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A STUDY OF DERIVATIONS AND LINEAR MAPPINGS ON SKEW GENERALIZED POWER SERIES MODULES

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ABSTRACT

This paper investigates the structure of skew generalized power series modules over skew generalized power series rings, emphasizing the extension of derivations in this context. We define and study additive mappings that generalize classical derivations with respect to module homomorphisms and ring derivations. Under suitable compatibility conditions, we construct corresponding derivations on skew generalized power series modules and establish their fundamental properties. These findings contribute to a broader understanding of how derivations can be systematically extended from classical module theory to generalized algebraic frameworks.

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1. INTRODUCTION

In [1], a ring is described as a non-empty set equipped with two binary operations satisfying certain axioms. In 1990, Ribenboim [2] introduced the Generalized Power Series Ring (GPSR), extending the notions of semigroup rings [3], polynomial rings [4], and power series rings, by requiring the underlying monoid to be both Artinian and narrow. A partially ordered set [5] is considered Artinian if every strictly decreasing sequence of its elements terminates. It is considered narrow if every subset of mutually incomparable elements is finite [6]. The fundamental properties of GPSRs are further elaborated in [7], [8], [9], [10], [11], [12], [13], [14].

A ring can naturally be viewed as a module over itself. Building upon this, in 2001, Varadarajan [15] developed the Generalized Power Series Module (GPSM), a module constructed over a GPSR. This construction generalizes earlier work by Varadarajan on polynomial modules [16]. Subsequent studies, such as those by Faisol et al. [17], explored the Noetherian properties of polynomial modules. Additional results concerning the structure of GPSMs can be found in [18] and [19].

Later, in 2008, Mazurek and Ziembowski [20] extended the GPSR structure by introducing a monoid homomorphism, leading to the formulation of the Skew Generalized Power Series Ring (SGPSR). Various algebraic properties of SGPSRs have since been studied [21], [22], [23], [24], [25], with additional advancements related to their module structures discussed in [26], [27], [28], [29], [30]. Building upon both the GPSM and SGPSR frameworks, Faisol et al. [31], [32] introduced the concept of the Skew Generalized Power Series Module (SGPSM), which serves as a module over SGPSR.

In general, an additive map Δ on a ring R is called a derivation if it satisfies $\Delta(xy) = \Delta(x)y + x\Delta(y)$ for all $x, y \in R$. Extending this, if Δ is a derivation on a ring R, and M and N are R-modules, with $\vartheta: M \to N$ an R-linear map, then an additive map $\rho: M \to N$ is referred to as a (Δ, ϑ)-derivation if $\rho(mr) = \rho(m)r + \vartheta(m)\Delta(r)$ for all $m \in M$ and $r \in R$ [33], [34].

Research concerning derivations has been extensive, covering both ring derivations [35], [36], [37] and module derivations [38], [39], [40], [41], [42]. For instance, Fitriani et al. [43], [44] investigated derivations of polynomial modules over polynomial rings. Given that a polynomial module is a special case of an SGPSM, this naturally motivates the exploration of derivations in the context of SGPSMs over SGPSRs. Thus, this study aims to analyze additive maps from an SGPSM $M[[S, \leq, \omega]]$ to another SGPSM $N[[S, \leq, \omega]]$ over a common SGPSR $R[[S, \leq, \omega]]$, satisfying the definition of an (Δ, ϑ)-derivation, where Δ is a derivation on R and ϑ is an R-linear map from M to N.

2. RESEARCH METHODS

Let R be a commutative ring with identity, and M be an R-module. Denote by $End(R)$ the set of all endomorphisms of R, forming a monoid under composition. Let S be a strictly ordered monoid with a compatible strict partial order.

2.1 Structure of SGPSR

Consider a monoid homomorphism $\omega: S \to End(R)$ satisfying:

$$\omega(s + t) = \omega(s) \circ \omega(t), \forall s, t \in S.$$

Define the set:

$$R[[S, \leq, \omega]] = \{g: S \to R \mid \text{the support of } g \text{ is Artinian and narrow}\}.$$

For $g, k \in R[[S, \leq, \omega]]$, the addition is given pointwise:

$$(g + k)(s) = g(s) + k(s), \forall s \in S, \tag{1}$$

and the multiplication is defined by:

$$(gk)(s) = \sum_{u+v=s} g(u) \cdot \omega(u)(k(v)), \forall s, u, v \in S. \tag{2}$$

Mazurek and Ziembowski [19] showed that equipping $R[[S, \leq, \omega]]$ with the operations defined in Equation (1) and Equation (2) yields a ring, now known as the Skew Generalized Power Series Ring (SGPSR).

2.2 Structure of SGPSM

Similarly, define:

$M[[S,\leq,\omega]] = \{\sigma: S \to M \mid$ the support of σ is Artinian and narrow$\}$.

For $\sigma,\tau \in M[[S,\leq,\omega]]$ and $g \in R[[S,\leq,\omega]]$, the module operations are defined as follows:

$$(\sigma+\tau)(s) = \sigma(s) + \tau(s), \forall s \in S, \tag{3}$$

and the scalar multiplication by:

$$(\sigma g)(s) = \textstyle\sum_{u+v=s} \sigma(u)\cdot\omega(u)(g(v)), \forall s,u,v \in S. \tag{4}$$

Equipped with the operations defined in **Equation (3)** and **Equation (4)**, $M[[S,\leq,\omega]]$ admits the structure of a module over the ring $R[[S,\leq,\omega]]$, known as the Skew Generalized Power Series Module (SGPSM), as introduced by Faisol et al. [30].

3. RESULTS AND DISCUSSION

We first introduce the notion of a $\tilde{\Delta}$-derivation on $R[[S,\leq,\omega]]$.

Definition 1. *Let $\boldsymbol{R}$ be a ring, $(\boldsymbol{S},\leq)$ a strictly ordered monoid, and $\boldsymbol{\omega}: \boldsymbol{S} \to \boldsymbol{End}(\boldsymbol{R})$ a monoid homomorphism. An additive map $\tilde{\Delta}: \boldsymbol{R}[[\boldsymbol{S},\leq,\boldsymbol{\omega}]] \to \boldsymbol{R}[[\boldsymbol{S},\leq,\boldsymbol{\omega}]]$ is called a derivation if:*

$$\tilde{\Delta}(\boldsymbol{gk}) = \tilde{\Delta}(\boldsymbol{g})\boldsymbol{k} + \boldsymbol{g}\tilde{\Delta}(\boldsymbol{k}), \text{ for all } \boldsymbol{g},\boldsymbol{k} \in \boldsymbol{R}[[\boldsymbol{S},\leq,\boldsymbol{\omega}]].$$

Next, we establish that given a derivation on R, one can construct a corresponding derivation on $R[[S,\leq,\omega]]$.

Lemma 1. *Let $(S,\leq)$ be a strictly ordered monoid, R a commutative ring with identity, and $\omega: S \to End(R)$ a monoid homomorphism. Suppose $\Delta: R \to R$ is a derivation on R satisfying*

$$\omega(s)\circ\Delta = \Delta\circ\omega(s), \text{ for all } s \in S.$$

Then there exists an additive map $\tilde{\Delta}: R[[S,\leq,\omega]] \to R[[S,\leq,\omega]]$ such that $\tilde{\Delta}$ is a derivation on SGPSR $R[[S,\leq,\omega]]$.

Proof. Assume Δ is a derivation on R. Define $\tilde{\Delta}$ on $R[[S,\leq,\omega]]$ by:

$$\tilde{\Delta}(g)(s) = \Delta(g(s))$$

for all $g \in R[[S,\leq,\omega]]$ and $s \in S$.

For any $g,k \in R[[S,\leq,\omega]]$, and for each $s \in S$, we compute:

$$\begin{aligned}
\tilde{\Delta}(gk)(s) &= \Delta\big((gk)(s)\big) \\
&= \Delta\left(\sum_{u+v=s} g(u)\cdot\omega(u)(k(v))\right) = \sum_{u+v=s} \Delta\big(g(u)\cdot\omega(u)(k(v))\big) \\
&= \sum_{u+v=s} \Big(\Delta\big(g(u)\big)\cdot\omega(u)(k(v)) + g(u)\cdot\Delta\big(\omega(u)(k(v))\big)\Big) \\
&= \sum_{u+v=s} \Delta\big(g(u)\big)\cdot\omega(u)(k(v)) + \sum_{u+v=s} g(u)\cdot\Delta\big(\omega(u)(k(v))\big) \\
&= \sum_{u+v=s} \Delta(g(u))\cdot\omega(u)(k(v)) + \sum_{u+v=s} g(u)\cdot\omega(u)(\Delta(k(v))) \\
&= \sum_{u+v=s} \tilde{\Delta}(g)(u)\cdot\omega(u)(k(v)) + \sum_{u+v=s} g(u)\cdot\omega(u)\left(\tilde{\Delta}(k)(v)\right) \\
&= \big(\tilde{\Delta}(g)k\big)(s) + \big(g\tilde{\Delta}(k)\big)(s)
\end{aligned}$$

By the linearity of Δ and properties of ω, it follows that:

$$\tilde{\Delta}(gk) = \tilde{\Delta}(g)k + g\tilde{\Delta}(k).$$

Thus, $\tilde{\Delta}$ is indeed a derivation on $R[[S,\leq,\omega]]$. ■

Now, we extend the concept of (Δ,ϑ)-derivation to SGPSMs.

Definition 2. *Let M, N be modules over a ring R, $(S,\leq)$ a strictly ordered monoid, and $\omega: S \to End(R)$ a monoid homomorphism. Suppose $\tilde{\Delta}: R[[S,\leq,\omega]] \to R[[S,\leq,\omega]]$ is a derivation and $\tilde{\vartheta}: M[[S,\leq,\omega]] \to N[[S,\leq,\omega]]$ is an $R[[S,\leq,\omega]]$-linear map. An additive map $\tilde{\rho}: M[[S,\leq,\omega]] \to N[[S,\leq,\omega]]$ is called a $(\tilde{\Delta},\tilde{\vartheta})$-derivation if $\tilde{\rho}(\sigma g) = \tilde{\rho}(\sigma)g + \tilde{\vartheta}(\sigma)\tilde{\Delta}(g)$, for all $\sigma \in M[[S,\leq,\omega]]$ and $g \in R[[S,\leq,\omega]]$.*

The following lemma demonstrates the existence of an induced $R[[S,\leq,\omega]]$-linear mapping between two SGPSMs.

Lemma 2. *Let $(S,\leq)$ be a strictly ordered monoid, R a commutative ring with identity, M, N two R-modules, and $\omega: S \to End(R)$ a monoid homomorphism. If $\vartheta: M \to N$ is an R-linear map, then there exists an $R[[S,\leq,\omega]]$-linear map $\tilde{\vartheta}: M[[S,\leq,\omega]] \to N[[S,\leq,\omega]]$ defined by:*

$$\tilde{\vartheta}(\sigma)(s) = \vartheta\big(\sigma(s)\big)$$

for all $\sigma \in M[[S,\leq,\omega]]$ and $s \in S$.

Proof. Let $g \in R[[S,\leq,\omega]]$ and $\sigma \in M[[S,\leq,\omega]]$.

By the construction of $\tilde{\vartheta}$, for every $s \in S$, we have:

$$\begin{aligned}
\tilde{\vartheta}(\sigma+\tau)(s) &= \vartheta\big((\sigma+\tau)(s)\big) \\
&= \vartheta\big(\sigma(s)+\tau(s)\big) \\
&= \vartheta(\sigma(s)) + \vartheta\big(\tau(s)\big) \quad \text{(since } \vartheta \text{ is } R\text{-linear)} \\
&= \tilde{\vartheta}(\sigma)(s) + \tilde{\vartheta}(\tau)(s).
\end{aligned}$$

Thus, $\tilde{\vartheta}(\sigma+\tau) = \tilde{\vartheta}(\sigma) + \tilde{\vartheta}(\tau)$.

Similarly, for scalar multiplication, for every s ∈ S, we have:

$$\begin{aligned}
\tilde{\vartheta}(\sigma g)(s) &= \vartheta\big((\sigma g)(s)\big) \\
&= \vartheta\left(\sum_{u+v=s} \sigma(u)\cdot\omega(u)(g(v))\right) \\
&= \textstyle\sum_{u+v=s} \vartheta(\sigma(u))\cdot\omega(u)(g(v)) \quad \text{(since } \vartheta \text{ is } R\text{-linear)} \\
&= \sum_{u+v=s} \tilde{\vartheta}(\sigma)(u)\cdot\omega(u)(g(v)) = \big(\tilde{\vartheta}(\sigma)g\big)(s)
\end{aligned}$$

Therefore, $\tilde{\vartheta}(\sigma g) = \tilde{\vartheta}(\sigma)g$.

Hence, $\tilde{\vartheta}$ is an $R[[S,\leq,\omega]]$-linear map. ■

In order to generalize linear maps between skew generalized power series modules over different underlying ordered monoids, we consider the following setting.

Let $(S,\leq)$ be a strictly ordered monoid and $\omega: S \to End(R)$ a monoid homomorphism. Let $(S',\leq')$ and $(S'',\leq'')$ be strictly ordered monoids such that $S', S'' \subseteq S$.
We assume that:

1. There is a strictly additive bijection $\varphi: S' \to S''$,
2. The partial orders $\leq'$ and $\leq''$ are either finer than or compatible with the restriction of $\leq$ on S, in the sense that Artinian and narrow subsets of S remain Artinian and narrow in S' and S'',
3. The monoid homomorphisms satisfy $\omega''\big(\varphi(s')\big) = \omega'(s')$ for all $s' \in S'$.

Under these conditions, we establish the existence of a well-defined linear map between the corresponding modules.

Proposition 1. *Let M, N be R-modules, and $\vartheta: M \to N$ be an R-linear map. Under the setting described above, there exists an $R[[S, \leq, \omega]]$-linear map*

$$\tilde{\vartheta}: M[[S', \leq', \omega']] \to N[[S'', \leq'', \omega'']]$$

defined by

$$\tilde{\vartheta}(\sigma)(s'') = \vartheta\left(\sigma\left(\varphi^{-1}(s'')\right)\right),$$

for all $\sigma \in M[[S', \leq', \omega']]$, $s'' \in S''$.

Proof. Let $\sigma, \tau \in M[[S', \leq', \omega']]$ and $g \in R[[S, \leq, \omega]]$. For each $s'' \in S''$, we have:

For addition:

$$\begin{aligned} \tilde{\vartheta}(\sigma+\tau)(s'') &= \vartheta\left((\sigma+\tau)\left(\varphi^{-1}(s'')\right)\right) \\ &= \vartheta\left(\sigma\left(\varphi^{-1}(s'')\right) + \tau\left(\varphi^{-1}(s'')\right)\right) \\ &= \vartheta\left(\sigma\left(\varphi^{-1}(s'')\right)\right) + \vartheta\left(\tau\left(\varphi^{-1}(s'')\right)\right) \\ &= \tilde{\vartheta}(\sigma)(s'') + \tilde{\vartheta}(\tau)(s''). \end{aligned}$$

Thus, $\tilde{\vartheta}(\sigma+\tau) = \tilde{\vartheta}(\sigma) + \tilde{\vartheta}(\tau)$.

For scalar multiplication:

$$\begin{aligned} \tilde{\vartheta}(\sigma g)(s'') &= \vartheta\left((\sigma g)\left(\varphi^{-1}(s'')\right)\right) \\ &= \vartheta\left(\sum_{u+v=\varphi^{-1}(s'')} \sigma(u) \cdot \omega(u)(g(v))\right) \\ &= \sum_{u+v=\varphi^{-1}(s'')} \vartheta(\sigma(u)) \cdot \omega(u)(g(v)) \end{aligned}$$

Since $\sigma \in M[[S', \leq', \omega']]$, the scalar multiplication at $u \in S'$ uses $\omega'(u)$, and by compatibility, we have $\omega''\left(\varphi(u)\right) = \omega'(u)$ for all $u \in S'$. Now, we reindex the sum by setting $u' = \varphi(u) \in S''$ and $v' = \varphi(v) \in S''$. Since φ is strictly additive, $u + v = \varphi^{-1}(s'')$ implies $u' + v' = s''$.

The sum becomes:

$$\begin{aligned} &= \sum_{u'+v'=s''} \vartheta(\sigma(\varphi^{-1}(u'))) \cdot \omega(\varphi^{-1}(u'))(g(\varphi^{-1}(v'))) \\ &= \sum_{u'+v'=s''} \tilde{\vartheta}(\sigma)(u') \cdot \omega''(u')\left(g(v')\right) \end{aligned}$$

Thus,

$$\tilde{\vartheta}(\sigma g)(s'') = \left(\tilde{\vartheta}(\sigma) g\right)(s'').$$

Therefore, $\tilde{\vartheta}$ is $R[[S, \leq, \omega]]$-linear map. ■

Remark 1.

Although the partial orders $\leq'$ and $\leq''$ may differ from $\leq$, the constructions of the modules and linear maps remain valid, provided that the Artinian and narrow properties are preserved under these orders, and that the monoid homomorphisms satisfy the compatibility condition.

Building upon **Proposition 1**, in which an $R[[S, \leq, \omega]]$-linear map between skew generalized power series modules was constructed via a strictly additive bijection and compatible monoid homomorphisms, we now extend this construction to the setting of derivations. In particular, we show that an (Δ, ϑ)-derivation between R-modules induces a corresponding $(\tilde{\Delta}, \tilde{\vartheta})$-derivation between skew generalized power series modules. This result is formalized in the following theorem.

Theorem 1. *Let M, N be R-modules, and let $\vartheta: M \to N$ be an R-linear map. Let $(S, \leq)$ be a strictly ordered monoid with a monoid homomorphism $\omega: S \to End(R)$. Suppose $S', S'' \subseteq S$ are subsets equipped with partial orders $\leq'$ and $\leq''$ that are finer or compatible with $\leq$, and monoid homomorphism $\omega': S' \to End(R)$, $\omega'': S'' \to End(R)$ such that:*

1. *There is a strictly additive bijection $\varphi: S' \to S''$,*
2. *The homomorphisms satisfy $\omega''(\varphi(s')) = \omega'(s')$ for all $s' \in S'$.*

If $\rho: M \to N$ is a (Δ, ϑ)-derivation with respect to R-linear map ϑ and a derivation $\Delta: R \to R$ satisfying $\omega(s) \circ \Delta = \Delta \circ \omega(s)$ for all $s \in S$, then there exist a uniqe map

$$\tilde{\rho}: M[[S', \leq', \omega']] \to N[[S'', \leq'', \omega'']]$$

which is a $(\tilde{\Delta}, \tilde{\vartheta})$-derivation with respect to the linear map

$$\tilde{\vartheta}: M[[S', \leq', \omega']] \to N[[S'', \leq'', \omega'']]$$

*constructed as in **Proposition 1**, and the induced derivation*

$$\tilde{\Delta}: R[[S, \leq, \omega]] \to R[[S, \leq, \omega]]$$

extending Δ, and satisfying $\omega''(u) \circ \tilde{\Delta} = \tilde{\Delta} \circ \omega''(u)$, for all $u \in S''$.

The map $\tilde{\rho}$ is defined by:

$$\tilde{\rho}(\sigma)(s'') = \rho\left(\sigma\left(\varphi^{-1}(s'')\right)\right) \text{ for all } \sigma \in M[[S', \leq', \omega']],\ s'' \in S''.$$

Proof. Let $\Delta: R \to R$ be a derivation, and $\rho: M \to N$ be a (Δ, ϑ)-derivation respect to R-linear map $\vartheta: M \to N$. By **Lemma 1**, the derivation Δ on R extends naturally to a derivation

$$\tilde{\Delta}: R[[S, \leq, \omega]] \to R[[S, \leq, \omega]]$$

defined by

$$\tilde{\Delta}(g)(s) = \Delta(g(s)) \text{ for all } g \in R[[S, \leq, \omega]] \text{ and } s \in S.$$

Thus, $\tilde{\Delta}$ satiesfies the derivation property:

$$\tilde{\Delta}(gk) = \tilde{\Delta}(g)k + g\tilde{\Delta}(k) \text{ for all } g, k \in R[[S, \leq, \omega]].$$

By **Proposition 1**, under the assumptions of:

1. strictly additive bijection $\varphi: S' \to S''$,
2. compatibility of monoid homomorphisms $\omega''(\varphi(s')) = \omega'(s')$ for all $s' \in S'$,
3. partial orders $\leq', \leq''$ being compatible with $\leq$,

the R-linear map $\vartheta: M \to N$ induces an $R[[S, \leq, \omega]]$-linear map

$$\tilde{\vartheta}: M[[S', \leq', \omega']] \to N[[S'', \leq'', \omega'']]$$

defined by

$$\tilde{\vartheta}(\sigma)(s'') = \vartheta\left(\sigma\left(\varphi^{-1}(s'')\right)\right) \text{ for all } \sigma \in M[[S', \leq', \omega']],\ s'' \in S''.$$

Now, define

$$\tilde{\rho}: M[[S', \leq', \omega']] \to N[[S'', \leq'', \omega'']]$$

by

$$\tilde{\rho}(\sigma)(s'') = \rho\left(\sigma\left(\varphi^{-1}(s'')\right)\right) \text{ for all } \sigma \in M[[S', \leq', \omega']],\ s'' \in S''.$$

Let $\sigma \in M[[S', \leq', \omega']]$ and $g \in R[[S, \leq, \omega]]$.

For each $s'' \in S''$,

$$\tilde{\rho}(\sigma g)(s'') = \rho\left((\sigma g)(\varphi^{-1}(s''))\right)$$

$$= \rho\left(\sum_{u+v=\varphi^{-1}(s'')} \sigma(u)\cdot\omega(u)(g(v))\right)$$

$$= \sum_{u+v=\varphi^{-1}(s'')} \rho\big(\sigma(u)\cdot\omega(u)(g(v))\big) \quad \text{(additivity of } \rho)$$

$$= \sum_{u+v=\varphi^{-1}(s'')} \rho\big(\sigma(u)\big)\cdot\omega(u)\big(g(v)\big) + \vartheta\big(\sigma(u)\big)\cdot\Delta\Big(\omega(u)\big(g(v)\big)\Big)$$

(since ρ is an $(\Delta, \vartheta) -$ derivation)

Now, reindex the sum by setting $u' = \varphi(u) \in S''$ and $v' = \varphi(v) \in S''$, so that $u' + v' = s''$.

Thus, the expression becomes:

$$= \sum_{u'+v'=s''} \rho(\sigma(\varphi^{-1}(u')))\cdot\omega(\varphi^{-1}(u'))(g(\varphi^{-1}(v'))) + \vartheta\big(\sigma(\varphi^{-1}(u'))\big)\cdot\Delta\Big(\omega(\varphi^{-1}(u'))\big(g(\varphi^{-1}(v'))\big)\Big)$$

$$= \sum_{u'+v'=s''} \tilde{\rho}(\sigma)(u')\cdot\omega''(u')\big(g(v')\big) + \check{\vartheta}\big(\sigma(u)\big)\cdot\tilde{\Delta}\Big(\omega''(u')\big(g(v')\big)\Big)$$

(using compatibility $\omega''(u') = \omega(\varphi^{-1}(u'))$).

Separate the two sums:

$$\tilde{\rho}(\sigma g)(s'') = \sum_{u'+v'=s''} \tilde{\rho}(\sigma)(u')\cdot\omega''(u')\big(g(v')\big) + \sum_{u'+v'=s''} \check{\vartheta}\big(\sigma(u)\big)\cdot\omega''(u')\Big(\tilde{\Delta}\big(g(v')\big)\Big)$$

Thus,

$$\tilde{\rho}(\sigma g)(s'') = (\tilde{\rho}(\sigma)g)(s'') + \Big(\check{\vartheta}(\sigma)\tilde{\Delta}(g)\Big)(s'').$$

Therefore,

$$\tilde{\rho}(\sigma g) = \tilde{\rho}(\sigma)g + \check{\vartheta}(\sigma)\tilde{\Delta}(g),$$

showing that $\tilde{\rho}$ is a $\big(\tilde{\Delta}, \tilde{\vartheta}\big)$-derivation as given in **Definition 2**. ■

In order to exemplify the applicability of **Theorem 1**, we present an original construction that demonstrates how a module derivation, defined with respect to a ring derivation and a module homomorphism, can be systematically extended to the associated skew generalized power series module. This construction is provided independently and serves to underline the subtleties involved in handling the interaction between the monoid action and module structure.

Example 1. Consider the ring of integers $R = \mathbb{Z}$ and R-modules $M = N = \mathbb{Z}$. Let $S = \mathbb{N}_0$ be the strictly ordered additive monoid with its natural order. Define submonoids $S' = S'' = \mathbb{N}$ and a strictly additive bijection $\varphi: S' \to S''$ given by the identity map. Equip S' and S'' with monoid homomorphisms $\omega': S' \to End(R)$ and $\omega'': S'' \to End(R)$, respectively, where $\omega'(s')(r) = 2^{s'}r$ and $\omega''(s'')(r) = 2^{s''}r$ for all $r \in R$. These maps satisfy $\omega''\big(\varphi(s')\big) = \omega'(s')$ for all $s' \in S'$, ensuring the compatibility conditions required by the framework. Define a linear map $\rho: M \to N$ by $\rho(x) = 3x$, which is a (Δ, ϑ)-derivation where ϑ is the identity map and $\Delta: R \to R$ is the zero derivation. By **Theorem 1**, there exists a unique extension $\tilde{\rho}: M[[S', \leq', \omega']] \to N[[S'', \leq'', \omega'']]$, given explicitly by $\tilde{\rho}(\sigma)(s'') = 3\sigma\big(\varphi^{-1}(s'')\big)$ for each $\sigma \in M[[S', \leq', \omega']]$ and $s'' \in S''$. This example illustrates how the interaction between non-trivial weightings via ω', ω'' and formal series structures enables the construction of derivations within a generalized monoidal formalism.

We now consider a more general example in which the R-module M is not equal to R, highlighting the flexibility of the framework.

Example 2. Let $R = \mathbb{Z}$ and define $M = \mathbb{Z}^2, N = \mathbb{Z}^2$, which are $\mathbb{Z}$-modules under component-wise addition and scalar multiplication. Let $S = \mathbb{N}_0$, the additive monoid of non-negative integers, equipped with the usual strict order.

Define submonoids $S' = S'' = \mathbb{N}$, and let $\varphi: S' \to S''$ be the identity map, which is a strictly additive bijection. Define the monoid homomorphisms:

$$\omega'(s)(r) = 2^s r,\ \ \omega''(s)(r) = 2^s r, \text{ for all } s \in S', r \in R\ .$$

Then for all $s' \in S'$, we have:

$$\omega''\big(\varphi(s')\big) = \omega''(s') = \omega'(s'),$$

so, the compatibility condition is satisfied.

Now define the R-linear map $\vartheta: M \to N$ as the identity map:

$$\vartheta(x, y) = (x, y).$$

Let $\Delta: R \to R$ be the zero derivation, defined by $\Delta(r) = 0$ for all $r \in \mathbb{Z}$. Since Δ is identically zero, it trivially satisfies the compatibility condition:

$$\omega(s) \circ \Delta = \Delta \circ \omega(s), \text{ for all } s \in S.$$

Define a map $\rho: M \to N$ by:

$$\rho(x, y) = (0, x),$$

which is a (Δ, ϑ)-derivation. Indeed, for all $r \in R$ and $(x, y) \in M$,

$$\rho\big(r \cdot (x, y)\big) = \rho(rx, ry) = (0, rx) = r \cdot \rho(x, y) + \Delta(r) \cdot \vartheta(x, y),$$

since $\Delta(r) = 0$ and $\vartheta(x, y) = (x, y)$. Thus, the derivation condition is satisfied.

By **Theorem 1**, there exists a unique extension

$$\tilde{\rho}: M[[S', \leq', \omega']] \to N[[S'', \leq'', \omega'']]$$

which is a $\big(\tilde{\Delta}, \tilde{\vartheta}\big)$-derivation, and is explicitly given by

$$\tilde{\rho}(\sigma)(s'') = \rho(\sigma(\varphi^{-1}(s''))) = \rho(\sigma(s'')) = (0, x),$$

for each $\sigma \in M[[S', \leq', \omega']]$ such that $\sigma(s'') = (x, y) \in M$, and $s'' \in S''$.

This example demonstrates how **Theorem 1** applies when the underlying module M is not equal to R, and illustrates the behavior of derivations in the setting of skew generalized power series modules.

4. CONCLUSION

In this paper, we have investigated the extension of derivations and linear mappings within the framework of skew generalized power series modules (SGPSMs). The main results are summarized as follows:

1. Ordinary derivations on a base ring R were extended to derivations on the skew generalized power series ring $R[[S, \leq, \omega]]$, preserving the Leibniz rule pointwise.
2. $R[[S, \leq, \omega]]$-linear maps were constructed between SGPSMs defined over distinct subsets $S', S'' \subseteq S$, based on a strictly additive bijection under appropriate compatibility conditions on the monoid homomorphisms and ordering structures.
3. A lifting theorem was established, showing that any (Δ, ϑ)-derivation between R-modules can be systematically lifted to a $\big(\tilde{\Delta}, \tilde{\vartheta}\big)$-derivation between their associated SGPSMs, while preserving algebraic structures through a careful reindexing process.

4. The results generalize classical derivation theory and provide a theoretical foundation for further studies on module theory and algebraic structures over non-standard ring extensions.
5. The framework developed in this paper can be further extended to investigate derivations on SGPSMs over non-commutative rings or topological modules. Moreover, potential applications may arise in symbolic computation, coding theory, and discrete dynamic systems where formal series and non-standard ring extensions play a key role.

AUTHOR CONTRIBUTIONS

Ahmad Faisol: Conceptualization, Methodology, Supervision, Writing - Original Draft, Writing - Review and Editing. Fitriani: Funding Acquisition, Project Administration, Validation, Visualization, Writing - Review and Editing. All authors discussed the results and contributed to the final manuscript.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest to report in this study.

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