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ASYMPTOTIC DISTRIBUTIONS OF ESTIMATORS FOR THE MEAN AND THE VARIANCE OF A COMPOUND CYCLIC POISSON PROCESS

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ABSTRACT

A stochastic process has an important role in modeling various real phenomena. One special form of the stochastic process is a compound Poisson process. A compound Poisson process model can be extended by generalizing the corresponding Poisson process. One of them is using a cyclic Poisson process. Our goals in this research are to determine the asymptotic distribution of the estimator for the mean and the variance of this process. In this paper, the problems of estimating the mean function and the variance function of a compound cyclic Poisson process are considered. We do not assume any parametric form for the intensity function except that it is periodic. We also consider the case when only a single realization of the cyclic Poisson process is observed in a bounded interval. Consistent estimators for the mean and variance functions of this process have been proposed in respectively. This paper introduces a set of novel theorems that, to the best of our knowledge, are not available in the existing literature and contribute original results to the field. Asymptotic distributions of these estimators are established when the size of the observation interval indefinitely expands. Asymptotic distributions of $\hat{\psi}_n(t)$ and $\hat{V}_n(t)$ are, respectively $\sqrt{n}\left(\hat{\psi}_n(t)-\psi(t)\right)$ $N\left(0,\left(1+k_{t,\tau}\right)^2\Lambda(t_r)\tau\mu^2+{k_{t,\tau}}^2\Lambda^c(t_r)\tau\mu^2+\frac{\sigma^2\Lambda(t)^2}{\theta}\right)$ and $\sqrt{n}\left(\hat{V}_n(t)-V(t)\right)\xrightarrow{d}$ $N\left(0,\left(1+k_{t,\tau}\right)^2\Lambda(t_r)\tau{\mu_2}^2+{k_{t,\tau}}^2\Lambda^c(t_r)\tau{\mu_2}^2+\frac{{\sigma_2}^2\Lambda(t)^2}{\theta}\right)$ as $n\to\infty$.

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1. INTRODUCTION

The Poisson process is one of the fundamental stochastic models widely used to describe random events occurring over a given period [1]. In its basic form, the homogeneous Poisson process assumes that the average number of events per unit time (intensity) remains constant throughout the observation period. This model has been extensively applied across various fields due to its simplicity and analytical tractability, such as in the health sector [2]. The mathematical development of this model has also been studied [3], [4], [5].

Nevertheless, despite its practicality, the assumption of constant intensity becomes a significant limitation when modeling phenomena where the event rate fluctuates over time. In situations where the intensity is influenced by seasonal factors or other temporal trends, the homogeneous Poisson process fails to adequately capture the underlying dynamics [6], [7]. Other studies also emphasize that the inability of this model to represent time-varying behaviors has a substantial impact on the accuracy of estimation and prediction in various real-world applications [8]. This limitation becomes even more critical when dealing with phenomena characterized by strong periodic or seasonal patterns, such as network traffic, financial market cycles, or the spread of infectious diseases [9].

To address this limitation, researchers have developed the non-homogeneous Poisson process, in which the event intensity is allowed to vary as a function of time. This model offers greater flexibility and can accommodate temporal variations in data. Several studies have applied this model, such as device reliability [6], automotive warranty [10], health diseases [11], and road accidents [12]. In addition, the extension of the Poisson framework to phase-type mixed Poisson processes can further overcome some limitations of the classical Poisson model, particularly in the context of shock modeling and reliability engineering [13]. Despite its flexibility, the non-homogeneous Poisson process often requires more complex parameter estimation procedures and sufficient historical data to accurately model the intensity function [14]. Furthermore, if the chosen time-dependent intensity function does not effectively capture the real patterns of fluctuation, the model may yield poor estimators.

Motivated by these considerations, the present research focuses on the asymptotic distributional properties of a more generalized and flexible model known as the compound cyclic Poisson process. This process is designed to capture periodic fluctuations in event intensity while accommodating random event magnitudes through the compound structure. The model not only accounts for the cyclic behavior in event counts but also incorporates the randomness in the magnitude of each event. Integrating periodicity with a compound structure provides a more realistic representation of real-world phenomena with inherent cycles, while also handling dependency structures that traditional Poisson models cannot explain [15]. This makes the model highly relevant for applications in fields characterized by seasonal or cyclical trends, such as financial transaction volumes, pollution spikes, or insurance claim frequencies during peak seasons.

The main objective of this study is to derive and analyze the asymptotic behavior of the mean and variance function with a compound cyclic Poisson process, thereby contributing to the theoretical foundation for modeling periodic random phenomena. Such an analysis is crucial for understanding the limiting distributions and long-term statistical properties of this class of stochastic processes. These insights have significant practical value in applied fields where seasonality and cyclic trends are prominent. For instance, in finance, this model can help in better forecasting of transaction volumes or risk factors that follow market cycles. In environmental sciences, it can model recurring natural events as rainfall patterns or pollution spikes. Similarly, in telecommunications or healthcare analytics, where traffic or patient arrivals follow periodic patterns, this research provides more accurate tools for prediction and resource allocation. By offering a robust framework to handle both the cyclic nature and the random systems with inherent periodicity.

Additionally, this research aims to bridge gaps in the existing literature, which has predominantly focused on homogeneous and non-homogeneous Poisson processes but has not explicitly integrated both periodic fluctuations and random magnitudes within a unified framework. The theoretical contributions of this work are expected to enrich the development of stochastic process theory while offering significant practical implications for fields such as actuarial science, risk management, financial modeling, and environmental science.

2. RESEARCH METHODS

2.1 Conceptual Framework

To guide the theoretical investigation of compound cyclic Poisson process, this study focusses in two primary statistical characteristics, which are the expected value function and the variance function. For each characteristic, corresponding estimators are constructed based on the compound structure and periodic nature of the process. The asymptotic distribution of these estimators remains an open problem and is central to the ongoing theoretical exploration. This dual pathway centered on expectation and variance highlights the complexity and novelty of the model under study. The conceptual framework of this study is illustrated in Fig. 1.

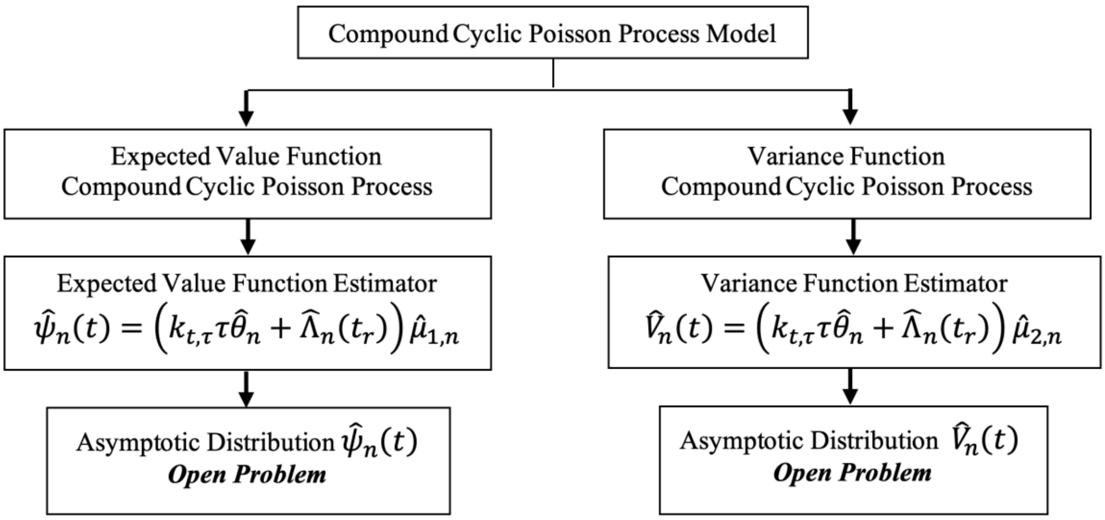

Figure 1. Conceptual Framework of The Study

The focus of this research lies in the development of stochastic process theory. The methodological approach consists of the following stages.

2.2 Preliminary Study

1. A comprehensive review and understanding of the compound cyclic Poisson process, including its structure, behavior, and relevance in real-world applications.

   This study begins with a thorough examination of the compound cyclic Poisson process, a stochastic model that captures both the cyclic or seasonal nature of event arrivals and the randomness in event magnitudes. The compound cyclic Poisson process generalizes the classical Poisson [1] and compound Poisson processes in [16] by introducing time time-periodic intensity function and integrating random valued, making it highly suitable for modeling aggregated phenomena such as insurance claims, financial transactions, and environmental occurrences with seasonal behavior. By analyzing its structural components, namely the periodic intensity function and the compounding random variables, the study contextualizes the model within existing literature and highlights its practical advantages in capturing real-world cyclic randomness. This foundational understanding sets the stage for addressing theoretical properties, especially in the context of long-term statistical behavior.

2. An in-depth mathematical investigation aimed at exploring the theoretical foundation necessary to construct a novel model by extending existing stochastic process theories.

   Building on this structural understanding, the study proceeds to conduct a rigorous mathematical analysis focused on deriving the asymptotic distributions of the estimators for the mean and variance function associated with the compound cyclic Poisson process. While prior studies have considered asymptotic properties of estimators in classical Poisson or compound Poisson frameworks, they typically assume stationary or non-periodic intensity structures [17]. In contrast, this work extends existing stochastic process theory by incorporating periodicity into the compounding framework and formulating consistent estimators that reflect the dual randomness of both frequency and magnitude. The key objective is to investigate the limiting behavior of these estimators, denoted as $\hat{\psi}_n(t)$ for the mean and $\hat{V}_n(t)$ for the variance, as the number of observations increases. This asymptotic analysis not only contributes to the theoretical enrichment of compound cyclic models but also lays the groundwork for future statistical inference in applied domains.

2.3 Core Research Analysis

The primary objective of this stage is to derive the asymptotics of estimators for both the mean function and the variance function of the compound cyclic Poisson process. The approach involves rigorous analysis and limit theorems to evaluate the statistical properties of the proposed estimators.

Let $\{N(t), t \geq 0\}$ be a non-homogeneous Poisson process with (unknown) locally integrable intensity function λ. The intensity function λ is assumed to be periodic with (known) period $\tau > 0$. We do not assume any (parametric) form of λ except that it is periodic, Eq. (1).

$$\lambda(s) = \lambda(s + k\tau), \tag{1}$$

holds for all $s \geq 0$ and $k \in \boldsymbol{Z}$, with $\boldsymbol{Z}$ denotes the set of integers. This condition of the intensity function is also considered in [16]. Let $\{Y(t), t \geq 0\}$ be a process with

$$Y(t) = \sum_{i=1}^{N(t)} X_i\,, \tag{2}$$

where $\{X_i, i \geq 1\}$ is a sequence of independent and identically distributed random variables with mean $\mu < \infty$ and variance $\sigma^2 < \infty$, which is also independent of the process $\{N(t), t \geq 0\}$. The process $\{Y(t), t \geq 0\}$ is called a compound cyclic Poisson process. The model presented in Eq. (2) is a generalization of the (well-known) compound Poisson process, which assumes that $\{N(t), t \geq 0\}$ is a homogeneous Poisson process.

Suppose that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the cyclic Poisson process $\{N(t), t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Furthermore, suppose that for each data point in the observed realization $N(\omega) \cap [0, n]$, say i-th data point, $i = 1,2, \ldots, N([0, n])$, its corresponding random variable X_i is also observed.

The mean function (expected value) of $Y(t)$, denoted by $\psi(t)$, is given by Eq. (3)

$$\psi(t) = E\big(N(t)\big)E(X_1) = \Lambda(t)\mu, \tag{3}$$

with $\Lambda(t) = \int_0^t \lambda(s)ds$.

Let $E[X_1^2] = \mu_2$. The variance function of $Y(t)$, denoted by $V(t)$, is given by Eq. (4).

$$V(t) = E\big(N(t)\big)E\big({X_1}^2\big) = \Lambda(t)\mu_2. \tag{4}$$

Let $t_r = t - \left\lfloor \frac{t}{\tau} \right\rfloor \tau$, where for any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, and let also $k_{t,\tau} = \left\lfloor \frac{t}{\tau} \right\rfloor$. Then, for any given real number $t \geq 0$, we can write $t = k_{t,\tau}\tau + t_r$, with $0 \leq t_r \leq \tau$. Let $\theta = \frac{1}{\tau}\int_0^\tau \lambda(s)ds$, that is the global intensity of the cyclic Poisson process $\{N(t), t \geq 0\}$. We assume that Eq. (5):

$$\theta > 0. \tag{5}$$

Then for any given $t \geq 0$, we have Eq. (6):

$$\Lambda(t) = k_{t,\tau}\tau\theta + \Lambda(t_r). \tag{6}$$

By substituting Eq. (6) into Eqs. (3) and (4), the mean and the variance functions of the compound cyclic Poisson process can be written as Eqs. (7) and (8).

$$\psi(t) = \Big(k_{t,\tau}\tau\theta + \Lambda(t_r)\Big)\mu, \tag{7}$$

$$V(t) = \left(k_{t,\tau}\tau\theta + \Lambda(t_r)\right)\mu_2. \tag{8}$$

Consistent estimators for the mean function $\psi(t)$ and the variance $V(t)$ of the process $\{Y(t), t \geq 0\}$ using the observed realization and asymptotic approximation to biases and variances of these estimators have been computed in [17]. Our goals in this paper are to establish asymptotic distributions of these estimators when the size of the observation interval indefinitely expands.

The rest of this paper is organized as follows. The estimators, main results, and some technical lemmas, which are needed in the proof of our theorems, are presented in Section 3. Some related works of compound Poisson process can be found in [16], [18], [19], [20], [21].

3. RESULTS AND DISCUSSION

3.1 The Estimators and Main Results

Let $k_{n,\tau} = \left\lfloor \frac{n}{\tau} \right\rfloor$. Estimators of the mean function $\psi(t)$ and the variance function $V(t)$ have been formulated in [17] as Eqs. (9) and (10).

$$\hat{\psi}_n(t) = \left(k_{t,\tau}\tau\hat{\theta}_n + \hat{\Lambda}_n(t_r)\right)\hat{\mu}_n, \tag{9}$$

$$\hat{V}_n(t) = \left(k_{t,\tau}\tau\hat{\theta}_n + \hat{\Lambda}_n(t_r)\right)\hat{\mu}_{2,n}, \tag{10}$$

with $\hat{\psi}_n(t) = 0$ and $\hat{V}_n(t) = 0$ when $N([0,n]) = 0$, where

$\hat{\theta}_n = \frac{1}{k_{n,\tau}\tau}\sum_{k=0}^{k_{n,\tau}-1} N([k\tau, k\tau+\tau]); \hat{\Lambda}_n(t_r) = \frac{1}{k_{n,\tau}}\sum_{k=0}^{k_{n,\tau}-1} N([k\tau, k\tau+t_r])\,; \hat{\mu}_n = \frac{1}{N([0,n])}\sum_{i=1}^{N([0,n])} X_i\,;$ and $\hat{\mu}_{2,n} = \frac{1}{N([0,n])}\sum_{i=1}^{N([0,n])} X_i{}^2.$

Since $\hat{\theta}_n$ and $\hat{\Lambda}_n(t_r)$ are not independent, to make it easier to formulate the asymptotic distribution of $\hat{\psi}_n(t)$ and $\hat{V}_n(t)$, it is needed to write $k_{t,\tau}\tau\hat{\theta}_n + \hat{\Lambda}_n(t_r)$ as a weighted sum of two independent random variables. To do this, first note that $\theta\tau = \int_0^\tau \lambda(s)ds$ can be written as $\Lambda(t_r) + \Lambda^c(t_r)$, with $\Lambda(t_r)$ as defined about and $\Lambda^c(t_r) = \int_{t_r}^\tau \lambda(s)ds$. Then, for any $t > 0$, we can write Eq. (11):

$$\Lambda(t) = (1 + k_{t,\tau})\Lambda(t_r) + k_{t,\tau}\Lambda^c(t_r). \tag{11}$$

Hence, the estimators of $\Lambda(t)$ can be written as Eq. (12):

$$\hat{\Lambda}_n(t) = (1 + k_{t,\tau})\hat{\Lambda}_n(t_r) + k_{t,\tau}\hat{\Lambda}_n{}^c(t_r), \tag{12}$$

where $\hat{\Lambda}_n(t_r)$ is defined as before and Eq. (13) as follows:

$$\hat{\Lambda}_n{}^c(t_r) = \frac{1}{k_{n,\tau}}\sum_{k=0}^{k_{n,\tau}-1} N([k\tau + t_r, k\tau + \tau]). \tag{13}$$

Note also that $\hat{\Lambda}_n(t_r)$ and $\hat{\Lambda}_n{}^c(t_r)$ are independent. Finally, our estimators of $\hat{\psi}_n(t)$ and $\hat{V}_n(t)$ now can be written as Eqs. (14) and (15).

$$\hat{\psi}_n(t) = \left((1 + k_{t,\tau})\hat{\Lambda}_n(t_r) + k_{t,\tau}\hat{\Lambda}_n{}^c(t_r)\right)\hat{\mu}_n, \tag{14}$$

$$\hat{V}_n(t) = \left((1 + k_{t,\tau})\hat{\Lambda}_n(t_r) + k_{t,\tau}\hat{\Lambda}_n{}^c(t_r)\right)\hat{\mu}_{2,n}. \tag{15}$$

3.2 Some Technical Lemmas

In this section, we present four lemmas that are needed in the proofs of our theorems.

Lemma 1. *Suppose that the intensity function* λ *satisfies* Eq. (1) *and is locally integrable. For all* $t > 0$, *then*

$$\sqrt{\frac{n}{\tau}}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right)\xrightarrow{d}\mathrm{N}\left(0,\left(k_{t,\tau}+1\right)^2\Lambda(t_r)+\left(k_{t,\tau}\right)^2\Lambda^c(t_r)\right)$$

as $n\to\infty$.

Proof. Using Eqs. (11) and (12), the left side of the equation can be written as

$$\sqrt{n}\left(\left((1+k_{t,\tau})\widehat{\Lambda}_n(t_r)+k_{t,\tau}\widehat{\Lambda}_n{}^c(t_r)\right)-\left((1+k_{t,\tau})\Lambda(t_r)+k_{t,\tau}\Lambda^c(t_r)\right)\right)=\sqrt{n}\,(1+k_{t,\tau})$$

$$\left(\widehat{\Lambda}_n(t_r)-\Lambda(t_r)\right)+\sqrt{n}\,k_{t,\tau}\left(\widehat{\Lambda}_n{}^c(t_r)-\Lambda^c(t_r)\right).$$

Since $\widehat{\Lambda}_n(t_r)$ and $\widehat{\Lambda}_n{}^c(t_r)$ are two independent random variables. To prove Lemma 1, it suffices to prove

$$\sqrt{n}\,(1+k_{t,\tau})\left(\widehat{\Lambda}_n(t_r)-\Lambda(t_r)\right)\to\mathrm{N}\left(0,\left(k_{t,\tau}+1\right)^2\Lambda(t_r)\tau\right)$$

or

$$\sqrt{n}\,\left(\widehat{\Lambda}_n(t_r)-\Lambda(t_r)\right)\to\mathrm{N}\left(0,\Lambda(t_r)\tau\right)\tag{16}$$

and

$$\sqrt{n}\,k_{t,\tau}\left(\widehat{\Lambda}_n{}^c(t_r)-\Lambda^c(t_r)\right)\to\text{Normal}\left(0,\left(k_{t,\tau}\right)^2\Lambda^c(t_r)\tau\right)$$

or

$$\sqrt{n}\,\left(\widehat{\Lambda}_n{}^c(t_r)-\Lambda^c(t_r)\right)\to\mathrm{N}\left(0,\Lambda^c(t_r)\tau\right)\tag{17}$$

as $n\to\infty$.

First, prove Eq. (16). The left side of Eq. (16) can be written as:

$$\sqrt{\frac{n\Lambda(t_r)}{k_{n,\tau}}}\left(\frac{\sum_{k=0}^{k_{n,\tau}-1}N([k\tau,k\tau+t_r])-k_{n,\tau}\Lambda(t_r)}{\sqrt{k_{n,\tau}\Lambda(t_r)}}\right).$$

Since $\sum_{k=0}^{k_{n,\tau}-1}N([k\tau,k\tau+t_r])$ is the accumulation of independent and identically distributed random variables with $E\left(N([k\tau,k\tau+t_r])\right)=Var\left(N([k\tau,k\tau+t_r])\right)=\Lambda(t_r)$. Then, based on the Central Limit Theorem $\left(\frac{\sum_{k=0}^{k_{n,\tau}-1}N([k\tau,k\tau+t_r])-k_{n,\tau}\Lambda(t_r)}{\sqrt{k_{n,\tau}\Lambda(t_r)}}\right)\to\mathrm{N}\,(0,1)$ as $n\to\infty$. To prove Eq. (16) just show

$$\sqrt{\frac{n\Lambda(t_r)}{k_{n,\tau}}}\to\sqrt{\Lambda(t_r)\tau}$$

as $n\to\infty$. The left side of this equation can be written as follows

$$\begin{aligned}\sqrt{\frac{n\Lambda(t_r)}{k_{n,\tau}}}&=\sqrt{\tau\Lambda(t_r)\left(\frac{n/\tau}{k_{n,\tau}}\right)}\\&=\sqrt{\tau\Lambda(t_r)\left(\frac{k_{n,\tau}-\left(k_{n,\tau}-n/\tau\right)}{k_{n,\tau}}\right)}\\&=\sqrt{\tau\,\Lambda(t_r)\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)}\\&=\sqrt{\tau\,\Lambda(t_r)}+o(1),\end{aligned}$$

as $n\to\infty$. Eq. (16) is proven. By using a similar way to the proof of Eq. (16), Eq. (17) can be proven. The proof of Lemma 1 is complete. ■

Lemma 2. *Suppose that the intensity function* λ *satisfies* Eq. (1) *and is locally integrable. If in addition,* Eq. (5) *holds, then with probability* $1, N([0,n]) \to \infty$ *as* $n \to \infty$.

Proof. Lemma 2 has been proved in [17]. ■

Lemma 3. *If* $X_1, X_2, \ldots$ *are independent and identically distributed random variables, each with means* $\mu < \infty$ *and finite variance* $\sigma^2 < \infty$*, we have*

$$\sqrt{\frac{n}{\tau}}\left(\hat{\mu}_{1,n} - \mu_1\right) \xrightarrow{d} \mathrm{N}\left(0, \frac{\theta}{{\sigma_1}^2}\right)$$

as $n \to \infty$.

Proof. From the left side, we have Eq. (18) as follows:

$$\sqrt{n}\left(\hat{\mu}_{1,n} - \mu_1\right) = \frac{\sigma_1 \sqrt{n}}{\sqrt{N([0,n])}} \frac{\sqrt{N([0,n])}}{\sigma_1}\left(\hat{\mu}_{1,n} - \mu_1\right), \tag{18}$$

where Eq. (19):

$$\frac{\sigma_1 \sqrt{n}}{\sqrt{N([0,n])}} = \frac{\sigma_1}{\sqrt{\frac{N([0,n])}{n}}} = \frac{\sigma_1}{\sqrt{\theta + o_p(1)}} = \frac{\sigma_1}{\sqrt{\theta}} + o_p(1), \tag{19}$$

and Eq. (20) :

$$\begin{aligned} \frac{\sqrt{N([0,n])}}{\sigma_1}\left(\hat{\mu}_{1,n} - \mu_1\right) &= \frac{\sqrt{N([0,n])}}{\sigma_1}\left(\frac{\sum_{i=1}^{N([0,n])} X_i}{\sqrt{N([0,n])}} - \mu_1\right) \\ &= \frac{\sum_{i=1}^{N([0,n])} X_i}{\sigma_1 \sqrt{N([0,n])}} - \sqrt{N([0,n])}\frac{\mu_1}{\sigma_1} \\ &= \frac{X_1 + X_2 + \cdots + X_{N([0,n])} - N([0,n])\mu_1}{\sigma_1 \sqrt{N([0,n])}}. \end{aligned} \tag{20}$$

Based on Central Limit Theorem and Lemma 2, we have this function $\frac{X_1+X_2+\cdots+X_{N([0,n])}-N([0,n])\mu_1}{\sigma_1\sqrt{N([0,n])}} \xrightarrow{d} \mathrm{N}(0,1)$ so $\frac{\sqrt{N([0,n])}}{\sigma_1}\left(\hat{\mu}_{1,n} - \mu_1\right) \xrightarrow{d} \mathrm{N}(0,1)$. Substituting Eqs. (19) into (20), and elaborating on Eq. (18), we have

$$\sqrt{n}\left(\hat{\mu}_{1,n} - \mu_1\right) \xrightarrow{d} \frac{\sigma_1}{\sqrt{\theta}} N(0,1),$$

$$\sqrt{n}\left(\hat{\mu}_{1,n} - \mu_1\right) \xrightarrow{d} N\left(0, \frac{\theta}{{\sigma_1}^2}\right).$$

This completes the proof of Lemma 3. ■

Lemma 4. *If* $X_1, X_2, \ldots$ *are non-negative sequences of independent and identically distributed random variables, each with means* $\mu < \infty$ *and finite variance* $\sigma^2 < \infty$*, we have*

$$\sqrt{n}\left(\hat{\mu}_{2,n} - \mu_2\right) \xrightarrow{d} \mathrm{N}\left(0, \frac{\theta}{{\sigma_2}^2}\right)$$

as $n \to \infty$.

Proof. Since $\mathrm{X}_1, \mathrm{X}_2, \ldots$ are non-negative sequences, similarly to Lemma 3, Lemma 4 can be obtained. ■

3.3 Some Technical Theorems

Our main results are presented in the following two theorems. The first theorem is about the asymptotic distribution of $\hat{\psi}_n(t)$ and the second theorem is about the asymptotic distribution of $\hat{V}_n(t)$.

Theorem 1. *Suppose that the intensity function λ satisfies Eq. (1) and is locally integrable. If in addition, $Y(t)$ satisfies the condition in Eq. (2), then*

$$\sqrt{\frac{n}{\tau}}\left(\hat{\psi}_n(t)-\psi(t)\right)\xrightarrow{d}\mathrm{N}\left(0,\left(1+k_{t,\tau}\right)^2\Lambda(t_r){\mu_1}^2+{k_{t,\tau}}^2\Lambda^c(t_r){\mu_1}^2+\frac{\theta}{\tau{\sigma_1}^2}\right)$$

as $n\to\infty$.

Proof. To check Theorem 1, using Eqs. (6) and (7), we have Eq. (21)

$$\begin{aligned}\sqrt{n}\left(\hat{\psi}_n(t)-\psi(t)\right)&=\sqrt{n}\left(\widehat{\Lambda}_n(t)\hat{\mu}_{1,n}-\Lambda(t)\mu_1\right)\\&=\sqrt{n}\left(\widehat{\Lambda}_n(t)\hat{\mu}_{1,n}-\Lambda(t)\hat{\mu}_{1,n}+\Lambda(t)\hat{\mu}_{1,n}-\Lambda(t)\mu_1\right)\\&=\sqrt{n}\left(\hat{\mu}_{1,n}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right)+\Lambda(t)\left(\hat{\mu}_{1,n}-\mu_1\right)\right)\\&=\hat{\mu}_{1,n}\sqrt{n}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right)+\Lambda(t)\sqrt{n}\left(\hat{\mu}_{1,n}-\mu_1\right).\end{aligned}\tag{21}$$

Based on Lemma 1 and the weak law of large numbers, where $\hat{\mu}_{1,n}\xrightarrow{P}\mu_1$ and with the Slutsky characteristics, we can obtain Eq. (22).

$$\hat{\mu}_{1,n}\sqrt{n}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right)\xrightarrow{d}\mathrm{N}\left(0,\left(1+k_{t,\tau}\right)^2\Lambda(t_r)\tau{\mu_1}^2+{k_{t,\tau}}^2\Lambda^c(t_r)\tau{\mu_1}^2\right).\tag{22}$$

Then, based on Lemma 3, we have Eq. (23):

$$\Lambda(t)\sqrt{n}\left(\hat{\mu}_{1,n}-\mu_1\right)\xrightarrow{d}\mathrm{N}\left(0,\frac{\theta\Lambda(t)^2}{{\sigma_1}^2}\right).\tag{23}$$

If $\hat{\mu}_{1,n}\sqrt{n}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right)$ and $\Lambda(t)\sqrt{n}\left(\hat{\mu}_{1,n}-\mu_1\right)$ are jointly normally distributed random variables, then $\hat{\mu}_{1,n}\sqrt{n}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right)+\Lambda(t)\sqrt{n}\left(\hat{\mu}_{1,n}-\mu_1\right)=\sqrt{n}\left(\hat{\psi}_n(t)-\psi(t)\right)$ is still normally distributed with Eq. (24):

$$E\left(\sqrt{n}\left(\hat{\psi}_n(t)-\psi(t)\right)\right)=E\left(\hat{\mu}_{1,n}\sqrt{n}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right)\right)+E\left(\Lambda(t)\sqrt{n}\left(\hat{\mu}_{1,n}-\mu_1\right)\right)=0\tag{24}$$

and Eq. (25):

$$\begin{aligned}Var\left(\sqrt{n}\left(\hat{\psi}_n(t)-\psi(t)\right)\right)&=Var\left(\hat{\mu}_{1,n}\sqrt{n}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right)+\Lambda(t)\sqrt{n}\left(\hat{\mu}_{1,n}-\mu_1\right)\right)\\&=Var\left(\hat{\mu}_{1,n}\sqrt{n}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right)\right)+Var\left(\Lambda(t)\sqrt{n}\left(\hat{\mu}_{1,n}-\mu_1\right)\right)\\&\quad+2cov\left(\hat{\mu}_{1,n}\sqrt{n}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right),\Lambda(t)\sqrt{n}\left(\hat{\mu}_{1,n}-\mu_1\right)\right),\end{aligned}\tag{25}$$

where Eq. (26):

$$\begin{aligned}&cov\left(\hat{\mu}_{1,n}\sqrt{n}\left(\widehat{\Lambda}_n(t)-\Lambda(t)\right),\Lambda(t)\sqrt{n}\left(\hat{\mu}_{1,n}-\mu_1\right)\right)\\&\quad=cov\left(\left(\sqrt{n}\hat{\mu}_{1,n}\widehat{\Lambda}_n(t)-\sqrt{n}\Lambda(t)\hat{\mu}_{1,n}\right),\left(\sqrt{n}\Lambda(t)\hat{\mu}_{1,n}-\Lambda(t)\sqrt{n}\,\mu_1\right)\right)\\&\quad=n\Lambda(t)\,cov\left(\hat{\mu}_{1,n}\widehat{\Lambda}_n(t),\hat{\mu}_{1,n}\right)+0-n\Lambda^2(t)\,cov\left(\hat{\mu}_{1,n},\hat{\mu}_{1,n}\right)+0\\&\quad=n\Lambda(t)\,cov\left(\hat{\mu}_{1,n}\widehat{\Lambda}_n(t),\hat{\mu}_{1,n}\right)-n\Lambda^2(t)\,Var\left(\hat{\mu}_{1,n}\right)\end{aligned}\tag{26}$$

with Eq. (27).

$$\begin{aligned}Var\left(\hat{\mu}_{1,n}\right)&=Var\left(\hat{\mu}_{1,n}-\mu_1\right)\\&=Var\left(\frac{1}{\sqrt{n}}\sqrt{n}\left(\hat{\mu}_{1,n}-\mu_1\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n} Var\left(\sqrt{n}(\hat{\mu}_{1,n} - \mu_1)\right) \\
&= \frac{1}{n} \times \frac{\theta}{{\sigma_1}^2} \\
&= \frac{\theta}{{\sigma_1}^2 n}. \qquad (27)
\end{aligned}
$$

Let $X = \hat{\mu}_{1,n}$ and $Y = \widehat{\Lambda}_n(t)$ we have Eq. (28):

$$
\begin{aligned}
cov(XY, X) &= E(X^2Y) - (E(XY)E(X)) \\
&= E(X^2)E(Y) - \left(E(X)\right)^2 E(Y)) \\
&= E(Y)\left(E(X^2) - \left(E(X)\right)^2\right) \\
&= E(Y)Var(X) \\
&= E\left(\widehat{\Lambda}_n(t)\right) Var(\hat{\mu}_{1,n}) \\
&= \Lambda(t)\frac{\theta}{{\sigma_1}^2 n}. \qquad (28)
\end{aligned}
$$

Substituting Eqs. (27) and (28) into Eq. (26), we obtain Eq. (29).

$$
\begin{aligned}
cov\left(\hat{\mu}_{1,n}\sqrt{n}\left(\widehat{\Lambda}_n(t) - \Lambda(t)\right), \Lambda(t)\sqrt{n}\left(\hat{\mu}_{1,n} - \mu_1\right)\right) &= n\Lambda(t)cov\left(\hat{\mu}_{1,n}\widehat{\Lambda}_n(t), \hat{\mu}_{1,n}\right) - n\Lambda^2(t)Var\left(\hat{\mu}_{1,n}\right) \\
&= n\Lambda(t)\Lambda(t)\frac{\theta}{{\sigma_1}^2 n} - n\Lambda^2(t)\frac{\theta}{{\sigma_1}^2 n} \\
&= n\Lambda^2(t)\frac{\theta}{{\sigma_1}^2 n} - n\Lambda^2(t)\frac{\theta}{{\sigma_1}^2 n} \\
&= 0. \qquad (29)
\end{aligned}
$$

Substituting Eqs. (22), (23), and (29) into Eq. (25), we obtain Eq. (30).

$$
Var\left(\sqrt{n}\left(\hat{\psi}_n(t) - \psi(t)\right)\right) = \left(1 + k_{t,\tau}\right)^2 \Lambda(t_r)\tau{\mu_1}^2 + {k_{t,\tau}}^2 \Lambda^c(t_r)\tau{\mu_1}^2 + \frac{\theta}{{\sigma_1}^2}. \qquad (30)
$$

Based on Eqs. (24) and (30), we have:

$$
\sqrt{n}\left(\hat{\psi}_n(t) - \psi(t)\right) \xrightarrow{d} \mathrm{N}\left(0, \left(1 + k_{t,\tau}\right)^2 \Lambda(t_r)\tau{\mu_1}^2 + {k_{t,\tau}}^2 \Lambda^c(t_r)\tau{\mu_1}^2 + \frac{\theta}{{\sigma_1}^2}\right).
$$

This completes the proof of Theorem 1. ■

Theorem 2. *Suppose that the intensity function* λ *satisfies* Eq. (1) *and is locally integrable. If in addition,* $Y(t)$ *satisfies the condition in* Eq. (2) *and* $\{X_i, i \geq 1\}$ *is a non-negative sequence of independent and identically distributed random variables, then*

$$
\sqrt{\frac{n}{\tau}}\left(\hat{V}_n(t) - V(t)\right) \xrightarrow{d} \mathrm{N}\left(0, \left(1 + k_{t,\tau}\right)^2 \Lambda(t_r){\mu_2}^2 + {k_{t,\tau}}^2 \Lambda^c(t_r){\mu_2}^2 + \frac{\theta}{\tau{\sigma_2}^2}\right)
$$

as $n \to \infty$.

Proof. To check Theorem 2, using Eqs. (6) and (8) we have

$$
\begin{aligned}
\sqrt{n}\left(\hat{V}_n(t) - V(t)\right) &= \sqrt{n}\left(\widehat{\Lambda}_n(t)\hat{\mu}_{2,n} - \Lambda(t)\mu_2\right) \\
&= \hat{\mu}_{2,n}\sqrt{n}\left(\widehat{\Lambda}_n(t) - \Lambda(t)\right) + \Lambda(t)\sqrt{n}\left(\hat{\mu}_{2,n} - \mu_2\right).
\end{aligned}
$$

Based on Lemma 4, then

$$\Lambda(t)\sqrt{n}\left(\hat{\mu}_{2,n}-\mu_2\right) \xrightarrow{d} \mathrm{N}\left(0,\frac{\theta\Lambda(\mathrm{t})^2}{{\sigma_2}^2}\right).$$

Similarly to Theorem 1, Theorem 2 can be obtained. ■

3.4 Implications and Future Work

The theoretical findings presented in this article not only contribute to the advancement of stochastic process theory, particularly in the context of compound cyclic Poisson processes, but also lay a solid foundation for future applied research. The models and asymptotic results derived herein have the potential to inspire new methodologies in real case applications, such as seasonal risk modeling in insurance, environmental event prediction, and other domains characterized by cyclic stochastic behavior. Therefore, this work may serve as a stepping stone for the development of innovative frameworks that bridge mathematical theory and practical implementation.

One of the practical areas where the compound cyclic Poisson process can be applied is in modeling seasonal insurance claims. In many types of insurance, such as agricultural insurance or vehicle damage insurance, the number of claims exhibits cyclic behavior throughout the year. For instance, agricultural claims may peak during certain months affected by climate risks, such as floods during the rainy season or drought during dry months, while vehicle claims may increase during holiday seasons when traffic intensity rises. To model this, the intensity function of the claim arrivals can be assumed to follow a cyclic structure that repeats annually. The claim sizes, which can vary significantly from one claim to another, make the compound structure appropriate for each event (claim) contributes a random amount to the total cost.

The proposed compound cyclic Poisson process allows us to capture both the randomness in the number of claims frequency. Using the asymptotic distribution of the estimators for the mean and variance functions ensures can make long-term predictions and assess the risk of unusually high total claims during peak seasons. This application is not only relevant for actuarial science and risk management, but it also provides an example where theoretical developments in stochastic processes offer significant practical value in modeling real-life phenomena characterized by cyclic and aggregated random events.

The results of this study build upon and extend previous models of non-homogeneous and seasonal Poisson processes by introducing a compound structure that allows for variability not just in event arrival times but also in event magnitudes. While prior works have often focused on cyclic Poisson processes or compound processes separately, this study synthesizes both aspects, offering a more robust and flexible framework. It complements existing research in applied probability and stochastic modeling by providing new insights into the long-term statistical properties of systems affected by both periodic trends and random fluctuations. In doing so, it addresses gaps in earlier models that could not simultaneously account for time-varying intensity and variable event sizes, contributing a novel and theoretically grounded approach to modeling complex systems in fields such as insurance, finance, and environmental science.

4. CONCLUSION

Each theorem proposed in this paper contributes not only to theoretical novelty but also provides a foundation for future applications in modeling periodic stochastic phenomena, which were not addressed in prior research. Asymptotic distributions of $\hat{\psi}_n(t)$ and $\hat{V}_n(t)$ are, respectively

$$\sqrt{n}\ \left(\hat{\psi}_n(t)\ -\ \psi(t)\right) \xrightarrow{d} N\left(0,\left(1+k_{t,\tau}\right)^2\Lambda(t_r)\tau\mu^2\ +{k_{t,\tau}}^2\Lambda^c(t_r)\tau\mu^2\ +\ \frac{\sigma^2\Lambda(t)^2}{\theta}\right)$$

and

$$\sqrt{n}\left(\hat{V}_n(t)-V(t)\right) \xrightarrow{d} N\left(0,\left(1+k_{t,\tau}\right)^2\Lambda(t_r)\tau{\mu_2}^2+{k_{t,\tau}}^2\Lambda^c(t_r)\tau{\mu_2}^2+\frac{{\sigma_2}^2\Lambda(t)^2}{\theta}\right)$$

as $n \to \infty$.

While this study lays a theoretical foundation for the compound cyclic Poisson process and asymptotic properties, several avenues remain open for future investigation. First, future research could explore simulation-based or numerical approaches to approximate these distributions in practical settings. Second, extending the model to accommodate multivariate or spatially distributed cyclic processes could enhance its applicability in complex systems such as climate modeling, network traffic, or financial contagion. Finally, empirical validation using real datasets, particularly from insurance or environmental sectors, would help evaluate the model's predictive performance and refine its assumptions.

Author Contributions

Ika Reskiana Adriani: Formal Analysis, Funding Acquisition, Investigation, Developing the Methodology, Writing the Original Draft, Reviewing and Editing. I Wayan Mangku: Conceptualization, Formal Analysis, Supervision, Validated the Results, Reviewing and Editing. Retno Budiarti: Development of Methodology, Validated the Findings, Review and Editing. All authors discussed the results and contributed to the final manuscript

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Declarations

The authors declare that there are no competing interests.

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