{n} d(a_i)x^i,
\end{aligned}$$

with $a_i \in R, i = 0,1, \dots, n$. We will show that $\hat{d}$ is a nil derivation. Let d be a nil derivation, meaning that for every $a_i \in R$ there is a positive integer n_i such that $d^{n_i}(a_i) = 0$. Define $N = max(n_i)$ which is the largest positive integer of all n_i. By Lemma 1, we have:

$$\hat{d}^N\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d^N(a_i)x^i.$$

Since $N = max(n_i)$ and for every n_i holds $d^{n_i}(a_i) = 0$, then $d^N(a_i) = 0$ for all i. Thus, $d^N\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d^N(a_i)x^i = 0$. Hence, $\hat{d}$ is a nil derivation on $R[x]$. $\blacksquare$

The following example is an illustration of Theorem 2.

Example 1. Given a ring $R = \mathbb{Z}_2$, a nil derivation $d : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ with $d(x) = 2x \bmod 2$ for every $x \in \mathbb{Z}_2$. We define the derivation of $\hat{d} : R[x] \longrightarrow R[x]$, which is a polynomial ring with integer coefficients modulo 2. Thus, every element in $R[x]$ is of the form:

$$\sum_{i=0}^{n} a_i x^i = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where $a_i \in \mathbb{Z}_2$for every i. Next, it will be shown $\hat{d}$ is a nil derivation.

$$\begin{aligned}
\hat{d}\left(\sum_{i=0}^{n} a_i x^i\right) &= \hat{d}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)\\
&= d(a_0) + d(a_1)x + d(a_2)x^2 + \cdots + d(a_n)x^n\\
&= \textstyle\sum_{i=0}^{n} d(a_i)x^i, \text{ for every } a_i \in \mathbb{Z}_2.
\end{aligned}$$

It is known that d is a nil derivation, and hence for every $a_i \in \mathbb{Z}_2$ there is a positive integer n_i such that $d^{n_i}(a_i) = 0$, that is:

$$d(\bar{0}) = 2.0 \, mod \, 2 = 0$$

$$d(\bar{1}) = 2.1 \, mod \, 2 = 0.$$

By choosing $n = 1$, we obtain:

$$\hat{d}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d(a_i) x^i = 0.$$

Since $d(a_i) = 0$ for every i. Therefore, $\hat{d}$ is a nil derivation on $R[x]$ with $R = \mathbb{Z}_2$.

In general, on the polynomial ring $R[x]$, the value of n needed for $f^n(p(x)) = 0$ as in Theorem 2 depends on the highest degree of the polynomial $p(x) \in R[x]$. The following theorem gives the relationship between the highest degree of the polynomial and the index of nilpotency of the derivation.

Theorem 3. *Given a derivation of f on the polynomial ring $R[x]$. If we define a mapping $f(p(x)) = p'(x)$ for every $p(x) \in R[x]$, then the index of nilpotency of f with respect to $p(x)$ is $n = nil(f, p(x)) = k + 1$ where k is the highest degree of $p(x)$.*

Proof. Given any $p(x) \in R[x]$ and a derivation f defined by $f(p(x)) = p'(x)$. We will show that $f^n(p(x)) = 0$, where $n = k + 1$. Given any $p(x) = \sum_{i=0}^{k} a_i x^i$ with $a_i \in R$, we have:

$$f(p(x)) = f\left(\sum_{i=0}^{k} a_i x^i\right) = \sum_{i=0}^{k} a_i f(x^i),$$

by using the definition $f(x^i) = i x^{i-1}$:

$$f(p(x)) = \sum_{i=0}^{k} a_i i x^{i-1}.$$

Next, we will calculate $f^2(p(x))$ as follows:

$$f\left(\sum_{i=0}^{k} a_i i x^{i-1}\right) = \sum_{i=0}^{k} a_i i f(x^{i-1}) = \sum_{i=0}^{k} a_i i (i-1) x^{i-2}.$$

In general, $f^n(p(x))$ is:

$$f^n(p(x)) = \sum_{i=0}^{n} a_i i (i-1)(i-2) \dots (i-(n-1)) x^{i-n}.$$

If $i = k = n - 1$, then $i(i-1)(i-2) \dots (i-(n-1)) = 0$, meaning $f^n(p(x)) = 0$ with $i = k$ being the highest degree of $p(x)$. Thus, $f^n(p(x)) = 0$ when it reaches the $n = k + 1$th iteration. In other words, it is proved that the index of nilpotency of f with respect to $p(x)$ is $n = nil(f, p(x)) = k + 1$, where k is the highest degree of $p(x)$. ■

3.2 Properties of Nil Derivation on Ring

To better understand the properties of nil derivation, we will first discuss the concept of composition and linear combination of derivations in general. If d is a derivation, then the composition of $d \circ d = d^2$ is not always a derivation. Here is an example of a problem regarding the composition of two derivations that are not derivations.

Example 2. Given a ring $R = \mathbb{Z}[x]$, a polynomial ring with integer coefficients and a derivation d defined by $d(p(x)) = p'(x)$ for every $p(x) \in \mathbb{Z}[x]$. Choose $p(x) = x^3 + 3$ and $q(x) = 2x^2$, since $d^2(p(x) + q(x)) = d^2(p(x)) + d^2(q(x)) = 6x + 4$ then d satisfies the additive property, but d does not satisfy Leibniz's rule because $d^2(p(x)q(x)) = 40x^3 + 12 \neq d^2(p(x))q(x) + p(x)d^2(q(x)) = 16x^3 + 12$. Thus, d^2 is not a derivation on ring $\mathbb{Z}[x]$.

Based on **Example 2**, since the composition of a derivation d^2 is not always a derivation, it is not possible to define the composition as a nil derivation. However, there is a special condition that a derivation composition can be a derivation if the derivation defined is a trivial derivation. This trivial derivation is also a nil derivation, since $d^n(x) = 0$ for every $n \geq 1$ and $x \in R$. Next, we will discuss the linear combination of derivations in general.

Theorem 4. Given a ring R. *If $d_1, d_2, \dots, d_n$ is a derivation on the ring R, then $d = c_1 d_1 + c_2 d_2 + \cdots + c_n d_n$ with $c_1, c_2, \dots, c_n \in R$ is a derivation on R.*

Proof. We define: $d(r) = c_1 d_1(r) + c_2 d_2(r) + \cdots + c_n d_n(r)$, for every $r \in R$ and $c_1, c_2, \dots, c_n \in R$. We will show that d is a derivation. Given any $r, s \in R$ and $c_1, c_2, \dots, c_n \in R$, we have:

1. $d(r+s) = c_1 d_1(r+s) + c_2 d_2(r+s) + \cdots + c_n d_n(r+s)$

$$= c_1 d_1(r) + c_1 d_1(s) + c_2 d_2(r) + c_2 d_2(s) + \cdots + c_n d_n(r) + c_n d_n(s)$$
$$= c_1 d_1(r) + c_2 d_2(r) + \ldots + c_n d_n(r) + c_1 d_1(s) + c_2 d_2(s) + \cdots + c_n d_n(s)$$
$$= d(r) + d(s).$$

2. $d(rs) = c_1 d_1(rs) + c_2 d_2(rs) + \cdots + c_n d_n(rs)$

$$= c_1(d_1(r)s + r d_1(s)) + c_2(d_2(r)s + r d_2(s)) + \cdots + c_n(d_n(r)s + r d_n(s))$$
$$= c_1 d_1(r)s + c_1 r d_1(s) + \cdots + c_n d_n(r)s + c_n r d_n(s)$$
$$= \big(c_1 d_1(r) + c_2 d_2(r) + \cdots + c_n d_n(r)\big)s + r\big(c_1 d_1(s) + c_2 d_2(s) + \cdots + c_n d_n(s)\big)$$
$$= d(r)s + r d(s).$$

So, it is proven that d is a derivation. ∎

Based on Theorem 4, this inspires to define the derivation to be a nil derivation and produces the following theorem.

Theorem 5. *Given a ring R. If $d_1, d_2, \dots, d_n$ are nil derivations in the ring $R[x]$ then $d = \sum_{i=1}^{n} c_i d_i = c_1 d_1 + c_2 d_2 + \cdots + c_n d_n$ with $c_1, c_2, \dots, c_n \in R$ being nil derivations.*

Proof. Since $d_1, d_2, \dots, d_n$ are nil derivations, it means that for every $d_i(p(x))$ there exists $k \in \mathbb{N}$ such that $d_i^k\big(p(x)\big) = 0$ with $p(x) \in R[x]$. Defined: $d(p(x)) = \sum_{i=1}^{n} c_i d_i\big(p(x)\big) = c_1 d_1(p(x)) + c_2 d_2(p(x)) + \cdots + c_n d_n(p(x))$ for every $p(x) \in R[x]$ and $c_1, c_2, \dots, c_n \in R$. It will be shown that d is a nil derivation on $R[x]$.

First iteration: $d\big(p(x)\big) = \sum_{i=1}^{n} c_i d_i\big(p(x)\big) = c_1 d_1\big(p(x)\big) + c_2 d_2\big(p(x)\big) + \cdots + c_n d_n\big(p(x)\big)$.

Second iteration: $d^2(p(x)) = d(d(p(x)))$

$$= d\left(\sum_{i=1}^{n} c_i d_i(p(x))\right)$$
$$= \sum_{i=1}^{n} c_i d_i\big(d(p(x))\big)$$
$$= \sum_{i=1}^{n} c_i d_i\left(\sum_{i=1}^{n} c_i d_i(p(x))\right).$$

m-th iteration: $d^m(p(x)) = d(d^{m-1}(p(x)))$

$$= \underbrace{\sum_{i=1}^{n} c_i d_i\left(\sum_{i=1}^{n} c_i d_i\right) \ldots \left(\sum_{i=1}^{n} c_i d_i(p(x))\right)}_{m\ times}$$

The general form of $d^m(p(x))$ will involve a combination of derivations $d_1, d_2, \dots, d_n$. If $d_1, d_2, \dots, d_n$ are nil derivation, meaning that these derivations will be zero at the k-th iteration, then there exists $m = k$ such that $d^m(p(x)) = 0$. Thus, d is a nil derivation. ∎

Next, we give an example of a linear combination of nil derivation.

Example 3. Given a ring $R = \mathbb{Z}[x]$ and we define:

$$d_1\big(p(x)\big) = p'(x)$$
$$d_2\big(p(x)\big) = 3p'(x),$$

for every $p(x) \in \mathbb{Z}[x]$. Defined $d = c_1d_1 + c_2d_2$ with $c_1 = 3$ and $c_2 = 2$, obtained:

$$d(p(x)) = c_1d_1(p(x)) + c_2d_2(p(x)) = 3p'(x) + 6p'(x) = 9p'(x).$$

It will be shown that d is a nil derivation. Given any d_1 and d_2 is a nil derivation, by Theorem 3 it is proved that the mapping d is a nil derivation at the $n = k + 1$th iteration where k is the highest degree of the polynomial $p(x)$. Thus, d is a nil derivation.

Remark 1. If one of the derivations is not a nil derivation, then d is not always a nil derivation.

The following is an example of applying the **Remark 1**.

Example 4. Given a ring $R = \mathbb{Z}[x]$, choose $p(x) = x^2 + 1, c_1 = 3$ and $c_2 = 2$. d_1 and d_2 are defined as $d_1(p(x)) = p'(x)$ and $d_2(p(x)) = xp'(x)$. The linear combination of them: $d(p(x)) = 3p'(x) + 2xp'(x)$, we have $d(p(x)) = 6x + 4x^2, d^2(p(x)) = 18 + 36x + 16x^2, d^3(p(x)) = 108 + 168x + 64x^2$. After d is iterated 3 times, it can be seen that the result of the derivation will not be close to zero. Thus, it is proven that d is not always a nil derivation.

Remark 2. If the derivations $d_1, d_2, \dots, d_n$ are not nil derivation, then d is also not always a nil derivation.

The following is an example of applying the **Remark 2**.

Example 5. Given a ring $R = \mathbb{Z}[x]$, choose $p(x) = x^2 + 1, c_1 = 1$ and $c_2 = 2$. d_1 and d_2 are defined as $d_1(p(x)) = xp'(x)$ and $d_2(p(x)) = xp'(x)$. The linear combination of them: $d(p(x)) = xp'(x) + 2xp'(x) = 3xp'(x)$, we have $d(p(x)) = 6x^2, d^2(p(x)) = 36x^2, d^3(p(x)) = 216x^2$. After d is iterated 3 times, it can be seen that the result of the derivation will not be close to zero. Thus, it is proven that d is not always a nil derivation.

3.3 d-Ideal with d being a Nil Derivation

Based on Definition 3 of the concept of d-ideal, that is an ideal I on ring R is called d-ideal if $d(I) \subseteq I$. The concept of d-ideal on the polynomial ring $R[x]$ is highly dependent on the type of derivation used. Some ideal I on the polynomial ring $R[x]$ will be a d-ideal if the derivation of d used is not a nil derivation. On the other hand, if the derivation of d used is a nil derivation, then the ideal I is not an d- ideal. Here is an example that illustrates the statement.

Example 6. Given an ideal $I = \langle x^3 \rangle$ is an ideal constructed by the polynomial x^3 in the polynomial ring $\mathbb{Z}[x]$. If the derivation of d is defined as a nil derivation, i.e. $d(p(x)) = p'(x)$, choose $p(x) = x^2 + 1$, meaning $d(I) = d(x^3(x^2 + 1)) = d(x^5 + x^3) + 5x^4 + 3x^2 \notin I$. Thus, the ideal $I = \langle x^3 \rangle$ is not an d-ideal if d is a nil derivation. If the derivation of d is not a nil derivation, suppose $d(p(x)) = xp'(x)$, the result will be different. In this case, it is obtained $d(I) = d(x^3p(x)) = x(3x^2)p(x) + x^3xp'(x) = 3x^3p(x) + x^4p'(x) = x^3(3p(x) + xp'(x)) = x^3h(x) \in I$ for every $h(x) \in \mathbb{Z}[x]$. Thus, the ideal $I = \langle x^3 \rangle$ is an d-ideal if d is not a nil derivation.

Here is another example of an ideal on the polynomial ring $R[x]$ that is also not a d-ideal, with d being a nil derivation and not a nil derivation.

Example 7. Given an ideal $I = \langle x^2 + 1 \rangle$ in the polynomial ring $\mathbb{Z}[x]$. The derivation d is a nil derivation defined as $d(p(x)) = p'(x)$ obtained $d(I) = 2xp(x) + (x^2 + 1)p'(x) \notin I$. If we define d is not a nil derivation, i.e. $d(p(x)) = xp'(x)$ we get $d(I) = x(2x)p(x) + (x^2 + 1)xp'(x) = 2x^2p(x) + x(x^2 + 1)p'(x) \notin I$. Thus, the ideal $I = \langle x^2 + 1 \rangle$ is not an d-ideal, with d being a nil derivation and not a nil derivation.

One example of an ideal that is an d-ideal with d being a nil derivation on the polynomial ring $\mathbb{Z}[x]$ is $I = \langle x^n, n \rangle$ and $I = \langle x^n, m \rangle$ provided n is a multiple of m.

3.4 Nil Derivation on the Quotient Ring

Furthermore, a nil derivation can be formed on the quotient ring R/I of the d-ideal concept based on the Theorem 1 which will be explained in the following theorem.

Theorem 6. *Given a ring R and an ideal I in R. If $d : R \to R$ is a nil derivation on ring R and ideal I is d-ideal, then $\bar{d} : R/I \to R/I$ by definition $\bar{d}(r + I) = d(r) + I$ for every $r + I \in R/I$ is a nil derivation on the quotient ring R/I.*

Proof. Given any nil derivation $d : R \to R$ with $d(r) = r$ for every $r \in R$. An ideal I is an d-ideal, meaning $d(r) \subseteq I$ if and only if $r \in I$. We define derivation $\bar{d} : R/I \to R/I$ by $\bar{d}(r + I) = d(r) + I$ for every $r + I \in R/I$. It will be shown that $\bar{d}$ is a nil derivation on the quotient ring R/I and it will be determined that $\bar{d}$ will be nil at which iteration. It will be reviewed in two cases as follows.

1. Case 1: $r \in I$.

   Suppose I is a d-ideal, then $d(r) \in I$. Consequently, for every $r \in I$, we have $\bar{d}(r + I) = d(r) + I = 0 + I$. In other words, the derivation of $\bar{d}$ will be a nil derivation at the first iteration for every $r \in I$.

2. Case 2: $r \notin I$.

   Given $r \notin I$, then $d(r) \notin I$. In this case, the iteration of $\bar{d}$ follows the iteration of the derivation of d on R, viz $\bar{d}^n(r + I) = d^n(r) + I$. It is known that d is a nil derivation, meaning that for every $r \in I$ there is $n \in \mathbb{N}$ such that $d^n(r) = 0$. Hence it is obtained $\bar{d}^n(r + I) = d^n(r) + I = 0 + I$. This shows that the derivation of $\bar{d}$ becomes a nil derivation at the n-th iteration, which is the same as the iteration of d on R. Thus, $\bar{d}$ is also a nil derivation on the quotient ring R/I. ■

4. CONCLUSION

A derivation on the polynomial ring $R[x]$ can be a nil derivation with the index of nilpotency $n = k + 1$, where k is the highest degree of the polynomial $p(x) \in R[x]$. Furthermore, this research also shows that nil derivations can be used to form and analyse linear combinations of n nil derivations that are also nil derivations. In addition, this study found that the linear combination of derivations that are not nil derivations does not always result in nil derivations, and if one of the derivations in the combination is not a nil derivation, then the result is also not always a nil derivation.The nil derivation can also be developed in the concept of d-ideal on polynomial ring. Ideal I will be an d-ideal if the derivation of each element forming the ideal remains in the ideal. In addition, the nil derivation can also be applied to the quotient ring by definition $\bar{d}(r + I) = d(r) + I$ for every $r + I \in R/I$ of the d-ideal concept.

However, this study has several limitations. The construction and analysis are currently restricted to commutative rings with identity and focus only on polynomial rings in one variable. The behavior of nil derivations in non-commutative rings, multivariate polynomial rings, or rings with additional algebraic structure remains unexplored. Furthermore, the general criteria or characterization for when linear combinations of arbitrary derivations yield nil derivations have not been fully formalized, leaving room for deeper algebraic investigation.

Author Contributions

Ditha Lathifatul Mursyidah: Investigation, Formal Analysis, Writing-Original Draft. Fitriani: Conceptualization, Methodology, Validation, Supervision, Formal Analysis, Writing-Review and Editing. Bernadhita Herindri Samodera Utami: supervision, methodology, validation. Ahmad Faisol: validation, Formal Analysis, Writing - Review & Editing. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare no competing interest.

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