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ON CONDITIONS FOR MATRICES T SUCH THAT $T - I$ AND $I - T^{-1}$ ARE INVERSE H-MATRICES*

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ABSTRACT

In this article, we study an analogue of a classical result for M-matrices: if $T - I$ is an invertible M-matrix, then both T and $I - T^{-1}$ are also invertible M-matrices. We extend this implication to a broader class—inverse H-matrices. The expressions $T - I$ and $I - T^{-1}$ commonly arise in the analysis of matrix stability, convergence of iterative methods, and spectral transformations, making their structural properties important for numerical analysis. We demonstrate that this implication does not generally hold for inverse H-matrices. However, we derive some conditions under which it remains valid. Specifically, we prove that under certain conditions, if $T - I$ is an inverse H-matrix, then T and $I - T^{-1}$ are also inverse H-matrices. Additionally, we investigate the result in the context of group inverses, showing that it does not hold for group inverse M-matrices and H-matrices.

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1. INTRODUCTION

Matrix theory is central in pure and applied mathematics, with uses as varied as numerical analysis and optimization, economics, physics, and the social sciences. Among the various classes of matrices, nonnegative matrices (matrices whose entries are all nonnegative) are a fundamental concept. The Z-matrix is another closely related, widely studied class, a real matrix with nonpositive off-diagonal entries. Alexander Ostrowski formulated 1937 a significant subclass of Z-matrices known as M-matrices which have become fundamental due to their applications. M-matrices are used in eigenvalue approximation, convergence analysis of iterative methods for solving linear systems, Markov chain analysis, probability, and operations research.

The M-matrix theory was extended later on to complex field matrices, leading to the definition of the H-matrix. The new class has some beneficial properties of M-matrices and has been the focus of considerable theoretical research. For instance, Bru (2015) analyzed classes of H-matrices and Hadamard products [1]. Zhao proposed a new subclass of the class of H-matrices in 2018 [2]. Chatterjee et al. (2019) investigated inverse H-matrices [3], while McDonald (2020) investigated generalizations of M-matrices [4]. Chatterjee (2021) investigated the relationship between interval H-matrices and inverse M-matrices [5], and Mondal in 2022 provided new results for inverse M-matrices and extended it to H-matrices [6]. Recently, Cvetković (2024) and Luo (2024) investigated subclasses of H-matrices, and they also applied M-matrices to the global exponential stability of neural networks [7], [8]. Encinas (2025) later contributed by studying the group inverse of tridiagonal M-matrices and generalized singular M-matrix properties extended to Lyapunov and Stein operators [9], [10]. Another study also applied M-matrices to stability analysis of a class of quaternion-valued memristor [11], studied Riccati equations with irreducible singular M-matrices [12], and studied nonsymmetric complex algebraic Riccati equations involving H-matrices [13]. Moreover, recent studies have continued to expand the application of M-matrices. For example, Guo [14] examined absolute value equations involving M-matrices and H-matrices and established conditions under which solutions exist and are unique. Guan and Wang [15] analyzed a class of quadratic matrix equations that share structural similarities with complementarity problems where M-matrices naturally arise. Zhong [16] derived new lower bounds for the minimal eigenvalue of M-matrices, which is crucial for spectral analysis and matrix stability. On the H-matrix side, Wang et al. [17] proposed improved matrix-splitting methods for solving horizontal implicit complementarity problems, where convergence criteria rely on properties of M- and H-matrices. Nedović and Arsić [18] introduced new scaling techniques to identify H-matrices, enhancing classification within general matrix classes. Liu et al. [19] improved convergence theorems for modulus-based splitting methods, often depending on whether the involved matrices satisfy H-matrix conditions. Ma et al. [20] developed Newton-based iterative techniques for generalized absolute value equations, many of which can be formulated using H-matrices. Kolotilina [21] studied the relationship between strictly diagonally dominant matrices and nonsingular H-matrices, while Zeng et al. [22] constructed scaling matrices to characterize and decompose S^k-SDD matrices, a subclass closely related to H-matrix structures.

Fan's result is a nicely referenced work towards this end, in which matrix inequalities with expressions like $A - I$ and $I - A^{-1}$ are investigated. Fan's result states that if $A - I$ is an invertible M-matrix, then A and $I - A^{-1}$ are also invertible M-matrices [23]. These expressions are important because they appear naturally when analyzing iterative methods, stability criteria, and matrix perturbation theory. For instance, $A - I$ is a transformed version of the original matrix that often preserves diagonal dominance or positivity properties. At the same time, $I - A^{-1}$ arises in contexts where inverse-based transformations are considered, i.e., preconditioning techniques or spectral radius estimations. Structural matrix behavior under these transformations helps determine principal matrix subclasses and offers convergence assurances in numerical solutions.

Fan's result has been a core topic of investigation in matrix theory due to its structural and analytic significance. The converse of Fan's result has also been shown to be true in M-matrices by Kalauch et al. in 2019 [23]. In the same year, Chatterjee also extended both the original and the converse of Fan's result to the class of invertible H-matrices with certain additional conditions, as stated in **Theorem 3** and **Theorem 4** [24]. Furthermore, Kalauch et al. later showed that Fan's result and its converse are also valid for inverse M-matrices [25].

Motivated by the depth and variety of research on Fan's result and its generalizations, this paper investigates the analogue of Fan's result for the class of inverse H-matrices. We identify some conditions in which the analogue and its converse hold. Our findings include counterexamples where the analogue fails and some conditions under which it is preserved.

2. RESEARCH METHODS

This research used a literature study approach to investigate an analogue of Fan's result for inverse H-matrices and extend it using group inverse concepts. The steps of this work were organized as follows: first, some related articles were collected and studied to understand the development and characterization of inverse M-matrices, H-matrices, and their generalized inverses. Secondly, the focus was whether Fan's result, established for class inverse M-matrices, is extendable to the inverse H-matrices and group inverse H-matrices. The review highlighted gaps in the literature concerning such analogues. Based on this research, we formulated theorems and theoretical results for inverse H-matrices and group inverse H-matrices. The last phase was to prove mathematically these findings and to support them with illustrative examples to demonstrate the validity of our claims.

Next, we begin with some basic concepts by defining the following notations. Let $\mathbb{R}^{k\times k}$ and $\mathbb{C}^{k\times k}$ denote the set of all real and complex $k\times k$ matrices. Suppose $A=(a_{ij})\in\mathbb{R}^{k\times k}$, then it is called a nonnegative or positive matrix if all its entries are nonnegative or positive, i.e., $a_{ij}\geq 0$ or $a_{ij}>0$ for all i,j. We denote a nonnegative matrix as $A\geq 0$ and a positive matrix as $A>0$. Additionally, if there are matrices A and B such that $A-B$ is nonnegative or positive, then it is denoted by $A\geq B$ or $A>B$ respectively. Similarly, if there are vectors $\bar{s}=(s_1,s_2,\dots,s_k)^T$ and $\bar{t}=(t_1,t_2,\dots,t_k)^T$, then $\bar{s}\geq 0$ or $\bar{s}>0$ denote vectors with all nonnegative or positive entries and if $\bar{s}-\bar{t}$ is nonnegative or positive, then it is denoted by $\bar{s}\geq\bar{t}$ or $\bar{s}>\bar{t}$ respectively. Next, we define the absolute matrix as $|A|:=(|a_{ij}|)$, absolute vector as $|\bar{s}|:=(|s_1|,|s_2|,\dots,|s_k|)^T$, and we denote vector e as the all-ones vector, i.e., $e=(1,1,1,\dots,1)^T\in\mathbb{R}^k$. Then, we denote $\rho(A)$ as the spectral radius of A, i.e., $\rho(A)=\max\{|v|: v\in\sigma(A)\}$ and $\sigma(A)$ is the set of eigenvalues of A. Lastly, we denote D_A as a diagonal matrix with entries a_{ii}.

A matrix $T\in\mathbb{R}^{k\times k}$ is called a Z-matrix if all its off-diagonal entries are nonpositive. Any Z-matrix can be written in the form $T=tI-D$ where $t\in\mathbb{R}$ and $D\geq 0$. If, in addition, $t\geq\rho(D)$, then T is called an M-matrix. Moreover, according to the Perron-Frobenius theorem, such a matrix T is invertible if and only if $t>\rho(D)$. A nonnegative matrix $T\geq 0$ is said to be convergent if $\lim_{h\to\infty}(T^h)_{ij}=0$. It is well known that T is convergent if and only if $\rho(T)<1$, or equivalently, if $(I-T)^{-1}$ exists and $(I-T)^{-1}\geq 0$. Finally, matrix T is a nonsingular or an invertible M-matrix if T is an M-matrix and T^{-1} exists. We recall some equivalent conditions for an invertible M-matrix.

Theorem 1. [24] *If $T\in\mathbb{R}^{k\times k}$, then these statements are equivalent:*

1. *T is an invertible M-matrix.*
2. *T^{-1} exists and $T^{-1}\geq 0$.*
3. *There exists $\bar{v}\in\mathbb{R}^k$ with $\bar{v}>0$ such that $T\bar{v}>0$.*

We now introduce the concept of the comparison matrix, defined for a complex matrix $T=(t_{ij})\in\mathbb{C}^{k\times k}$. The comparison matrix of T, denoted by $\mathcal{M}(T)=(m_{ij})$, is defined as follows:

$$m_{ij}:=\begin{cases}|t_{ij}| & ,i=j\\ -|t_{ij}| & ,i\neq j\end{cases}$$

The comparison matrix of any matrix is a Z-matrix. Now, matrix T is called an H-matrix if the $\mathcal{M}(T)$ is an M-matrix. Also, matrix T is an invertible H-matrix if T is an H-matrix and T^{-1} exists.

We now define the group inverse. Let $G\in\mathbb{C}^{k\times k}$, then matrix $N\in\mathbb{C}^{k\times k}$ is called the group inverse of G if $GNG=G, NGN=N,$ and $GN=NG$ and we denote it by $G^{\#}$. If the group inverse exists, then it is unique. In other words, matrix G has the group inverse if and only if $rank(G)=rank(G^2)$. Another equivalent condition is using full-rank factorization. This is a technique to factorize a nonzero matrix into a product of a matrix of full column rank and a matrix of full row rank. Let $G\in\mathbb{C}^{k\times k}$ be written as $G=PQ$ with P and Q as full-rank factorization, then G has a group inverse if and only if QP is invertible, in this case we can find the group inverse as $G^{\#}=P(QP)^{-2}P$.

The main result of this work is to extend the analogue of Fan's result for inverse H-matrices. First of all, the class inverse H-matrix is different from the invertible H-matrix. We define an inverse H-matrix as a matrix $T \in \mathbb{R}^{k\times k}$ such that its inverse T^{-1} is an H-matrix (or equivalently, an M-matrix). In invertible H-Matrix, the matrix T must be an H-matrix, but in inverse H-matrix, the matrix T must not be an H-matrix. Now, we have Fan's result regarding invertible M-matrices, which is a foundation for our proposed analogue involving inverse H-matrices.

Theorem 2. [23]. *Let $T \in \mathbb{R}^{k\times k}$. If $T - I$ is an invertible M-matrix, then T and $I - T^{-1}$ are invertible M-matrices.*

In [23], it was shown that the converse of **Theorem 2** also holds. Moreover, we obtained a related result for another class of M-matrices, i.e., singular irreducible M-matrices and those with "property c". More recently, an analogue of Fan's result and its converse was established for inverse M-matrices [25].

The analogue for H-matrices has also been found. In [24], it was shown that a similar result holds for invertible H-matrices under certain additional conditions.

Theorem 3. [24]. *Let $H \in \mathbb{R}^{k\times k}$ with $D_H \geq I$ and $|H^{-1}|e < e$. If $H - I$ is an invertible H-matrix, then both H and $I - H^{-1}$ are invertible H-matrices.*

If we put on a stronger condition, we also ensure $H - I$ and it stated as theorem belows

Theorem 4. [24]. *Let $H \in \mathbb{R}^{k\times k}$ with $h_{ii} \geq 0$ and $\mathcal{M}(H)e > e$. Then $H, H - I$, and $I - H^{-1}$ are invertible H-matrices.*

Before stating the main results, we state a helpful condition for determining whether a matrix is an H-matrix. Let us first define a matrix $T = (t_{ij}) \in \mathbb{C}^{k\times k}$ is called strictly row diagonally dominant (SDD) if

$$|t_{ii}| > \sum_{j=1, j\neq i}^{k} |t_{ij}|$$

Note that, the symbol $>$ means the absolute value of t_{ii} being more than the sigma of the absolute value of t_{ij}. Moreover, it is called a generalized strictly row diagonally dominant (GSDD) matrix if there exists a positive diagonal matrix V such that TV is an SDD matrix.

The following theorem provides a tool for identifying whether a matrix is an invertible H-matrix. This theorem will be used later in some examples of this article.

Theorem 5. *Let $T = (t_{ij}) \in \mathbb{C}^{k\times k}$. If T is GSDD, then T is an invertible H-matrix.*

Proof. Let $T = (t_{ij})$ be a GSDD matrix, then $V = diag(v_1, v_2, \dots, v_k)$ exists, where $v_i > 0$, such that TV is an SDD matrix. That is,

$$|t_{ii}|v_i > \sum_{j=1, j\neq i}^{k} |t_{ij}|v_j$$

By the definition of the comparison matrix, $\mathcal{M}(T)V$ has the same diagonal and off diagonal magnitudes as TV, thus $\mathcal{M}(T)V$ is also SDD. It follows from classical results that a Z-matrix whose product with a positive diagonal matrix is SDD must be an invertible M-matrix. Therefore, $\mathcal{M}(T)$ is invertible. Let $D = diag(|t_{ii}|)$, then

$$D^{-1}\mathcal{M}(T) = I - W$$

where

$$W = \begin{pmatrix} 0 & \frac{|t_{1,2}|}{|t_{1,1}|} & \cdots & \frac{|t_{1,k-1}|}{|t_{1,1}|} & \frac{|t_{1,k}|}{|t_{1,1}|} \\ \frac{|t_{2,1}|}{|t_{2,2}|} & 0 & \cdots & \frac{|t_{2,k-1}|}{|t_{2,2}|} & \frac{|t_{2,k}|}{|t_{2,2}|} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{|t_{k-1,1}|}{|t_{k-1,k-1}|} & \frac{|t_{k-1,2}|}{|t_{k-1,k-1}|} & \cdots & 0 & \frac{|t_{k-1,k}|}{|t_{k-1,k-1}|} \\ \frac{|t_{k,1}|}{|t_{k,k}|} & \frac{|t_{k,2}|}{|t_{k,k}|} & \cdots & \frac{|t_{k,k-1}|}{|t_{k,k}|} & 0 \end{pmatrix}$$

Note that $I - W$ is invertible and $W \geq 0$. Since $\mathcal{M}(T)$ is SDD, then

$$|t_{ii}| > \sum_{j=1, j\neq i}^{k} |t_{ij}| \Leftrightarrow 1 > \sum_{j=1, j\neq i}^{k} \frac{|t_{ij}|}{|t_{ii}|}$$

and by spectral radius properties, we have

$$\rho(W) \leq \max_{1\leq i\leq k} \sum_{j=1}^{k} |w_{ij}| = \max_{1\leq i\leq k} \sum_{j=1, j\neq 1}^{k} \frac{|t_{ij}|}{|t_{ii}|} < 1$$

Since $\rho(W) < 1$, then W is convergent and $(I - W)^{-1} \geq 0$. We have

$$\left(D^{-1}\mathcal{M}(T)\right)^{-1} \geq 0 \Leftrightarrow \mathcal{M}(T)^{-1} D \geq 0$$

Note that since $D \geq 0$, then $\mathcal{M}(T)^{-1} \geq 0$. By **Theorem 1** (ii), we have $\mathcal{M}(T)$ is an invertible M-matrix. Therefore, T is an invertible H-matrix. ■

This article extends **Theorem 3** and **Theorem 4** to get the analogue of Fan's result for inverse H-matrices. The results are provided in **Theorem 6** and **Theorem 7**. In addition, several illustrative examples are included to support the theoretical results. Furthermore, we show that the analogue of Fan's result does not generally hold for group inverse M-matrices. Lastly, we demonstrate that group inverse H-matrices contain the group inverse M-matrices. This confirms that the analogue result also fails for the group inverse H-matrices.

3. RESULTS AND DISCUSSION

In this part, we show that the analogue of Fan's result holds for inverse H-matrices under certain additional conditions. Now, we investigate whether the analogue of Fan's result holds in the context of inverse H-matrices. In general, this analogue does not hold. Specifically, even if both T and $(I - T^{-1})$ are inverse H-matrices, it does not follow that $T - I$ is also an inverse H-matrix. Conversely, if $T - I$ is known to be an inverse H-matrix, this does not imply that T is an inverse H-matrix.

Example 1. These examples illustrate that the analogue of Fan's result does not generally hold for inverse H-matrices.

1. Let $H = \begin{pmatrix} 3/8 & -3/8 & 9/8 \\ -1/4 & 1/4 & 5/4 \\ 3/2 & 5/2 & -23/2 \end{pmatrix}$. We compute

$$(H - I)^{-1} = \begin{pmatrix} -50/23 & 15/23 & -3/23 \\ 10/23 & -49/23 & -4/23 \\ -4/23 & -8/23 & -3/23 \end{pmatrix}$$

Then, the inverse of its comparison matrix is:

$$\left(\mathcal{M}((H - I)^{-1})\right)^{-1} = \begin{pmatrix} 5/8 & 3/8 & 9/8 \\ 1/4 & 3/4 & 5/4 \\ 3/2 & 5/2 & 25/2 \end{pmatrix}$$

By **Theorem 1**(ii), since $\mathcal{M}((H-I)^{-1})$ is an invertible M-matrix, we conclude that $(H-I)^{-1}$ is an invertible H-matrix. Hence, $H-I$ is an inverse H-matrix. Now consider

$$H^{-1} = \begin{pmatrix} 2 & 1/2 & 1/4 \\ 1/3 & 2 & 1/4 \\ 1/3 & 1/2 & 0 \end{pmatrix}$$

However, it can be verified that $\left(\mathcal{M}(H^{-1})\right)^{-1} \ngeq 0$, so by **Theorem 1** (ii), H^{-1} is not an invertible H-matrix. Therefore, H is not an inverse H-matrix, even though $H-I$ is.

2. Let $X = \begin{pmatrix} 3/2 & 1/2 \\ -1/2 & 3/2 \end{pmatrix}$. Then $X^{-1} = \begin{pmatrix} 3/5 & -1/5 \\ 1/5 & 3/5 \end{pmatrix}$. Since X^{-1} is SDD, then by **Theorem 5**, X^{-1} is an invertible H-matrix. Therefore, X is an inverse H-matrix. Next,

$$I - X^{-1} = \begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix}, (I-X^{-1})^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Since $(I-X^{-1})^{-1}$ is SDD, then it is also an invertible H-matrix. Therefore, $I-X^{-1}$ is an inverse H-matrix. However,

$$X - I = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}, \left(\mathcal{M}((X-I)^{-1})\right)^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Since there is no vector $\bar{v} \in \mathbb{R}^k$ *with* $\bar{v} > 0$ *such that* $\mathcal{M}((X-I)^{-1})\bar{v} > 0$, then by **Theorem 1** (iii), $\mathcal{M}((X-I)^{-1})$ is not an invertible M-matrix. So, $X-I$ is not an inverse H-matrix.

Motivated by **Theorem 3**, which provides a condition such that the analogue of Fan's result holds for invertible H-matrices, then by adding some additional conditions, we obtained an analogue for inverse H-matrices.

Theorem 6. *Let $T \in \mathbb{R}^{k\times k}$ with $D_{(T-I)^{-1}+I} \geq I$ and $|((T-I)^{-1}+I)^{-1}|e < e$. If $T-I$ is an inverse H-matrix, then both T and $I-T^{-1}$ are inverse H-matrices.*

Proof. Suppose that T is a $k \times k$ real matrix with $D_{(T-I)^{-1}+I} \geq I$ and $|((T-I)^{-1}+I)^{-1}|e < e$ and $T-I$ is an inverse H-matrix. Then, $(T-I)^{-1}$ is an invertible H-matrix; by definition, $\mathcal{M}((T-I)^{-1})$ is an invertible M-matrix.

We aim to show that T and $I-T^{-1}$ are inverse H-matrices. Let us define

$$S \coloneqq (T-I)^{-1} + I$$

$$S - I = (T-I)^{-1}$$

which is an invertible H-matrix.

Since $D_{(T-I)^{-1}+I} \geq I$ and $|((T-I)^{-1}+I)^{-1}|e < e$, then we have $D_S \geq I$ and $|S^{-1}|e < e$, **Theorem 3** implies that both S and $I-S^{-1}$ are invertible H-matrices. Now observe that:

$$T = I + (S-I)^{-1} = (S-I)^{-1}\left((S-I)+I\right) = \left(S(I-S^{-1})\right)^{-1}S = (I-S^{-1})^{-1}$$

So, we have

$$T^{-1} = I - S^{-1}$$

Since S^{-1} is an invertible H-matrix, so is $I-S^{-1}$, and hence T is an inverse H-matrix. Finally, since $S^{-1} = I-T^{-1}$, then

$$(I-T^{-1})^{-1} = S,$$

which is also an invertible H-matrix. Thus, $I-T^{-1}$ is an inverse H-matrix. ■

Example 2. Here is an example to illustrate the theorem. Let

$$T = \begin{pmatrix} 38/17 & -9/17 & -3/17 \\ -6/17 & 39/17 & -4/17 \\ -4/17 & -8/17 & 37/17 \end{pmatrix}$$

Then

$$(T-I)^{-1} + I = \begin{pmatrix} 2 & 1/2 & 1/4 \\ 1/3 & 2 & 1/4 \\ 1/3 & 1/2 & 2 \end{pmatrix}$$

$$((T-I)^{-1} + I)^{-1} = \begin{pmatrix} 93/176 & -21/176 & -9/176 \\ -7/88 & 47/88 & -5/88 \\ -3/44 & -5/44 & 23/44 \end{pmatrix}$$

Clearly, $D_{(T-I)^{-1}+I} \geq I$ and $|((T-I)^{-1}+I)^{-1}|e < e$.
Moreover,

$$(T-I)^{-1} = \begin{pmatrix} 1 & 1/2 & 1/4 \\ 1/3 & 1 & 1/4 \\ 1/3 & 1/2 & 1 \end{pmatrix},$$

which is SDD, and by **Theorem 5**, we have $(T-I)^{-1}$ is an invertible H-matrix. Hence, $(T-I)$ is an inverse H-matrix. Next,

$$T^{-1} = \begin{pmatrix} 83/176 & 21/176 & 9/176 \\ 7/88 & 41/88 & 5/88 \\ 3/44 & 5/44 & 21/44 \end{pmatrix},$$

which is also SDD and therefore an invertible H-matrix. Thus, T is an inverse H-matrix.

Finally,

$$I - T^{-1} = \begin{pmatrix} 93/176 & -21/176 & -9/176 \\ -7/88 & 47/88 & -5/88 \\ -3/44 & -5/44 & 23/44 \end{pmatrix}, (I-T^{-1})^{-1} = \begin{pmatrix} 2 & 1/2 & 1/4 \\ 1/3 & 2 & 1/4 \\ 1/3 & 1/2 & 2 \end{pmatrix}$$

Since $(I-T^{-1})^{-1}$ is also SDD, then by **Theorem 5** $(I-T^{-1})^{-1}$ is an invertible H-matrix. Therefore, $I - T^{-1}$ is an inverse H-matrix.

Now, motivated by **Theorem 4**, we demonstrate that the converse also holds under certain additional conditions.

Theorem 7. *Let $T \in \mathbb{R}^{k\times k}$ with $\mathcal{M}((I-T^{-1})^{-1})e > e$ and all diagonal entries of $(I-T^{-1})^{-1}$ are nonnegative. If both T and $I-T^{-1}$ are inverse H-matrices, then $T-I$ is also an inverse H-matrix.*

Proof. Assume that T and $I-T^{-1}$ are inverse H-matrices. Consequently, T^{-1} and $(I-T^{-1})^{-1}$ are invertible H-matrices. Define

$$S \coloneqq (I-T^{-1})^{-1}$$

It follows that S and $I - S^{-1} = T^{-1}$ are invertible H-matrices. Furthermore, if $\mathcal{M}(S)e > e$ and the diagonal entries of S are nonnegative, then by **Theorem 4**, $S - I$ is an invertible H-matrix. Next, we observe that

$$S - I = (I-T^{-1})^{-1} - I = (I-T^{-1})^{-1}\big(I - (I-T^{-1})\big) = (I-T^{-1})^{-1}T^{-1}$$

Thus,

$$S - I = \big(T(I-T^{-1})\big)^{-1} = (T-I)^{-1},$$

which is an invertible H-matrix. Therefore, $T-I$ is an inverse H-matrix.■

Example 3. Here is an example to illustrate the theorem. Let

$$T = \begin{pmatrix} 11/9 & 5/36 \\ 1/9 & 43/36 \end{pmatrix}$$

Then, we have

$$T^{-1} = \begin{pmatrix} 43/52 & -5/52 \\ -1/13 & 11/13 \end{pmatrix}, (I - T^{-1})^{-1} = \begin{pmatrix} 8 & -5 \\ -4 & 9 \end{pmatrix}$$

Verifying that $\mathcal{M}((I - T^{-1})^{-1})\, e > e$ and all the diagonal entries of $(I - T^{-1})^{-1}$ are nonnegative is straightforward. Since both T^{-1} and $(I - T^{-1})^{-1}$ are SDD, then by **Theorem 5**, they are invertible H-matrices. Therefore, T and $I - T^{-1}$ are inverse H-matrices. Now compute:

$$T - I = \begin{pmatrix} 2/9 & 5/36 \\ 1/9 & 7/36 \end{pmatrix}, (T - I)^{-1} = \begin{pmatrix} 7 & -5 \\ -4 & 8 \end{pmatrix}$$

Since $(T - I)^{-1}$ is also SDD, it follows from **Theorem 5** that $(T - I)^{-1}$ is an invertible H-matrix. So, $T - I$ is an inverse H-matrix.

We now observe a different class of matrices. In this part, we show that the analogue of Fan's result and its converse does not hold in general for the class of group inverse M-matrices. Additionally, we demonstrate that the class of group inverse H-matrices contains all group inverse M-matrices. Consequently, the analogue of Fan's result and its converse also fail for group inverse H-matrices. We begin by defining the relevant matrix classes. A real matrix $T \in \mathbb{R}^{k\times k}$ is called a group inverse M-matrix, if its group inverse $T^{\#}$ is an M-matrix. Similarly, a complex matrix $T \in \mathbb{C}^{k\times k}$ is called a group inverse H-matrix if $T^{\#}$ is an H-matrix.

As previously discussed, the analogue of Fan's result does not generally hold for group inverse M-matrices. In particular, the matrix $X - I$ need not be a group inverse M-matrix, even if both X and $(I - X^{\#})$ are group inverse M-matrices. Conversely, X need not be a group inverse M-matrix, even if $X - I$ is. These facts are supported by examples originally discussed in [25], and we include them here with additional explanations and details.

Example 4. Let $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Using full rank factorization, write $B = FG$ where $F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $G = (0 \quad 1)$. Then $B^{\#} = F(GF)^{-2}\, G = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We observe that $B^{\#}$ can be written in the form $B^{\#} = sI - C$ where $C \geq 0$ and $s \geq \rho(C)$ for $s = 1$ and then $B^{\#}$ is an M-matrix. Hence, B is a group inverse M-matrix. Next, consider $I - B^{\#} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which also admits a full rank factorization and satisfies the M-matrix condition. Thus, $(I - B^{\#})$ is a group inverse M-matrix. Now, consider $B - I = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. It is clear that $(B - I)^{\#} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. Suppose $(B - I)^{\#}$ is an M-matrix. Then, there exists $t > 0$ such that $(B - I)^{\#} = tI - D$ where $D \geq 0$ and $t \geq \rho(D)$. Note that $t \geq 1$ is required due to the diagonal entry -1. However, this implies that $\rho(D) = 1 + t > t$, contradicts the M-matrix condition $t \geq \rho(D)$. Therefore, $(B - I)^{\#}$ is not an M-matrix, and $B - I$ is not a group inverse M-matrix.

Example 5. Let $P = \frac{1}{10}\begin{pmatrix} 11 & 0 \\ -3 & 10 \end{pmatrix}$. Note that $P - I = \begin{pmatrix} 1/10 & 0 \\ -3/10 & 0 \end{pmatrix}$. Using full rank factorization, we obtain $(P - I)^{\#} = \begin{pmatrix} 10 & 0 \\ -30 & 0 \end{pmatrix}$. Since it can be expressed as $(P - I)^{\#} = sI - X$, with $X = \begin{pmatrix} 0 & 0 \\ 30 & 11 \end{pmatrix}$, $s = 10$, and $s \geq \rho(X)$, then it is clear that $(P - I)^{\#}$ is an M-matrix. Hence, $P - I$ is a group inverse M-matrix. Since P is invertible, its group inverse is $P^{\#} = P^{-1}$, which equals $P^{\#} = \frac{1}{11}\begin{pmatrix} 10 & 0 \\ 3 & 11 \end{pmatrix}$. Observe that $P^{\#}$ is not even a Z-matrix, then $P^{\#}$ is not an M-matrix. Therefore, P is not a group inverse M-matrix.

Now, let us consider the analogue for group inverse H-matrices. Let $\mathcal{GM}$ be the class of all group inverse M-matrices, and $\mathcal{GH}$ be the class of all group inverse H-matrices. Recall that any arbitrary M-matrix is also an H-matrix. Therefore, if $A \in \mathcal{GM}$, then by definition, $A^{\#}$ is an M-matrix and then it is also an H-matrix. This implies that $A \in \mathcal{GH}$, so we have the relation:

$$\mathcal{GM} \subseteq \mathcal{GH}$$

From this relation and the counterexamples given earlier, we conclude that the analogue of Fan's result and its converse also fail for group inverse H-matrices. In particular, if Y and $(I - Y^{\#})$ are group inverse H-matrices, then $Y - I$ need not be a group inverse H-matrix. Conversely, even if $Y - I$ is a group inverse H-matrix, Y need not be a group inverse H-matrix.

4. CONCLUSION

In this study, we established the analogue of Fan's result and its converse for inverse H-matrices. We further revealed that the analogue does not generally hold for the group inverse H-matrices, highlighting the need for exact structural characterizations. The impact of this work lies in providing criteria to design efficient preconditioners in iterative solvers and advancing numerical linear algebra. Future work will extend the concepts developed here to other matrix classes, such as singular irreducible H-matrices and H-matrices with "property c", thereby further developing the understanding of inverse-related matrix structures.

AUTHOR CONTRIBUTIONS

Jeriko Gormantara: Conceptualization, Funding Acquisition, Supervision, Writing - Review and Editing. Hanni Garminia: Investigation, Methodology. Amir Kamal Amir: Project Administration, Validation. Evan Ramdan: Writing - Original Draft. All authors discussed the results and contributed to the final manuscript.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest to report.

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