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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN MODULAR B-METRIC SPACES

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ABSTRACT

This paper explores multivalued mappings in modular b-metric spaces, with particular emphasis on contraction-type mappings. It introduces the concept of a Hausdorff distance adapted to this setting and investigates fixed point theorems associated with these mappings. The existence of fixed points is established under the assumptions that the space is complete, the considered subset is closed, and the modular b-metric satisfies the Δ_2-condition. These results not only extend classical fixed point theory but also provide a theorem that guarantees the existence of solutions to integral equations, with potential applications in mathematical modeling.

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1. INTRODUCTION

The fixed point theorem (briefly FP) is a fundamental result in mathematical analysis, with extensive applications across various fields. This broad utility has inspired significant research into extending and generalizing the theorem. The Banach Contraction Principle (briefly BCP), introduced by S. Banach in 1922, serves as the foundation of FP theory, and modern studies have explored more complex mappings and spaces. Researchers have investigated FP theorems in generalized structures such as b-metric spaces (briefly bMS), as seen in [1]-[3]. In bMS, the classical triangle inequality from metric spaces (briefly MS) is relaxed by incorporating a constant $s \geq 1$, leading to a generalized inequality that still preserves the essential properties of distance.

Another generalization of MS in which FP theorems have been studied is the modular metric space (briefly MMS), introduced in [4]-[5]. As defined in [6], a modular is a functional on vector spaces. The concept of MMS generalizes both MS and modular spaces by extending the theory of MS, and a FP theorem for contractions in the framework of MMS was obtained in [7] as the basis of their structure. Further developments of this theorem in the same framework were presented in [8]-[12]. Moreover, various studies on FP theorems in generalized MMS have been conducted in [13]-[21]. In [22], the notion of complex valued MMS was introduced, and a generalization of The BCP was established. Building on this, [23] extended the framework by proving Meir-Keeler's FP theorem.

In [24], a space more general than both bMS and MS, known as a modular b-metric space (briefly MbMS), was introduced. Similar to the concept of MMS, this space extends the theory of bMS by incorporating a modular. In the same work, some FP theorems for ordinary contraction mappings were established, along with their applications to systems of linear equations. Subsequently, some common FP theorems for two self-mappings, as well as results for a self-mapping in the framework of MbMSs were established in [25]. Furthermore, there are developments related to FP theorems in extended MbMSs [26]. In addition, studies have also explored Generalized $\mathfrak{F}$-Contraction mappings, both in the context of MS [27] and MbMSs [28]. On the other hand, [29] introduced the concept of MbMSs also with a different formulation of the triangle inequality axiom compared to [24]. In the same work, FP theorems for mappings satisfying a contractive type condition were also established. Building on this space, [30] proposed the notion of modular b-gauge spaces, which are more general than MbMSs, and obtained several FP theorems to set-valued mappings. Subsequently, [31] presented some common FP theorems for set-valued mappings in modular b-gauge spaces, thereby generalizing the result in [30].

Along with the emergence of various generalizations of MS, the development of more general types of mappings than contraction mappings has also influenced the advancement of FP theory. Notable examples include non-expansive mappings discussed in [32] and pointwise contraction mappings presented in [33]. Another important generalization is the concept of multivalued contraction mappings, introduced in [34], where some FP theorems for such mappings in MS were also established. Building on this, in [35], generalized multivalued non-expansive mappings are presented along with the corresponding FP theorems. In the framework of bMS, a generalization of set-valued contraction mappings with their FP theorems was established in [36], while a generalized common FP for such mappings was provided in [37]. Moreover, multivalued mappings satisfying a nonlinear quasi-contractive condition in bMS were investigated in [38], where a corresponding FP theorem was also established. These advancements aim to adapt the theory to increasingly complex mathematical and real-world problems, demonstrating its enduring significance and versatility.

Despite various advancements in the study of multivalued mappings, research on FP theorems for such mappings within MbMSs remain limited. For instance, [30] and [31] established FP and common FP theorems for set-valued mappings in modular b-gauge spaces, which generalize the MbMSs introduced in [29]. However, no prior work has investigated FP theorems for multivalued mappings using the Hausdorff distance within the specific framework of MbMSs introduced in [24]. Previous studies have predominantly focused on multivalued mappings in other frameworks, such as MS in [39], bMS in [40]-[42], and MMS in [43]-[45]. In this paper, we extend FP theorems for multivalued mappings previously established in bMS [42] and MMS [44] to the more general framework of MbMSs [24]. The paper aims to develop and prove new FP theorems for multivalued contraction mappings within this context. To support and illustrate the applicability of our results, we also provide a concrete example and an illustrative application.

2. RESEARCH METHODS

This research utilizes a literature study method, starting with an examination of articles on FP theorems for multivalued mappings in bMS. Subsequently, we explore the concepts of MMS and MbMSs as a foundation for developing FP theorems for multivalued mappings in MbMSs.

We outline the preliminary concepts and definitions necessary for establishing the main results of this paper. We begin with the definition of MbMSs, as introduced in [24], which extends the classical notion of bMS by incorporating modular structures. This generalization provides a flexible framework for analyzing a broader class of mappings and FP theorems.

Definition 1. [24] Let $\mathfrak{X}$ denote an arbitrary nonempty set, and let $\mathfrak{s} \in \mathbb{R}$ with $\mathfrak{s} \geq 1$. A function $\varpi: (0,\infty) \times \mathfrak{X} \times \mathfrak{X} \to [0,\infty]$ is called a modular b-metric (briefly MbM) on $\mathfrak{X}$ if the following three axioms hold:

(B1) $\varpi_\lambda(\mathfrak{x},\mathfrak{y}) = 0$ for all $\mathfrak{x},\mathfrak{y} \in \mathfrak{X}$ and $\lambda > 0$ if and only if $\mathfrak{x} = \mathfrak{y}$;

(B2) $\varpi_\lambda(\mathfrak{x},\mathfrak{y}) = \varpi_\lambda(\mathfrak{y},\mathfrak{x})$ for all $\mathfrak{x},\mathfrak{y} \in \mathfrak{X}$ and $\lambda > 0$;

(B3) $\varpi_{\lambda+\mu}(\mathfrak{x},\mathfrak{y}) \leq \mathfrak{s}[\varpi_\lambda(\mathfrak{x},\mathfrak{z}) + \varpi_\mu(\mathfrak{y},\mathfrak{z})]$ for all $\mathfrak{x},\mathfrak{y},\mathfrak{z} \in \mathfrak{X}$ and $\lambda,\mu > 0$.

Later, the pair $(\mathfrak{X},\varpi,\mathfrak{s})$ is called a MbMS. Furthermore, if condition (B1) is replaced by

(B1') $\varpi_\lambda(\mathfrak{x},\mathfrak{x}) = 0$ for all $\lambda > 0$ and $\mathfrak{x} \in \mathfrak{X}$,

then ϖ is called a pseudomodular b-metric on $\mathfrak{X}$.

It is clear that every MbM ϖ is also a pseudomodular b-metric. Further, if ϖ is a pseudomodular b-metric on $\mathfrak{X}$ and $0 < \mu < \lambda$, then based on Axioms (B1'), (B2), and (B3), it follows that for all $\mathfrak{x},\mathfrak{y} \in \mathfrak{X}$,

$$\varpi_\lambda(\mathfrak{x},\mathfrak{y}) \leq \mathfrak{s}\big[\varpi_{\lambda-\mu}(\mathfrak{x},\mathfrak{x}) + \varpi_\mu(\mathfrak{y},\mathfrak{x})\big] = \mathfrak{s}\,\varpi_\mu(\mathfrak{y},\mathfrak{x}) = \mathfrak{s}\,\varpi_\mu(\mathfrak{x},\mathfrak{y}), \tag{1}$$

which shows that w is increasing with respect to $\mathfrak{s}$, as described in Eq. (1).

To illustrate the concept of a MbMS, consider the following example.

Example 1. Let $\mathfrak{X} = [0,1]$ and $\varpi: (0,\infty) \times \mathfrak{X} \times \mathfrak{X} \to [0,\infty]$ be a function defined by

$$\varpi_\lambda(\mathfrak{x},\mathfrak{y}) = \frac{|\mathfrak{x}-\mathfrak{y}|^2}{\lambda}$$

for all $\mathfrak{x},\mathfrak{y} \in \mathfrak{X}$ and $\lambda > 0$. Then, $(\mathfrak{X},\varpi,2)$ is a MbMS.

Next, let $(\mathfrak{X},\varpi,\mathfrak{s})$ be a MbMS and $\mathfrak{x}^0 \in \mathfrak{X}$. The modular set in [24] is defined as

$$\mathfrak{X}_\varpi(\mathfrak{x}^0) = \mathfrak{X}_\varpi = \left\{\mathfrak{x} \in \mathfrak{X}: \lim_{\lambda\to\infty} \varpi_\lambda(\mathfrak{x},\mathfrak{x}^0) = 0\right\}.$$

The pair $(\mathfrak{X}_\varpi,\varpi,\mathfrak{s})$ is also a MbMS. For convenience, instead of writing $(\mathfrak{X}_\varpi,\varpi,\mathfrak{s})$, we denote the space simply as $\mathfrak{X}_\varpi$. In the following discussion, we focus on MbMSs of the form $\mathfrak{X}_\varpi$. We now present the relevant definitions within the framework of $\mathfrak{X}_\varpi$. The notions of convergence, Cauchy sequences, and complete spaces are established based on [46], which are more general than those in [24], thereby enabling the framework to support broader theoretical development and applications. Moreover, the definition of closed sets is constructed following the framework presented in [46], while the definition of bounded sets is developed based on [47].

Definition 2. Let $\mathfrak{X}_\varpi$ be an arbitrary MbMS.

1. A sequence $\{\mathfrak{x}_n\} \subseteq \mathfrak{X}_\varpi$ is said to be ϖ-convergent to $\mathfrak{x} \in \mathfrak{X}_\varpi$ if there exists $\lambda > 0$ such that $\varpi_\lambda(\mathfrak{x}_n,\mathfrak{x}) \to 0$ as $n \to \infty$. In this case, $\mathfrak{x}$ is called the ϖ-limit of $\{\mathfrak{x}_n\}$.
2. A sequence $\{\mathfrak{x}_n\} \subseteq \mathfrak{X}_\varpi$ is called a ϖ-Cauchy (briefly ϖ-C) sequence if there exists $\lambda > 0$ such that $\varpi_\lambda(\mathfrak{x}_n,\mathfrak{x}_m) \to 0$ as $n,m \to \infty$.
3. A MbMS is said to be ϖ-complete if every ϖ-C sequence in $\mathfrak{X}_\varpi$ is ϖ-convergent, and its ϖ-limit belongs to $\mathfrak{X}_\varpi$. Specifically, for some $\lambda > 0$, if $\varpi_\lambda(\mathfrak{x}_n,\mathfrak{x}_m) \to 0$ as $n,m \to \infty$, then there exists $\mathfrak{x} \in \mathfrak{X}_\varpi$ such that $\varpi_\lambda(\mathfrak{x}_n,\mathfrak{x}) \to 0$ as $n \to \infty$.
4. A set $C \subseteq \mathfrak{X}_\varpi$ is called ϖ-closed if C contains all ϖ-limits of ϖ-convergent sequences in C.
5. A set $C \subseteq \mathfrak{X}_\varpi$ is said to be ϖ-bounded if

$$\delta_{\varpi}(C) = \sup\{\varpi_{\lambda}(x, y): x, y \in C\} < \infty.$$

Now, let us recall the concept of a multivalued mapping. Let X be an arbitrary nonempty set. A mapping $T: X \to 2^{X}$ is called a multivalued mapping if each element in X is mapped to a subset of X rather than a single point. From [48] and [49], the FP for such a mapping is defined as follows.

Definition 3. Let X denote an arbitrary nonempty set, and let 2^{X} the collection of all subsets of X. A mapping $T: X \to 2^{X}$ is said to have a FP if there exists $x \in X$ such that $x \in T(x)$. In this case, x is called a FP of T.

3. RESULTS AND DISCUSSION

Let X_{ϖ} be an arbitrary MbMS, and let $M \subseteq X_{\varpi}$. The following notations represent specific collections of subsets of X:

$$P(M) = \{Y \subseteq M: Y \neq \emptyset\},$$

$$P_b(M) = \{Y \in P(M) \ : Y \text{ is } \varpi\textit{-bounded}\},$$

$$P_{cl}(M) = \{Y \in P(M) \ : Y \textit{ is } \varpi\textit{-closed}\},$$

$$CB(M) = P_b(M) \cap P_{cl}(M).$$

The following definitions extend several well-known concepts from bMS, particularly those related to the Hausdorff distance, into the framework of MbMSs, based on the formulations presented in [42] and [50]. These extensions adapt the notions of proximity and separation between subsets in the context of MbMSs.

Definition 4. Let X_{ϖ} be a MbMS and $M \subseteq X_{\varpi}$.

1. The mapping $D: (0, \infty) \times CB(M) \times CB(M) \to [0, \infty)$ is defined by
$$D_{\lambda}(A, B) = \inf\{\varpi_{\lambda}(x, y): x \in A, y \in B\},$$
for all $A, B \in CB(M)$ and $\lambda > 0$. Furthermore, if $x_0 \in X_{\varpi}$, we denote $D_{\lambda}(\{x_0\}, B)$ by $\varpi_{\lambda}(x_0, B)$.
2. The mapping $\rho: (0, \infty) \times CB(M) \times CB(M) \to [0, \infty)$ is defined by
$$\rho_{\lambda}(A, B) = \sup\{\varpi_{\lambda}(x, B): x \in A\},$$
for all $A, B \in CB(M)$ and $\lambda > 0$.
3. The mapping $H: (0, \infty) \times CB(M) \times CB(M) \to [0, \infty)$ is defined by
$$H_{\lambda}(A, B) = \max\{\rho_{\lambda}(A, B), \rho_{\lambda}(B, A)\},$$
for all $A, B \in CB(M)$ and $\lambda > 0$.

The mappings established in Definition 4 yield the following properties. First, for all $A, B \in CB(M)$ and $\lambda > 0$, we have

$$D_{\lambda}(A, B) \leq \rho_{\lambda}(A, B) \leq H_{\lambda}(A, B). \tag{2}$$

This further implies the inequality

$$\varpi_{\lambda}(x, B) \leq \rho_{\lambda}(A, B) \leq H_{\lambda}(A, B), \tag{3}$$

for all $x \in A$ and $\lambda > 0$. Next, when $B \in CB(M)$ and $\lambda > 0$, we have

$$\varpi_{\lambda}(x, B) = \inf\{\varpi_{\lambda}(x, y): y \in B\} \leq \varpi_{\lambda}(x, y), \tag{4}$$

for all $x \in X_{\varpi}$. Lastly, we find the symmetry property of H, that is

$$H_{\lambda}(A, B) = H_{\lambda}(B, A), \tag{5}$$

for all $A, B \in CB(M)$ and $\lambda > 0$.

Based on [42], [44], and [51], these results establish fundamental relationships among the distance functions D, ρ, and H in MbMSs, which will be useful in proving the following results.

Lemma 1. *Let X_{ϖ} be an arbitrary MbMS and $M \subseteq X$. Then*

1. *for all* $ꭓ, ƴ \in Ꭓ_{\varpi}$, $\lambda, \mu > 0$, *and* $Ⱥ \in CB(Ɱ)$,
$$\varpi_{\lambda+\mu}(ꭓ, Ⱥ) \leq ꞩ\big[\varpi_{\lambda}(ꭓ, ƴ) + \varpi_{\mu}(ƴ, Ⱥ)\big], \tag{6}$$
2. *for all* $ꭓ \in Ꭓ_{\varpi}$, $\lambda, \mu > 0$, *and* $Ⱥ, Ƀ \in CB(Ɱ)$,
$$\varpi_{\lambda+\mu}(ꭓ, Ⱥ) \leq ꞩ\big[\varpi_{\lambda}(ꭓ, Ƀ) + Ħ_{\mu}(Ƀ, Ⱥ)\big]. \tag{7}$$

Proof.

1. Let $ꭓ, ƴ \in Ꭓ_{\varpi}$, $\lambda, \mu > 0$, and $Ⱥ \in CB(Ɱ)$. Based on Axiom (B3) in Definition 1, it follows that
$$\varpi_{\lambda+\mu}(ꭓ, ⱥ) \leq ꞩ[\varpi_{\lambda}(ꭓ, ƴ) + \varpi_{\mu}(ƴ, ⱥ)]$$
for all $ⱥ \in Ⱥ$. As a result, we have
$$\inf\{\varpi_{\lambda+\mu}(ꭓ, ⱥ) : ⱥ \in Ⱥ\} \leq \inf\{ꞩ\,[\varpi_{\lambda}(ꭓ, ƴ) + \varpi_{\mu}(ƴ, ⱥ)] : ⱥ \in Ⱥ\},$$
$$\Leftrightarrow \inf\{\varpi_{\lambda+\mu}(ꭓ, ⱥ) : ⱥ \in Ⱥ\} \leq ꞩ\,\varpi_{\lambda}(ꭓ, ƴ) + ꞩ\inf\{\varpi_{\mu}(ƴ, ⱥ) : ⱥ \in Ⱥ\},$$
$$\Leftrightarrow \varpi_{\lambda+\mu}(ꭓ, Ⱥ) \leq ꞩ\,[\varpi_{\lambda}(ꭓ, ƴ) + \varpi_{\mu}(ƴ, Ⱥ)].$$
2. Let $ꭓ \in Ꭓ_{\varpi}$, $\lambda, \mu > 0$, and $Ⱥ, Ƀ \in CB(Ɱ)$. From point (1), we have
$$\varpi_{\lambda+\mu}(ꭓ, Ⱥ) \leq ꞩ\big[\varpi_{\lambda}(ꭓ, ƴ) + \varpi_{\mu}(ƴ, Ⱥ)\big],$$
for all $ƴ \in Ƀ$. Thus, based on Eq. (3), it follows that
$$\varpi_{\lambda+\mu}(ꭓ, Ⱥ) \leq ꞩ[\varpi_{\lambda}(ꭓ, ƴ) + Ħ_{\mu}(Ƀ, Ⱥ)].$$
As a result, we have
$$\begin{aligned}\varpi_{\lambda+\mu}(ꭓ, Ⱥ) &= \inf\{\varpi_{\lambda+\mu}(ꭓ, ƴ, Ⱥ) : ƴ \in Ƀ\} \\ &\leq \inf\{ꞩ\,[\varpi_{\lambda}(ꭓ, ƴ) + Ħ_{\mu}(Ƀ, Ⱥ)] : ƴ \in Ƀ\} \\ &= ꞩ[\varpi_{\lambda}(ꭓ, Ƀ) + Ħ_{\mu}(Ƀ, Ⱥ)]. \ \blacksquare\end{aligned}$$

Lemma 2. *Let* $Ꭓ_{\varpi}$ *be an arbitrary MbMS,* $\lambda > 0$, *and* $Ɱ \subseteq Ꭓ_{\varpi}$. *If* $Ⱥ \in CB(Ɱ)$ *and* $ꭓ \in Ꭓ$, *then*
$$\varpi_{\lambda}(ꭓ, Ⱥ) = 0 \Leftrightarrow ꭓ \in Ⱥ.$$

Proof. If $ꭓ \in Ⱥ$, then $\varpi_{\lambda}(ꭓ, Ⱥ) \leq \varpi_{\lambda}(ꭓ, ꭓ) = 0$, so $\varpi_{\lambda}(ꭓ, Ⱥ) = 0$. Conversely, if $\varpi_{\lambda}(ꭓ, Ⱥ) = 0$, then for all $n \in \mathbb{N}$, there exists $ⱥ_n \in Ⱥ$ such that $\varpi_{\lambda}(ꭓ, ⱥ_n) < \frac{1}{n}$. Thus, $\varpi_{\lambda}(ꭓ, ⱥ_n) \to 0$ as $n \to \infty$. In other words, $\{ⱥ_n\}$ is ϖ-convergent to $ꭓ$. Since $Ⱥ$ is ϖ-closed, we conclude that $ꭓ \in Ⱥ$. ■

The following lemmas provide some properties of the function $Ħ$. The lemmas show that $Ħ$ satisfies the conditions required to be a M*b*M.

Lemma 3. *Let* $Ꭓ_{\varpi}$ *be an arbitrary MbMS and* $Ɱ \subseteq Ꭓ_{\varpi}$. *Then for all* $Ⱥ, Ƀ \in CB(Ɱ)$ *and* $\lambda > 0$, *we have*
$$Ħ_{\lambda}(Ⱥ, Ƀ) = 0 \Leftrightarrow Ⱥ = Ƀ.$$

Proof. Let $Ⱥ, Ƀ \in CB(Ɱ)$ and $\lambda > 0$. If $Ⱥ = Ƀ$, then by Lemma 2, we have $\varpi_{\lambda}(ꭓ, Ƀ) = 0$ and $\varpi_{\lambda}(ꭓ, Ⱥ) = 0$ for all $ꭓ \in Ⱥ = Ƀ$. As a result, we have $\rho_{\lambda}(Ⱥ, Ƀ) = 0$ and $\rho_{\lambda}(Ƀ, Ⱥ) = 0$, so that
$$Ħ_{\lambda}(Ⱥ, Ƀ) = 0.$$
Conversely, if $Ħ_{\lambda}(Ⱥ, Ƀ) = 0$, then $\rho_{\lambda}(Ⱥ, Ƀ) = \rho_{\lambda}(Ƀ, Ⱥ) = 0$. We consider the following two cases

1. **Case 1**. Let $ꭓ \in Ⱥ$. Since $\rho_{\lambda}(Ⱥ, Ƀ) = 0$, it follows that $\varpi_{\lambda}(ꭓ, Ƀ) = 0$. By Lemma 2, we have $ꭓ \in Ƀ$, so $Ⱥ \subseteq Ƀ$.
2. **Case 2**. Let $ꭓ \in Ƀ$. Since $\rho_{\lambda}(Ƀ, Ⱥ) = 0$, it follows that $\varpi_{\lambda}(ꭓ, Ⱥ) = 0$. By Lemma 2, we have $ꭓ \in Ⱥ$, so $Ƀ \subseteq Ⱥ$.

Therefore, we conclude that $Ⱥ = Ƀ$. ■

Lemma 4. *Let* $Ꭓ_{\varpi}$ *be an arbitrary MbMS and* $Ɱ \subseteq Ꭓ_{\varpi}$. *Then*
$$Ħ_{\lambda+\mu}(Ⱥ, Ȼ) \leq ꞩ\big[Ħ_{\lambda}(Ⱥ, Ƀ) + Ħ_{\mu}(Ƀ, Ȼ)\big], \tag{8}$$

for all $A, B, C \in CB(M)$ *and* $\lambda, \mu > 0$.

Proof. Let $A, B, C \in CB(M)$ and $\lambda, \mu > 0$. Based on Lemma 1, we have

$$\varpi_{\lambda+\mu}(a, C) \leq s[\varpi_{\lambda}(a, B) + H_{\mu}(B, C)] \tag{9}$$

for all $a \in A$ and

$$\varpi_{\lambda+\mu}(c, A) \leq s[\varpi_{\lambda}(c, B) + H_{\mu}(B, A)] \tag{10}$$

for all $c \in C$. As a consequence of Eqs. (3) and (9), we obtain

$$\begin{aligned} \rho_{\lambda+\mu}(A, C) &= \inf\{\varpi_{\lambda+\mu}(a, C): a \in A\} \\ &\leq \inf\{s[\varpi_{\lambda}(a, B) + H_{\mu}(B, C)]: a \in A\} \\ &= s \inf\{\varpi_{\lambda}(a, B): a \in A\} + s\, H_{\mu}(B, C) \\ &= s[\rho_{\lambda}(A, B) + H_{\mu}(B, C)] \\ &\leq s[H_{\lambda}(A, B) + H_{\mu}(B, C)]. \end{aligned}$$

Similarly, using Eqs. (3), (5), and (10), we derive

$$\begin{aligned} \rho_{\lambda+\mu}(C, A) &= \inf\{\varpi_{\lambda+\mu}(c, A): c \in C\} \\ &= \inf\{\varpi_{\mu+\lambda}(c, A): c \in C\} \\ &\leq \inf\{s[\varpi_{\mu}(c, B) + H_{\lambda}(B, A)]: c \in C\} \\ &= s\,[\rho_{\mu}(C, B) + H_{\lambda}(B, A)] \\ &\leq s[H_{\mu}(C, B) + H_{\lambda}(B, A)] \\ &= s[H_{\lambda}(A, B) + H_{\mu}(B, C)]. \end{aligned}$$

Thus, we have

$$H_{\lambda+\mu}(A, C) = \max\{\rho_{\lambda+\mu}(A, C), \rho_{\lambda+\mu}(C, A)\} \leq s[H_{\lambda}(A, B) + H_{\mu}(B, C)]. \blacksquare$$

Lemma 5. *If* X_{ϖ} *is a MbMS,* $M \subseteq X_{\varpi}$, $\lambda > 0$, *and* $A, B \in CB(M)$, *then for all* $\varepsilon > 0$ *and* $y \in B$, *there exists* $a \in A$ *such that*

$$\varpi_{\lambda}(a, y) \leq H_{\lambda}(A, B) + \varepsilon. \tag{11}$$

Proof. Let $\varepsilon > 0$ and $y \in B$. Using the infimum property, there exists $a \in A$ such that

$$\varpi_{\lambda}(a, y) = \varpi_{\lambda}(y, a) \leq \varpi_{\lambda}(y, A) + \varepsilon.$$

Based on Eq. (3), we have

$$\varpi_{\lambda}(a, y) \leq H_{\lambda}(A, B) + \varepsilon. \blacksquare$$

Therefore, based on Lemma 3 satisfies condition (B1), Eq. (5) satisfies condition (B2), and Lemma 4 satisfies condition (B3), we have the following theorem.

Theorem 1. *If* X_{ϖ} *is a MbMS and* $M \subseteq X_{\varpi}$, *then* $(CB(M), H, s)$ *is also a MbMS.*

Note that if a sequence is ϖ-convergent for some $\lambda > 0$, it does not necessarily converge for all $\lambda > 0$. Therefore, the following definition is introduced, based on [47].

Definition 5. A M*b*M ϖ on a M*b*MS X_{ϖ} is said to satisfy the Δ_2-condition if

$$\lim_{n\to\infty} \varpi_{\lambda}(x_n, x) = 0,$$

for some $\lambda > 0$ implies

$$\lim_{n\to\infty} \varpi_{\lambda}(x_n, x) = 0$$

for all $\lambda > 0$.

Example 2. Let $(Ҳ, \varpi, 2)$ be the modular b-metric space given in Example 1. It can be shown that ϖ satisfies the Δ_2-condition.

The following discussion focuses on contraction-type multivalued mappings in MbMSs and their corresponding FP theorems. We begin by defining multivalued mappings in this space, based on [42], [44], and [45].

Definition 6. Let $Ҳ_\varpi$ be an arbitrary MbMS, $Ɱ \subseteq Ҳ_\varpi$, and H be the mapping defined in Definition 4. A mapping $Ʈ: Ɱ \to CB(Ɱ)$ is called a ϖ-contraction mapping if there exist $\lambda > 0$ and $\kappa \in \left[0, \frac{1}{s}\right)$ such that for all $ҳ, y \in Ɱ$, the following inequality holds:

$$H_\lambda\big(Ʈ(ҳ), Ʈ(y)\big) \le \kappa\, \varpi_\lambda(ҳ, y). \tag{12}$$

To illustrate this concept, we provide an example demonstrating a ϖ-contraction mapping in a MbMS.

Example 3. Let $(Ҳ, \varpi, 2)$ be a MbMS given in Example 1. Let $Ʈ: Ҳ \to CB(Ҳ)$ be a mapping defined by

$$Ʈ(ҳ) = \left[\frac{ҳ}{2}, \frac{ҳ+1}{2}\right]$$

for all $ҳ \in Ҳ$. Note that

$$H_\lambda\big(Ʈ(ҳ), Ʈ(y)\big) = \max\left\{\varpi_\lambda\left(\frac{ҳ}{2}, \frac{y}{2}\right), \varpi_\lambda\left(\frac{ҳ+1}{2}, \frac{y+1}{2}\right)\right\} = \frac{1}{4}\frac{|ҳ - y|^2}{\lambda}.$$

Thus, by taking $\kappa = \frac{1}{4}$, we have $\kappa \in \left[0, \frac{1}{s}\right)$ and

$$H_\lambda\big(Ʈ(ҳ), Ʈ(y)\big) \le \kappa\, \varpi_\lambda(ҳ, y).$$

Hence, $Ʈ$ is a ϖ-contraction mapping.

Based on [42], [44], and [45], we extend the following theorem. The theorem is the main result concerning multivalued mappings in the context of MbMSs. It establishes a FP theorem for ϖ-contraction mappings defined on nonempty subsets of X_ϖ.

Theorem 2. *Let $Ҳ_\varpi$ be a MbMS such that ϖ satisfies the Δ_2-condition, $Ҳ_\varpi$ is ϖ-complete, and $Ɱ \subseteq Ҳ_\varpi$ is a ϖ-closed nonempty set. If $Ʈ: Ɱ \to CB(Ɱ)$ is a ϖ-contraction mapping and there exists $ҳ_0 \in Ɱ$* such that

$$\varpi_\lambda(ҳ_0, y) < \infty, \text{ for all } y \in Ʈ(x_0) \text{ and } \lambda > 0, \tag{13}$$

then $Ʈ$ has a fixed point.

Proof. Since $Ʈ: Ɱ \to CB(Ɱ)$ is a ϖ-contraction mapping, there exist $\lambda > 0$ and $\kappa \in \left[0, \frac{1}{s}\right)$ such that

$$H_\lambda\big(Ʈ(ҳ), Ʈ(y)\big) \le \kappa\, \varpi_\lambda(ҳ, y)$$

for all $ҳ, y \in Ɱ$. Note that $0 < \kappa < \frac{1}{s}$; it follows that $1 < \frac{1}{\kappa s}$. Consequently, there exists $q \in \mathbb{R}$ such that

$$1 < q < \frac{1}{\kappa s}.$$

Let $ҳ_0 \in Ɱ$ be a point satisfying Eq. (13). Since $Ʈ(ҳ_0) \subseteq CB(Ɱ) \subseteq P(Ɱ)$, it follows that $Ʈ(ҳ_0) \neq \emptyset$. Thus, there exists $ҳ_1 \in Ʈ(ҳ_0)$. If $Ʈ(ҳ_0) = Ʈ(ҳ_1)$, then $ҳ_1 \in Ʈ(ҳ_1)$ and we are done. Suppose that $Ʈ(ҳ_0) \neq Ʈ(ҳ_1)$. By applying Lemma 3, we obtain $H_\lambda\big(Ʈ(ҳ_0), Ʈ(ҳ_1)\big) > 0$. Since $(q-1)H_\lambda\big(Ʈ(ҳ_0), Ʈ(ҳ_1)\big) > 0$, Lemma 5 guarantees the existence of $ҳ_2 \in Ʈ(ҳ_1)$ such that

$$\varpi_\lambda(ҳ_1, ҳ_2) \le H_\lambda\big(Ʈ(ҳ_0), Ʈ(ҳ_1)\big) + (q-1)H_\lambda\big(Ʈ(ҳ_0), Ʈ(ҳ_1)\big) = q\, H_\lambda\big(Ʈ(ҳ_0), Ʈ(ҳ_1)\big).$$

Similarly, suppose that $Ʈ(ҳ_1) \neq Ʈ(ҳ_2)$. Following the same reasoning as before, there exists $ҳ_3 \in Ʈ(ҳ_2)$ such that

$$\varpi_\lambda(ҳ_2, ҳ_3) \le q\, H_\lambda\big(Ʈ(ҳ_1), Ʈ(ҳ_2)\big).$$

By continuing this process, we construct a sequence $\{ҳ_n\}$ such that $ҳ_{n+1} \in Ʈ(ҳ_n)$ and

$$\varpi_\lambda(ҳ_{n+1}, ҳ_n) \le q\, H_\lambda\big(Ʈ(ҳ_n), Ʈ(ҳ_{n-1})\big)$$

for all $n \in \mathbb{N}$. Since T satisfies Eq. (12), it follows that

$$\varpi_\lambda(x_{n+1}, x_n) \leq q\kappa\, \varpi_\lambda(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$. By iterating this process, we obtain

$$\begin{aligned} \varpi_\lambda(x_{n+1}, x_n) &\leq q\kappa\, \varpi_\lambda(x_n, x_{n-1}) \\ &\leq (q\kappa)^2\, \varpi_\lambda(x_{n-1}, x_{n-2}) \\ &\leq (q\kappa)^3\, \varpi_\lambda(x_{n-2}, x_{n-3}) \\ &\vdots \\ &\leq (q\kappa)^n \varpi_\lambda(x_1, x_0) \end{aligned} \tag{14}$$

for all $n \in \mathbb{N}$. Next, we will show that $\{x_n\}$ is a ϖ-C sequence. By applying Axiom (B3) and Eq. (1), we obtain the following inequality for all $n, i \in \mathbb{N}$:

$$\begin{aligned} \varpi_\lambda(x_n, x_{n+i}) \leq{}& s\, \varpi_{\frac{\lambda}{2}}(x_n, x_{n+1}) + s^2\, \varpi_{\frac{\lambda}{2^2}}(x_{n+1}, x_{n+2}) + s^3\, \varpi_{\frac{\lambda}{2^3}}(x_{n+2}, x_{n+3}) + \cdots \\ &+ s^i\, \varpi_{\frac{\lambda}{2^i}}(x_{n+i-1}, x_{n+i}) \end{aligned} \tag{15}$$

We illustrate the validity of this inequality by considering the first few cases explicitly.

For $i = 1$, we have

$$\varpi_\lambda(x_n, x_{n+1}) \leq s\left[\varpi_{\frac{\lambda}{2}}(x_n, x_n) + \varpi_{\frac{\lambda}{2}}(x_n, x_{n+1})\right] = s\varpi_{\frac{\lambda}{2}}(x_n, x_{n+1}).$$

For $i = 2$, we obtain

$$\begin{aligned} \varpi_\lambda(x_n, x_{n+2}) &\leq s\left[\varpi_{\frac{\lambda}{2}}(x_n, x_{n+1}) + \varpi_{\frac{\lambda}{2}}(x_{n+1}, x_{n+2})\right] \\ &\leq s\, \varpi_{\frac{\lambda}{2}}(x_n, x_{n+1}) + s^2 \varpi_{\frac{\lambda}{4}}(x_{n+1}, x_{n+2}) \\ &= s\, \varpi_{\frac{\lambda}{2}}(x_n, x_{n+1}) + s^2 \varpi_{\frac{\lambda}{2^2}}(x_{n+1}, x_{n+2}). \end{aligned}$$

For $i = 3$, it follows that

$$\begin{aligned} \varpi_\lambda(x_n, x_{n+3}) &\leq s\left[\varpi_{\frac{\lambda}{2}}(x_n, x_{n+1}) + \varpi_{\frac{\lambda}{2}}(x_{n+1}, x_{n+3})\right] \\ &\leq s\varpi_{\frac{\lambda}{2}}(x_n, x_{n+1}) + s^2\left[\varpi_{\frac{\lambda}{4}}(x_{n+1}, x_{n+2}) + \varpi_{\frac{\lambda}{4}}(x_{n+2}, x_{n+3})\right] \\ &\leq s\varpi_{\frac{\lambda}{2}}(x_n, x_{n+1}) + s^2\varpi_{\frac{\lambda}{4}}(x_{n+1}, x_{n+2}) + s^3\varpi_{\frac{\lambda}{8}}(x_{n+2}, x_{n+3}) \\ &= s\varpi_{\frac{\lambda}{2}}(x_n, x_{n+1}) + s^2\varpi_{\frac{\lambda}{2^2}}(x_{n+1}, x_{n+2}) + s^3\varpi_{\frac{\lambda}{2^3}}(x_{n+2}, x_{n+3}). \end{aligned}$$

By continuing this iterative process, Eq. (15) is established for all $n, i \in \mathbb{N}$.

Since for all $i \in \mathbb{N}$, we have

$$\frac{\lambda}{2^{i+1}} < \frac{\lambda}{2^i} < \ldots < \frac{\lambda}{4} < \frac{\lambda}{2}$$

using this fact along with Eqs. (1) and (15), we further obtain

$$\begin{aligned} \varpi_\lambda(x_n, x_{n+i}) \leq{}& s^2\, \varpi_{\frac{\lambda}{2^{i+1}}}(x_n, x_{n+1}) + s^3\varpi_{\frac{\lambda}{2^{i+1}}}(x_{n+1}, x_{n+2}) + s^4\, \varpi_{\frac{\lambda}{2^{i+1}}}(x_{n+2}, x_{n+3}) + \cdots \\ &+ s^{i+1}\varpi_{\frac{\lambda}{2^{i+1}}}(x_{n+i-1}, x_{n+i}) \end{aligned} \tag{16}$$

for all $n, i \in \mathbb{N}$. By utilizing Eqs. (14) and (16), it follows that

$$\begin{aligned}\varpi_\lambda(\mathfrak{x}_n,\mathfrak{x}_{n+i}) &\le s^2(q\kappa)^n\varpi_{\frac{\lambda}{2^{i+1}}}(\mathfrak{x}_0,\mathfrak{x}_1)+s^3(q\kappa)^{n+1}\varpi_{\frac{\lambda}{2^{i+1}}}(\mathfrak{x}_0,\mathfrak{x}_1)+s^4(q\kappa)^{n+2}\varpi_{\frac{\lambda}{2^{i+1}}}(\mathfrak{x}_0,\mathfrak{x}_1)\\ &+\cdots+s^{i+1}(q\kappa)^{\{n+i-1\}}\varpi_{\frac{\lambda}{2^{i+1}}}(\mathfrak{x}_0,\mathfrak{x}_1)\\ &= s^2(q\kappa)^n\varpi_{\frac{\lambda}{2^{i+1}}}(\mathfrak{x}_0,\mathfrak{x}_1)[1+sq\kappa+(sq\kappa)^2+\cdots+(sq\kappa)^{i-1}]\\ &\le s^2(q\kappa)^n\varpi_{\frac{\lambda}{2^{i+1}}}(\mathfrak{x}_0,\mathfrak{x}_1)[1+sq\kappa+(sq\kappa)^2+\cdots]\end{aligned}$$

for all $n, i \in \mathbb{N}$. Since $1 < q < \frac{1}{\kappa s}$, we have that $0 < sq\kappa < 1$. Therefore, for all $n, i \in \mathbb{N}$, we obtain

$$\varpi_\lambda(\mathfrak{x}_n,\mathfrak{x}_{n+i}) \le \frac{s^2(q\kappa)^n}{1-sq\kappa}\,\varpi_{\frac{\lambda}{2^{i+1}}}(\mathfrak{x}_0,\mathfrak{x}_1).$$

Now, observe that $0 < q\kappa < \frac{1}{s} \le 1$, so that $(q\kappa)^n \to 0$ as $n \to \infty$. Moreover, since $\mathfrak{x}_0 \in \mathcal{M}$ satisfies Eq. (13) and $\mathfrak{x}_1 \in \mathcal{T}(x_0)$, we have

$$\varpi_{\frac{\lambda}{2^{i+1}}}(\mathfrak{x}_0,\mathfrak{x}_1) < \infty.$$

Hence, we conclude that

$$\varpi_\lambda(\mathfrak{x}_n,\mathfrak{x}_{n+i}) \to 0$$

as $n, i \to \infty$. Thus, $\{x_n\}$ is a $\{\mathfrak{x}_n\}$ is a ϖ-C sequence. By the completeness of $\mathfrak{X}_\varpi$, there exists $\mathfrak{x} \in \mathfrak{X}_\varpi$ such that

$$\varpi_\lambda(\mathfrak{x}_n,\mathfrak{x}) \to 0$$

as $n \to \infty$. Since $\mathcal{M}$ is ϖ-closed, we have $\mathfrak{x} \in \mathcal{M}$. Then, applying Lemma 1, we obtain

$$\varpi_\lambda(\mathfrak{x},\mathcal{T}(\mathfrak{x})) = \lim_{n\to\infty}\varpi_\lambda(\mathfrak{x},\mathcal{T}(\mathfrak{x})) \le \lim_{n\to\infty} s\left[\varpi_{\frac{\lambda}{2}}(\mathfrak{x},\mathcal{T}(\mathfrak{x}_n)) + H_{\frac{\lambda}{2}}(\mathcal{T}(\mathfrak{x}_n),\mathcal{T}(\mathfrak{x}))\right]. \tag{17}$$

Since $\mathfrak{x}_{n+1} \in \mathcal{T}(\mathfrak{x}_n)$ for all $n \in \mathbb{N}$, it follows that

$$\varpi_{\frac{\lambda}{2}}(\mathfrak{x},\mathcal{T}(\mathfrak{x}_n)) = \inf_{y\in T(\mathfrak{x}_n)}\varpi_{\frac{\lambda}{2}}(\mathfrak{x},y) \le \varpi_{\frac{\lambda}{2}}(\mathfrak{x},\mathfrak{x}_{n+1}). \tag{18}$$

for all $n \in \mathbb{N}$. Using Eqs. (12), (17), and (18), we derive

$$\varpi_\lambda(\mathfrak{x},\mathcal{T}(\mathfrak{x})) \le \lim_{n\to\infty} s\left[\varpi_{\frac{\lambda}{2}}(\mathfrak{x},\mathfrak{x}_{n+1}) + k\,\varpi_{\frac{\lambda}{2}}(\mathfrak{x}_n,\mathfrak{x})\right].$$

Since ϖ satisfies the Δ_2-condition and $\varpi_\lambda(\mathfrak{x}_n,\mathfrak{x}) \to 0$ as $n \to \infty$, it follows that

$$\varpi_{\frac{\lambda}{2}}(\mathfrak{x}_n,\mathfrak{x}) \to 0$$

as $n \to \infty$. As a result, we obtain

$$\varpi_\lambda(\mathfrak{x},\mathcal{T}(\mathfrak{x})) \le \lim_{n\to\infty} s\left[\varpi_{\frac{\lambda}{2}}(\mathfrak{x},\mathfrak{x}_{n+1}) + k\,\varpi_{\frac{\lambda}{2}}(\mathfrak{x}_n,\mathfrak{x})\right] = 0.$$

Since $\varpi_\lambda(\mathfrak{x},\mathcal{T}(\mathfrak{x})) = 0$, applying Lemma 2, we conclude that $\mathfrak{x} \in \mathcal{T}(\mathfrak{x})$.■

Example 4. Let $(\mathfrak{X},\varpi,2)$ be a MbMS as defined in Example 1. From Example 2, we have that ϖ satisfies the Δ_2-condition. Based on the definition of $\mathfrak{X}_\varpi$, consider $\mathfrak{x}^0 = 0 \in \mathfrak{X}$. Then, we have

$$\mathfrak{X}_\varpi = \left\{\mathfrak{x} \in \mathfrak{X}: \frac{|\mathfrak{x}|^2}{\lambda} \to 0 \text{ as } \lambda \to \infty\right\}.$$

It is clear that $\mathfrak{X}_\varpi = \mathfrak{X}$, which implies that $\mathfrak{X}$ is ϖ-complete and ϖ-closed. Moreover, by taking $\mathfrak{x}_0 = 0 \in \mathfrak{X}_\varpi$, we obtain

$$\mathcal{T}(\mathfrak{x}_0) = \mathcal{T}(0) = \left[0,\frac{1}{2}\right],$$

and for all $y \in T(x_0)$ and $\lambda > 0$,

$$\varpi_\lambda(x_0, y) = \frac{|x_0 - y|^2}{\lambda} = \frac{y^2}{\lambda} < \infty.$$

This shows that $x_0 = 0$ satisfies Eq. (13). Now, let T be the ϖ-contractive mapping given in Example 3. Since

$$T(0) = \left[0, \frac{1}{2}\right] \text{ and } T(1) = \left[\frac{1}{2}, 1\right],$$

where $0 \in T(0)$ and $1 \in T(1)$, it follows that 0 and 1 are both FPs of T.

If $T: X_\varpi \to X_\varpi$ is a single-valued mapping on a MbMS X_ϖ, the following corollary directly follows as a consequence of Theorem 2.

Corollary 1. *Let X_ϖ be a MbMS, where ϖ satisfies the Δ_2-condition. Suppose that X_ϖ is ϖ-complete, $M \subseteq X_\varpi$ is a ϖ-closed nonempty set, and there exists $x_0 \in M$ that satisfies (13). Consider a mapping $T: M \to M$ for which there exists a constant $k \in \left[0, \frac{1}{s}\right)$ such that*

$$\varpi_\lambda\big(T(x), T(y)\big) \leq \kappa\, \varpi_\lambda(x, y),$$

for all $x, y \in M$. Then, there exists $x \in M$ such that $x \in T(x)$.

Application to Integral Equation

In the section, inspired by [52] and [53], we give a typical application of FP methods to the study of existence of solutions to integral equations. Briefly, we provide the background a notation. Let $X = C([0, a], \mathbb{R})$ be the set of all real-valued continuous functions defined on $[0, a]$, where $a > 0$, and define $d: X \times X \to [0, \infty)$ by

$$d(x, y) = \| |x - y|^2 \|_\infty = \sup_{t \in [0,a]} |x(t) - y(t)|^2$$

for all $x, y \in X$. Then $(X, d, 2)$ is a complete bMS. Next, let $\varpi: (0, \infty) \times X \times X \to [0, \infty]$ given by

$$\varpi_\lambda(x, y) = \frac{d(x, y)}{\lambda}$$

for all $x, y \in X$. Then $(X, \varpi, 2)$ is a ϖ-complete MbMS and ϖ satisfies the Δ_2-condition. We consider the following integral equation

$$x(t) = p(t) + \int_0^a S(t, u)\, f\big(u, x(u)\big)\, du$$

where $f: [0, a] \times \mathbb{R} \to \mathbb{R}$ and $p: [0, a] \to \mathbb{R}$ are two continuous functions. Moreover, $S: [0, a] \times [0, a] \to [0, \infty)$ is a function satisfying $S(t, .) \in L^1([0, a])$ for all $t \in [0, a]$, i.e.,

$$\int_0^a |S(t, u)|\, du < \infty$$

for all $t \in [0, a]$.

Defined the operator $T: X \to X$ by

$$T(x)(t) = p(t) + \int_0^a S(t, u) f\big(u, x(u)\big)\, du.$$

Then we prove the following existence result.

Theorem 3. *Let $X = C([0, a], \mathbb{R})$. Suppose that there exists $x_0 \in X$ satisfying Eq. (13) and a constant $\kappa \in \left[0, \frac{1}{s}\right)$ such that the following inequalities are satisfied:*

(i) $\left|f\big(u, x(u)\big) - f\big(u, y(u)\big)\right| \leq \sqrt[4]{\kappa}\,|x(u) - y(u)|$ *for all $x, y \in X$ and $u \in [0, a]$*

(ii) $\left\|\int_0^a S(t, u)\, du\right\|_\infty < \sqrt{\kappa}.$

Then the integral equation has a solution in X.

Proof. For all $ꭓ, ƴ \in Ӿ$ and $t \in [0, ⱥ]$, we have

$$
\begin{aligned}
|Ʈ(ꭓ)(t) - Ʈ(ƴ)(t)|^2 &= \left[\left|\int_0^{ⱥ} S(t,ư)\left[f\big(ư, ꭓ(ư)\big) - f\big(ư, ƴ(ư)\big)\right] dư\right|\right]^2 \\
&\le \left[\int_0^{ⱥ} S(t,ư) \left|f\big(ư, ꭓ(ư)\big) - f\big(ư, ƴ(ư)\big)\right| dư\right]^2 \\
&\le \left[\int_0^{ⱥ} S(t,ư) \sqrt[4]{\kappa}\, |ꭓ(ư) - ƴ(ư)|\, dư\right]^2 \\
&= \sqrt{\kappa}\left[\int_0^{ⱥ} S(t,ư) \sqrt{|ꭓ(ư) - ƴ(ư)|^2}\, dư\right]^2 \\
&\le \sqrt{\kappa}\left[\int_0^{ⱥ} S(t,ư) \sqrt{\||ꭓ - ƴ|^2\|_\infty}\, dư\right]^2 \\
&= \sqrt{\kappa}\||ꭓ - ƴ|^2\|_\infty \left[\int_0^{ⱥ} S(t,ư)\, dư\right]^2.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
\||Ʈ(ꭓ) - Ʈ(ƴ)|^2\|_\infty &\le \sqrt{\kappa}\||ꭓ - ƴ|^2\|_\infty \left\|\int_0^{ⱥ} S(t,ư)\, dư\right\|_\infty \\
&\le \kappa\||ꭓ - ƴ|^2\|_\infty.
\end{aligned}
$$

As a result, it follows that

$$
\varpi_\lambda\big(Ʈ(ꭓ), Ʈ(ƴ)\big) \le \kappa\, \varpi_\lambda(ꭓ, ƴ).
$$

By Corollary 1, $Ʈ$ has a FP. ■

4. CONCLUSION

This study introduces multivalued mappings in MbMSs, focusing on contraction-type mappings. By establishing the concept of Hausdorff distance in this framework, the study proves FP theorems for contraction-type multivalued mappings. These findings confirm that the existence of a FP is guaranteed under the following fundamental conditions: completeness of the space, closedness of the considered subset, and satisfaction of the Δ_2-condition of the MbM. Furthermore, the corollary of the main result is applied to guarantee the existence of solutions to an integral equation. These results open new possibilities for further developments in FP theory and its applications in various fields of applied mathematics.

Author Contibutions

Hartono: Conceptualization, Methodology, Writing-Original Draft, Validation. Lusi Harini: Data Curation, Resources, Draft Preparation, Investigation. Afifah Hayati: Formal Analysis, Investigation, Validation, Writing-Original Draft. Syamsudin Mas'ud: Formal Analysis, Investigation, Validation. Thesa Adi Saputra Yusri: Funding Acquisition, Project Administration, Writing-Review and Editing. Karyati: Supervision, Validation. Fitriana Yuli Saptaningtyas: Methodology, Writing-Review and Editing. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare no competing interest.

Declaration of Generative AI and AI-assisted Technologies

AI tools were used solely for language refinement (grammar, spelling, and clarity), while the scientific content, analysis, interpretation, and conclusions were entirely developed by the authors. All final text has been reviewed and approved by us.

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