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APPLICATION OF UNCERTAIN MAX PLUS LINEAR FOR SHIP SCHEDULE SAFETY ANALYSIS: A CASE STUDY OF KM. LAMBELU

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ABSTRACT

This study aims to analyze the safety of the KM. Lambelu passenger ship schedule on its Parepare-Balikpapan route using the uncertain Max-Plus Linear (uMPL) approach. The uMPL model is used to represent the dynamics by considering the uncertainty of travel times between ports. Forward reachability analysis is conducted to verify whether the ship scheduling system meets the established safety criteria. The analysis results show that the analysis indicates that the KM. Lambelu scheduling system has safety vulnerabilities. This finding indicates the presence of potential accident or incident risks and emphasizes the need for evaluation and improvement of scheduling system to ensure ship operation within safe limits. This study identifies potential problems and risks associated with these findings and provides recommendations for improving the ship schedule.

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1. INTRODUCTION

Regularity and efficiency in maritime transportation schedules, particularly for passenger ships in archipelagic nations like Indonesia, play a vital role in supporting inter-regional connectivity and economic growth. However, operational realities are often marked by various uncertainties and disturbances (noises) that can cause significant deviations from planned schedules. Factors such as adverse weather conditions, varying ship docking times, technical constraints, and the availability of resources are inherent challenges in maintaining punctuality [1], [2], [3]. Real-world incidents such as the grounding KM. Lambelu in Tarakan waters on Saturday, October 22, 2016 in the early morning [4] and the sinking of KM Journey due to a collision with KM. Lambelu on April 1, 2014 [5] further emphasizes the importance of safety and reliability in passenger ship operations.

The Max-Plus Linear (MPL) model has been recognized as an effective mathematical framework for modeling and analyzing discrete event systems with temporal constraints [6]. MPL has been successfully applied in various transportation systems such as railway network systems [7], trains [8], [9], [10], transportation networks [11], inland water transport systems [12], which also exhibit characteristics of synchronization and timing constraints. Nevertheless, the direct application of conventional MPL models often encounters limitations in accommodating the inherent uncertainties within complex maritime transportation systems. To address this, the development of models capable of explicitly handling uncertainty, such as uncertain Max Plus Linear (uMPL), becomes relevant. uMPL offers the potential to model systems by considering the variability of operational parameters [1].

This research focuses on the problem of passenger ship scheduling, specifically a case study in KM. Lambelu on its Parepare-Balikpapan route. The deliberate decision to focus on a single ship, rather than the more complex multi-ship collision problem, is a fundamental step in validating our formal verification methodology. This approach allows us to manage the exponential state space complexity that would arise from including multiple ships, enabling a meticulous analysis of the core model's effectiveness. By first verifying a single ship's ability to maintain a safe and punctual schedule, a logical prerequisite for any multi-ship analysis, we can ensure the integrity of the system's foundational safety properties. Given the importance of operational safety and punctuality, as well as incidents highlighting the vulnerability of schedules to disruptions, this study aims to answer the question: can the uncertain Max-Plus Linear (uMPL) approach analyze the ship schedule safety of KM. Lambelu? Reachability analysis in this context will be used as an initial step towards performing a mathematical safety verification of the schedule, with the hope of identifying potential unsafe operational conditions.

Although MPL models have been applied in various transportation scheduling contexts, as mentioned earlier, research specifically exploring the application of uMPL for reachability analysis in passenger ship scheduling, particularly with real-world case studies like KM. Lambelu, remains very limited. Most research using MPL in the maritime field tends to focus on railway traffic management optimization [13] or cargo scheduling [14]. Meanwhile, reachability analysis with uMPL has been applied in the context of control systems and safety verification of discrete event systems with uncertainty [15], [16]. However, its specific application to passenger ship scheduling, considering operational uncertainties and their implications for schedule safety, is still a relatively underexplored area.

The main objective of this research is to construct a dynamical affine representation of the KM. Lambelu scheduling system using uMPL. Through this model, forward reachability analysis will be conducted to understand the boundaries of the operational conditions that the system can reach and to evaluate the potential for undesirable deviations from a safe schedule [17], [18], [19], [20]. The primary contribution of this research lies in addressing a significant gap in the literature by applying the uMPL-based dynamical affine model to passenger ship scheduling, specifically demonstrated through a real-world case study of KM. Lambelu, which remains a relatively underexplored area. This application offers a distinct advantage over conventional approaches by explicitly handling the inherent uncertainties in maritime operations, thereby providing a more robust framework for mathematical safety verification and operational risk analysis. The findings of this study are expected to benefit ship operators in identifying potential risks, regulators in developing more effective safety policies, and the scientific community in expanding the application of uMPL in the field of maritime transportation.

2. RESEARCH METHODS

This chapter details the research methodology employed in this study to analyze the reachability of the KM. Lambelu passenger ship scheduling model. The discussion will cover the theoretical basis underpinning the research, including the relevant mathematical concepts. Subsequently, the chosen research design, the data sources used, and the development process of the uMPL model will be explained. Finally, the data analysis methods for testing reachability and the rationale for selecting KM. Lambelu as the case study will be outlined.

2.1 Theoretical Basis

This subsection outlines the theoretical foundations of this research. The discussion encompasses the concepts of Difference-Bound Matrices (DBM), Max-Plus Algebra, Interval Analysis, uncertain Max-Plus Linear Systems, Piecewise Affine (PWA) Representation, and the relevant principles of reachability analysis for uMPL systems.

2.1.1 Difference Bound Matrices (DBM)

The difference between two variables characterizes Difference-Bound Matrices (DBM). DBM is defined as follows [21]:

Definition 1. A DBM in $\mathbb{R}^{\boldsymbol{n}}$ is a square matrix that represents the intersection of a finite set defined by $\boldsymbol{y_i} - \boldsymbol{y_j} \bowtie_{\boldsymbol{i,j}} \boldsymbol{\alpha_{i,j}}$ where $\bowtie_{\boldsymbol{i,j}} \in \{<, \leq\}$ is the inequality sign and $\boldsymbol{\alpha_{i,j}} \in \mathbb{R} \cup \{+\infty\}$ is the upper bound, for $\boldsymbol{1} \leq \boldsymbol{i} \neq \boldsymbol{j} \leq \boldsymbol{n}$.

The value of $\boldsymbol{y_0}$ is always 0. This variable represents the set formed by a single variable, such as: $\boldsymbol{y_i} \bowtie_{\boldsymbol{i,i}} \boldsymbol{\alpha_{i,i}}$. A DBM can be represented in matrix form where the entries consist of an upper bound and an inequality sign. The writing of DBM uses the column-row rule, that is, the matrix element in row $\boldsymbol{i}$ and column $\boldsymbol{j}$ corresponds to $\boldsymbol{x_{j-1}} - \boldsymbol{x_{i-1}}$.

Example 2.1: Given a DBM $\boldsymbol{A} = \{\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{2}} : -\boldsymbol{1} \leq \boldsymbol{x_1} \leq \boldsymbol{5}, \boldsymbol{0} \leq \boldsymbol{x_2} \leq \boldsymbol{2}\}$ in $\mathbb{R}^{\boldsymbol{n}}$. The matrix representation of the DBM is obtained:

$$\boldsymbol{A} = \begin{bmatrix} (\boldsymbol{0}, \leq) & (\boldsymbol{5}, \leq) & (\boldsymbol{2}, \leq) \\ (\boldsymbol{1}, \leq) & (\boldsymbol{0}, \leq) & (+\infty, <) \\ (\boldsymbol{0}, \leq) & (+\infty, <) & (\boldsymbol{0}, \leq) \end{bmatrix}.$$

2.1.2 Max-Plus Algebra

Max-plus algebra is an idempotent semiring with two binary operations: maximum and addition. Given $\mathbb{R}_\varepsilon := \mathbb{R} \cup \{-\infty\}$ is a set equipped with two binary operations defined by [6], [22]:

$$a \oplus b := \max\{a, b\}, \tag{1}$$

$$a \otimes b := a + b, \tag{2}$$

for any $a, b \in \mathbb{R}_\varepsilon$. The semiring $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is defined with the neutral element $\varepsilon \stackrel{\text{def}}{=} -\infty$ and the identity element $e \stackrel{\text{def}}{=} 0$. The semiring $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is more compactly written as $\mathbb{R}_{max}$.

The $\oplus$ and $\otimes$ operations on matrices are defined as:

$$\begin{aligned} [\boldsymbol{A} \oplus \boldsymbol{B}]_{i,j} &= [\boldsymbol{A}_{i,j}] \oplus [\boldsymbol{B}_{i,j}], \\ [a \otimes \boldsymbol{A}]_{i,j} &= a \otimes [\boldsymbol{A}_{i,j}], \\ [\boldsymbol{C} \otimes \boldsymbol{D}]_{i,j} &= \bigoplus_{k=1}^{r} [\boldsymbol{C}]_{i,k} \otimes [\boldsymbol{D}]_{k,j}, \end{aligned} \tag{3}$$

for every matrix $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}_\varepsilon^{m \times n}$, $\boldsymbol{C} \in \mathbb{R}_\varepsilon^{m \times r}$, and $\boldsymbol{D} \in \mathbb{R}_\varepsilon^{r \times n}$, respectively.

The operation rules defined above have identical analogies to the operation rules in conventional algebra. The notation $[\boldsymbol{A}_{i,j}]$ is defined as the entry of matrix $\boldsymbol{A}$ in the i-th row and j-th column.

2.1.3 Interval Analysis

An interval is defined as [21]:

$$[x] = [\underline{x}, \overline{x}] = \{x \in \overline{\mathbb{R}}_{max}: \underline{x} \leq x \leq \overline{x}\}. \tag{4}$$

The intersection of two intervals $[x]$ and $[y]$ is empty or an interval, defined as:

$$[x] \cap [y] = \left[\max\{\underline{x}, \underline{y}\}, \min\{\overline{x}, \overline{y}\}\right]. \tag{5}$$

If the intersection of the intervals is not empty then the union is written as:

$$[x] \cup [y] = \left[\min\{\underline{x}, \underline{y}\}, \max\{\overline{x}, \overline{y}\}\right]. \tag{6}$$

The Max-Plus operations can be extended to intervals as follows:

$$[x] \oplus [y] = \{x \oplus y: x \in [x], y \in [y]\} = \left[\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}\right], \tag{7}$$

$$[x] \otimes [y] = \{x \otimes y: x \in [x], y \in [y]\} = \left[\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}\right]. \tag{8}$$

2.1.4 Uncertain Max-Plus Linear (uMPL) Systems

The uncertain Max-Plus Linear (uMPL) system is an extension of the Max-Plus Linear (MPL) system. MPL is defined as follows [23]:

$$\boldsymbol{x}(\boldsymbol{k}) = \boldsymbol{A} \otimes \boldsymbol{x}(\boldsymbol{k}-\boldsymbol{1}), \tag{9}$$

where $\boldsymbol{A} \in \mathbb{R}_{\varepsilon}^{\boldsymbol{m}\times\boldsymbol{n}}$ is a deterministic matrix, the variable k represents the event index, the vector $\boldsymbol{x}(\boldsymbol{k})$ is the event that occurs at time k. $\boldsymbol{x_i}(\boldsymbol{k})$ represents the k-th occurrence time of the i-th event.

If a number of matrix entries in Eq. (9) depend on k and lie within an interval then the MPL system is called an uMPL system. The uMPL system is defined as:

$$\boldsymbol{x}(\boldsymbol{k}) = \boldsymbol{A}(\boldsymbol{k}) \otimes \boldsymbol{x}(\boldsymbol{k}-\boldsymbol{1}), \tag{10}$$

where $\boldsymbol{A}(\boldsymbol{k}) \in [\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}}]$ is a non-deterministic matrix. Matrices $\underline{\boldsymbol{A}}$ and $\overline{\boldsymbol{A}}$ represent the lower and upper bound matrices, respectively. The interpretation of the state vector $\boldsymbol{x}$ in the uMPL system is the same as the interpretation in the MPL system.

2.1.5 Piecewise Affine (PWA) Representation

Max-Plus Linear (MPL) systems can be expressed as Piecewise Affine (PWA) systems. A PWA system is formed by a collection of regions. The dynamics of each region are affine, that is, linear plus a constant. Each region is formed by a finite coefficient $g = (g_1, g_2, \cdots, g_n) \in \{1, 2, \cdots, n\}^n$ where n is the dimension of the MPL system. The region corresponding to the coefficient g is [19]:

$$R_g = \bigcap_{i=1}^{k} \bigcap_{j=1, j\neq g_i}^{n} \{\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}: \boldsymbol{x_j} - \boldsymbol{x_{g_i}} \leq [\boldsymbol{A}]_{\boldsymbol{i},\boldsymbol{g_i}} - [\boldsymbol{A}]_{\boldsymbol{i},\boldsymbol{j}}\}, \tag{11}$$

where the corresponding affine dynamics are $\boldsymbol{x_i}(\boldsymbol{k}+\boldsymbol{1}) = \boldsymbol{x_{g_i}}(\boldsymbol{k}) + [\boldsymbol{A}]_{\boldsymbol{i},\boldsymbol{g_i}}$ for $i = \{1,2,\cdots,n\}$.

Uncertain MPL systems can be partitioned using the upper bound matrix $\overline{\boldsymbol{A}}$. In this case, each region is formed by a finite coefficient $g = (g_1, g_2, \cdots, g_n) \in \{1, 2, \cdots, n\}^n$. The region corresponding to the finite coefficient g is:

$$R_g^u = \bigcap_{i=1}^{k} \bigcap_{j=1, j\neq g_i}^{n} \left\{\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}: \boldsymbol{x_j} - \boldsymbol{x_{g_i}} \leq [\overline{\boldsymbol{A}}]_{\boldsymbol{i},\boldsymbol{g_i}} - [\overline{\boldsymbol{A}}]_{\boldsymbol{i},\boldsymbol{j}}\right\}, \tag{12}$$

where the corresponding dynamics are:

$$\bigcap_{i=1}^{n} \left\{\boldsymbol{x_i}(\boldsymbol{k}) - \boldsymbol{x_{g_i}}(\boldsymbol{k}-\boldsymbol{1}) \leq [\overline{\boldsymbol{A}}]_{\boldsymbol{i},\boldsymbol{g_i}}\right\} \cap \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} \left\{\boldsymbol{x_j}(\boldsymbol{k}-\boldsymbol{1}) - \boldsymbol{x_i}(\boldsymbol{k}) \leq -[\underline{\boldsymbol{A}}]_{\boldsymbol{i},\boldsymbol{j}}\right\}, \tag{13}$$

the dynamics in Eq. (13) can be represented as a DBM with respect to the state variables at times k and $k-1$.

2.1.6 Reachability Analysis of uMPL Systems

Reachability analysis for uncertain Max-Plus Linear (uMPL) systems has been conducted in [19]. Reachability analysis in this system can be performed using forward reachability and backward reachability methods. Given a DBM $\boldsymbol{X} \subseteq \mathbb{R}^{\boldsymbol{n}}$ and $\boldsymbol{A} \in [\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}}]$ is a max-plus interval matrix. The steps to compute the image of $\boldsymbol{X}$ corresponding to A are as follows [21]:

1. Construct a Piecewise Affine (PWA) system based on the upper bound of the max-plus interval matrix $\overline{\boldsymbol{A}}$;
2. Compute the cross product $(\boldsymbol{X} \cap \boldsymbol{R}_{\boldsymbol{g}}^{\boldsymbol{u}}) \times \mathbb{R}^{\boldsymbol{n}}$, for each finite coefficient g such that $\boldsymbol{X} \cap \boldsymbol{R}_{\boldsymbol{g}}^{\boldsymbol{u}}$ Is not empty;
3. Compute the intersection between the cross product result and the corresponding affine dynamics; and
4. Compute the projection of the intersection result onto the corresponding state variables in the previous step.

Given an uMPL system with a non-empty initial state set X_0, the set of reachable states at time k can be computed recursively:

$$\boldsymbol{X}_{\boldsymbol{k}} = \{\boldsymbol{A}_{\boldsymbol{k}} \otimes \boldsymbol{x} : \boldsymbol{A}_{\boldsymbol{k}} \in [\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}}], \boldsymbol{x} \in \boldsymbol{X}_{\boldsymbol{k-1}}\}, \tag{14}$$

where x_k is the set of states reachable at time k. Given a DBM $\boldsymbol{X} \subseteq \mathbb{R}^{\boldsymbol{n}}$ and $\boldsymbol{A} \in [\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}}]$ is a max-plus interval matrix. The steps to compute the inverse image of corresponding to A are as follows:

1. Construct PWA system based on the upper bound of the max-plus interval matrix $\overline{\boldsymbol{A}}$;
2. Compute the cross product $\mathbb{R}^{\boldsymbol{n}} \times \boldsymbol{X}$;
3. Compute the intersection between the corresponding affine dynamics and the result of $\mathbb{R}^{\boldsymbol{n}} \times \boldsymbol{X}$, for each finite coefficient $\boldsymbol{g}$ such that the intersection of the affine dynamics and $\mathbb{R}^{\boldsymbol{n}} \times \boldsymbol{X}$ is not empty; and
4. Compute the projection of the intersection result onto the corresponding state variables in the previous step.

Given an uMPL system with a non-empty final state set X_G, the set of states that can reach X_G at time k can be computed recursively:

$$X_{-k} = \{\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} : \exists \boldsymbol{A}_{\boldsymbol{k}} \in [\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}}] \Rightarrow \boldsymbol{A}_{\boldsymbol{k}} \otimes \boldsymbol{x} \in \boldsymbol{X}_{-\boldsymbol{k}+\boldsymbol{1}}\}, \tag{15}$$

where X_{-k} is the set of states that can reach the final state set at time k.

2.2 Research Design

This research employs a case study approach with the aim of analyzing the safety of the KM. Lambelu passenger ship schedule through mathematical verification. The research is quantitative, focusing on modeling and system analysis using the uncertain Max-Plus Linear (uMPL) approach. The research design involves a structured series of steps. First, operational schedule data of KM. Lambelu, historical delay data, and other relevant data will be collected from credible sources. Subsequently, an uMPL model will be developed to represent the ship scheduling dynamics, considering uncertainties in travel times between ports. A Piecewise Affine (PWA) representation of the uMPL model will be constructed to facilitate reachability analysis. Reachability analysis, using the methods described in the previous subsection, will then be applied to the PWA model to determine the boundaries of safe operational conditions. The results of this analysis will be used to verify whether the operational schedule of KM. Lambelu meets the established safety criteria. If potential safety violations are found, this research will provide recommendations to management for schedule improvements, with the goal of ensuring adherence to the predefined schedule. The scope of this research is limited to the operational route of KM. Lambelu, considering the duration of the ship's journey calculated from the difference between the scheduled departure and arrival times at each destination. This research also involves mathematical simulations to verify the safety of the schedule.

2.3 Data Sources

The data used in this research is secondary data obtained directly from official sources. The primary data includes the operational schedule of KM. Lambelu, which covers the ship's departure and arrival schedules. This data was obtained from the Kantor Kesyahbandaran dan Otoritas Pelabuhan (KSOP) and the PT. Pelni Parepare Branch Office. In addition, the shipping route data of KM. Lambelu was also obtained from the same sources. The data period used is from January to June 2023. The departure and arrival schedule data were then processed to calculate the travel time between ports on a daily basis. Since the data was obtained directly from authorized official agencies, namely KSOP and PT. Pelni, the data is considered to have a high degree of accuracy and reliability.

2.4 Model Development

The uncertain Max-Plus Linear (uMPL) model for KM. Lambelu's scheduling is constructed by defining the state variables as follows:

$\boldsymbol{x}(\boldsymbol{k})$: a matrix describing the departure time of the ship from each port at the k-th departure
$\boldsymbol{x}(\boldsymbol{k}-\mathbf{1})$: a matrix describing the departure time of the ship from each port at the $(k-1)$-th departure
$\boldsymbol{A}(\boldsymbol{k})$: a matrix containing intervals representing the travel time of the ship from one port to another at the k-th departure.

This uMPL model is based on the principle that the ship's departure time at a certain departure ($\boldsymbol{x}(\boldsymbol{k})$) is influenced by the departure time at the previous departure ($\boldsymbol{x}(\boldsymbol{k}-\mathbf{1})$) and the travel time between ports ($\boldsymbol{A}(\boldsymbol{k})$). Mathematically, as explained Subsection 2.1.4 regarding uMPL systems.

To represent the uncertainty in travel time between ports, each entry in the $\boldsymbol{A}(\boldsymbol{k})$ matrix is in the form of an interval. This interval reflects the variations that may occur in travel time due to factors such as weather conditions, sea currents, or non-constant ship speed. In this study, the lower and upper bounds of the interval are determined based on historical travel time data of KM. Lambelu during the period of January-June 2023. Specifically, for each route between ports, the lower bound of the interval is taken as the minimum recorded travel time value, while the upper bound of the interval is taken as the maximum recorded travel time value during that period. Details on the formation of these intervals will be further explained in the section discussing the overall uMPL model information.

Furthermore, this uMPL model is transformed into a Piecewise Affine (PWA) representation using the principles described in Eq. (12). This PWA representation allows for partitioning the system's state space into several regions, where the system dynamics in each region can be approximated by an affine function. The PWA representation is structured based on the formed uMPL model and the identified ship travel route. For each route, the relevant region g is determined, resulting in the corresponding affine dynamics. This process involves defining the constraints on the state variables that define each region, and then deriving the affine equations that describe the system's evolution within that region.

2.5 Data Analysis

Data analysis in this research aims to verify the safety of the KM. Lambelu's schedule through reachability analysis. To achieve this goal, this research will use the forward reachability analysis method. This method will be applied to the developed uMPL model to calculate the set of states that can be reached by the ship scheduling system from a given initial state.

Specifically, the reachability analysis will be carried out by calculating successive iterations of the system's state. In each iteration, the set of possible reachable states will be calculated based on the previous state and the possible variations in travel time between ports, represented as intervals. This process will be repeated until a relevant time horizon is reached.

In implementing this reachability analysis, this research will utilize the Piecewise Affine (PWA) representation of the uMPL model. As explained in the previous subsection, the PWA representation partitions the system's state space into several regions where the system dynamics can be approximated by affine functions. Reachability analysis will be performed on each affine region separately, and then the results from each region will be combined to obtain an overall picture of the system's reachability.

To facilitate the calculation and simulation of reachability analysis, this research will use the Python programming language. In particular, the Floyd-Warshall algorithm will be implemented in Python to find the canonical form of Difference-Bound Matrices (DBM), which are used to represent the set of states in reachability analysis. This canonical form will then be used to calculate the image of the previous state set.

The safety of the KM. Lambelu's schedule will be evaluated by calculating the intersection between the set of unsafe states (U_s), the initial state set (X_0), and the set of states reached at each iteration (X_1, X_2, and soon). If this intersection is empty, then the scheduling system is considered safe. Conversely, if this intersection is not empty, then the scheduling system is considered unsafe, because there is a possibility that an unsafe state can be reached.

In this study, the set of unsafe states is defined as $U_s = \boldsymbol{x} \in \mathbb{R}^{\mathbf{2}}: \mathbf{17} \leq \boldsymbol{x_1} - \boldsymbol{x_2} \leq \mathbf{24}$, where x_1 and x_2 represent the departure times of the ship from ports A and B, respectively. Thus, U_s includes states where the difference in departure times between the ship from ports A and B is in the range of 17 to 24 hours. This range is chosen as the U_s criterion because it reflects the worst conditions that may occur due to unexpected delays.

3. RESULTS AND DISCUSSION

This section presents the results of the safety analysis of the KM. Lambelu's schedule using the uncertain Max-Plus Linear (uMPL) approach. This analysis involves several stages, from system modeling to safety verification. For ease of understanding, this section is divided into several subsections which include: uMPL modeling, route identification, Piecewise Affine (PWA) and Difference-Bound Matrices Representation, Image X_k calculation, and system safety verification.

3.1 Uncertain Max-Plus (uMPL) Model

This subsection presents the uncertain Max-Plus Linear (uMPL) model developed to represent the scheduling dynamics of KM. Lambelu. The model is based on the ship's sailing route structure, travel times between ports, and potential uncertainties. KM. Lambelu serves 9 ports with 16 travel routes. Details of the travel routes and estimated travel times between ports are presented in Table 1.

Table 1. KM. Lambelu Travel Routes

No.	From	To	Estimated Travel Time (hours)
1	Parepare	Balikpapan	16-17
2	Balikpapan	Parepare	16-18
3	Makassar	Bau-bau	15-16
4	Bau-bau	Makassar	14-15
5	Makassar	Parepare	4-6
6	Parepare	Makassar	5-7
7	Bau-bau	Maumere	12
8	Maumere	Larantuka	5
9	Larantuka	Bau-bau	11-12
10	Balikpapan	Pantoloan	12-17
11	Pantoloan	Balikpapan	17-18
12	Pantoloan	Tarakan	18-21
13	Tarakan	Nunukan	6-7
14	Nunukan	Balikpapan	25-30
15	Balikpapan	Tarakan	21-26
16	Nunukan	Pantoloan	15-22

***Data source**: KSOP and the PT. Pelni Parepare Branch Office*

The sailing route of KM. Lambelu can also be visualized in Fig. 1, which shows the sequence of ports and the direction of the ship's travel. In this figure, each letter represents a port, with detail as follows: A (Makassar), B (Bau-bau), C (Maumere), D (Parepare), E (Balikpapan), F (Pantoloan), G (Tarakan), H (Nunukan), and I (Larantuka).

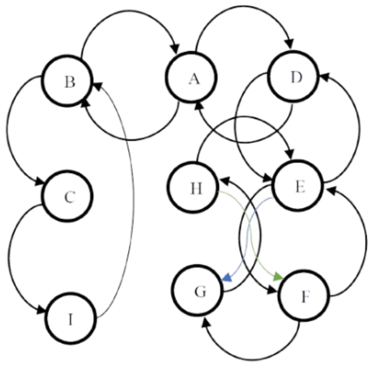

Figure 1. KM. Lambelu Route

The uMPL model for KM. Lambelu is based on the sailing route shown in Table 1 and Fig.1. This model describes the dynamics of the ship's movement between ports, considering the uncertainty in travel times. This model uses the following state variables:

1. x_k: a vector representing the departure time of the ship from each port at the k-th departure. $x_k = \left(x_1(k), x_2(k), \cdots, x_9(k)\right)'$, where $x_i(k)$ is the departure time from the i-th port at the k-th departure;
2. a_{ij}: travel time from port i to port j. This value is taken from Table 1 and can be a single value or an interval, depending on the recorded travel time variations; and
3. ε: represents the route a_{ij} is not available.

Using the variable above, the scheduling dynamics of KM. Lambelu is modeled by the following equation:

$$\boldsymbol{x}(k) = A(k) \otimes \boldsymbol{x}(k-1), \tag{16}$$

where

$$\boldsymbol{A}(\boldsymbol{k}) = \begin{bmatrix} \varepsilon & [14,15] & \varepsilon & [5,7] & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ [15,16] & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & [11,12] \\ \varepsilon & 12 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ [4,6] & \varepsilon & \varepsilon & \varepsilon & [16,18] & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & [16,17] & \varepsilon & [17,18] & \varepsilon & [25,30] & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & [12,17] & \varepsilon & \varepsilon & [15,22] & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & [21,26] & [18,21] & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & [6,7] & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}.$$

This equation describes how the ship's departure time at the k-th departure is influenced by the departure time at the previous departure $(k-1)$ and the travel times between ports. The symbol $\otimes$ denotes the max-plus operation. Matrix $\boldsymbol{A}(\boldsymbol{k})$ represents the travel times between ports, considering uncertainties. In this matrix, each element a_{ij} shows the travel time from port i to port j. Interval values, such as [14,16], indicate variations in travel time. The symbol ε indicates that there is no direct route from port i to port j.

3.2 Route Identification

This subsection identifies the shipping route of KM. Lambelu, that is the focus of the analysis in this study. This study focuses on the Parepare-Balikpapan route. The selection of this route is illustrative for the purpose of demonstrating the proposed analysis methodology. The estimated travel time and uncertainty for the Parepare-Balikpapan route is 16-17 hours, while for the Balikpapan-Parepare, it is 16-18 hours. This route is represented in the uMPL model by the following submatrix of the $\boldsymbol{A}(\boldsymbol{k})$ matrix defined in Subsection 3.1:

$$\boldsymbol{A}^*(\boldsymbol{k}) = \begin{bmatrix} \varepsilon & [16,18] \\ [16,17] & \varepsilon \end{bmatrix}. \tag{17}$$

This matrix shows the travel time from Parepare to Balikpapan and vice versa, taking into account the uncertainty. The element ε indicates that the ship does not return to its origin port. This identified route will be used in the following subsection for further analysis using the Piecewise Affine (PWA) and Difference-Bound Matrices (DBM) Approach.

3.3 Piecewise Affine Identification

Referring to Eqs. (12), (16), and (17), given the region $R_g \in \bar{R}^2_{max}$ for $\boldsymbol{g} \in \{1,2\}^2$. Based on the calculation in the previous subsection, for Parepare-Balikpapan route, only the region $R_{(2,1)}$ is not empty, with affine dynamics as follows:

$$x(k) = \left\{\begin{bmatrix}[\varepsilon \otimes x_1(k-1) \oplus 16 \otimes x_2(k-1), 16 \otimes x_2(k-1)]\\ [16 \otimes x_1(k-1) \oplus \varepsilon \otimes x_2(k-1), 16 \otimes x_1(k-1)]\end{bmatrix}, x(k-1) \in R^u_{(2,1)}\right\}. \quad (18)$$

This equation describes how the departure time from each port at the k-th departure is influenced by the departure time at the previous departure $(k-1)$ and the travel time between ports.

3.4 Difference-Bound Matrices Representation

Region $R_{(2,1)}$ and its corresponding dynamics are represented as a Difference-Bound Matrix (DBM). The DBM variables consist of the state variables at time t, the state variables at the next step, and the variable x_0. The transformation of the dynamics for $R_{(2,1)}$ in the KM. Lambelu schedulling is as follows:

$$\boldsymbol{x'_1 - x_2 \leq 1} \text{ and } \boldsymbol{6x'_2 - x_1 \leq 16}. \quad (19)$$

The DBM is generated from the affine dynamics: $\{[x_0, x_1, x_2, x'_1, x_2{}']^T : x'_1 - x_2 \leq 16, x'_2 - x_1 \leq 16\}$. The intersection between the DBM from the affine dynamics and $\boldsymbol{R}_{(2,1)} \times \mathbb{R}^2$ produces the DBM:

$$\boldsymbol{D}_{(2,1)} = \begin{bmatrix} (\mathbf{0}, \leq) & (+\infty, <) & (+\infty, <) & (+\infty, <) & (+\infty, <) \\ (+\infty, <) & (\mathbf{0}, \leq) & (+\infty, <) & (+\infty, <) & (\mathbf{6}, \leq) \\ (+\infty, <) & (+\infty, <) & (\mathbf{0}, \leq) & (\mathbf{4.75}, \leq) & (+\infty, <) \\ (+\infty, <) & (+\infty, <) & (+\infty, <) & (\mathbf{0}, \leq) & (+\infty, <) \\ (+\infty, <) & (+\infty, <) & (+\infty, <) & (+\infty, <) & (\mathbf{0}, \leq) \end{bmatrix}.$$

3.5 Image X_k Calculation

This subsection presents the calculation of the image of the state set X_0 iteratively to determine the set of states that can be reached by the system at each time step. The image calculation is done using the forward reachability approach. The set X_0 is defined as $X_0 = \{\boldsymbol{x} \in \mathbb{R}^2 : \mathbf{0} \leq \boldsymbol{x_1} \leq \mathbf{6}; \mathbf{2} \leq \boldsymbol{x_2} \leq \mathbf{5}\}$ and $U_s = \{\boldsymbol{x} \in \mathbb{R}^2 : \mathbf{17} \leq \boldsymbol{x_1} - \boldsymbol{x_2} \leq \mathbf{24}\}$ with $N = 2$. The set X_0 in the region $R_{(2,1)}$ is expressed in DBM as follows:

$$\boldsymbol{D}^{X_0} = \begin{bmatrix} (\mathbf{0}, \leq) & (\mathbf{6}, \leq) & (\mathbf{5}, \leq) \\ (\mathbf{0}, \leq) & (\mathbf{0}, \leq) & (+\infty, <) \\ (\mathbf{-2}, \leq) & (+\infty, <) & (\mathbf{0}, \leq) \end{bmatrix}.$$

The cross product of $\boldsymbol{D}^{X_0}$ and $\mathbb{R}^2$ produces:

$$\boldsymbol{D}^{\mathbb{R}^2 \times X_0} = \begin{bmatrix} (\mathbf{0}, \leq) & (\mathbf{6}, \leq) & (\mathbf{5}, \leq) & (+\infty, <) & (+\infty, <) \\ (\mathbf{0}, \leq) & (\mathbf{0}, \leq) & (+\infty, <) & (+\infty, <) & (+\infty, <) \\ (\mathbf{-2}, \leq) & (+\infty, <) & (\mathbf{0}, \leq) & (+\infty, <) & (+\infty, <) \\ (+\infty, <) & (+\infty, <) & (+\infty, <) & (\mathbf{0}, \leq) & (+\infty, <) \\ (+\infty, <) & (+\infty, <) & (+\infty, <) & (+\infty, <) & (\mathbf{0}, \leq) \end{bmatrix}.$$

The intersection of $\boldsymbol{D}^{\mathbb{R}^2 \times X_0}$ and DBM $\boldsymbol{D}(\mathbf{2}, \mathbf{1})$ results in $\boldsymbol{D}^{\mathbb{R}^2 \times X_0}_{(2,1)}$:

$$\boldsymbol{D}^{\mathbb{R}^2 \times X_0}_{(2,1)} = \begin{bmatrix} (\mathbf{0}, \leq) & (\mathbf{6}, \leq) & (\mathbf{5}, \leq) & (+\infty, <) & (+\infty, <) \\ (\mathbf{0}, \leq) & (\mathbf{0}, \leq) & (+\infty, <) & (+\infty, <) & (\mathbf{6}, \leq) \\ (\mathbf{-2}, \leq) & (+\infty, <) & (\mathbf{0}, \leq) & (\mathbf{4.75}, \leq) & (+\infty, <) \\ (+\infty, <) & (+\infty, <) & (+\infty, <) & (\mathbf{0}, \leq) & (+\infty, <) \\ (+\infty, <) & (+\infty, <) & (+\infty, <) & (+\infty, <) & (\mathbf{0}, \leq) \end{bmatrix}.$$

The canonical form of $\boldsymbol{D}^{\mathbb{R}^2 \times X_0}_{(2,1)}$ is obtained using the Floyd-Warshall algorithm [21]:

$$\boldsymbol{cf}\left(\boldsymbol{D}_{(\mathbf{2},\mathbf{1})}^{\mathbb{R}^{2}\times \boldsymbol{X}_{0}}\right)=\begin{bmatrix} (\mathbf{0},\le) & (\mathbf{6},\le) & (\mathbf{5},\le) & (\mathbf{9.75},\le) & (\mathbf{12},\le) \\ (\mathbf{0},\le) & (\mathbf{0},\le) & (\mathbf{5},\le) & (\mathbf{9.75},\le) & (\mathbf{6},\le) \\ (-\mathbf{2},\le) & (\mathbf{4},\le) & (\mathbf{0},\le) & (\mathbf{4.75},\le) & (\mathbf{10},\le) \\ (+\infty,<) & (+\infty,<) & (+\infty,<) & (\mathbf{0},\le) & (+\infty,<) \\ (+\infty,<) & (+\infty,<) & (+\infty,<) & (+\infty,<) & (\mathbf{0},\le) \end{bmatrix}.$$

The projection of DBM $\boldsymbol{D}_{(\mathbf{2},\mathbf{1})}^{\mathbb{R}^{2}\times \boldsymbol{X}_{0}}$ on the variables $\{x_1', x_2'\}$ is:

$$\boldsymbol{D}^{\boldsymbol{X}_1^{(2,1)}}=\begin{bmatrix} (\mathbf{0},\le) & (\mathbf{9.75},\le) & (\mathbf{12},\le) \\ (+\infty,<) & (\mathbf{0},\le) & (+\infty,<) \\ (+\infty,<) & (+\infty,<) & (\mathbf{0},\le) \end{bmatrix}.$$

The image of X_0 corresponding to the dynamics in the region $R_{(2,1)}^{u}$ is given:

$$X_1^{(2,1)}=\{\boldsymbol{x'}_{\mathbf{1}}\le \mathbf{9.75}, \boldsymbol{x}_{\mathbf{2}}'\le \mathbf{12}, -\infty<\boldsymbol{x'}_{\mathbf{1}}-\boldsymbol{x'}_{\mathbf{2}}<+\infty\}.$$

The procedure for calculating the finite set in the second step $(N=2)$ is done in the same way as before, i.e. finding the image of $X_1^{(2,1)}$ with respect to the dynamics of the region $R_{(2,1)}^{u}$ so we obtain:

$$X_2^{(2,1)}=\{\boldsymbol{x'}_{\mathbf{1}}\le \mathbf{16.75}, \boldsymbol{x}_{\mathbf{2}}'\le \mathbf{15.75}, -\infty<\boldsymbol{x'}_{\mathbf{1}}-\boldsymbol{x'}_{\mathbf{2}}<+\infty\}.$$

Based on the results obtained, the intersection between U_s and the union of X_0, X_1, and X_2 is non-empty, i.e. there are intersection between X_1 and U_s and intersection between X_2 and U_s.

3.6 System Safety Verification

This subsection presents the results of the safety verification of the KM. Lambelu's scheduling system. The safety of the system is evaluated by calculating the intersection between the set of unsafe states (U_s), the initial state set (X_0), and the set of states reached at each iteration (X_k). If this intersection is not empty, then the scheduling system is considered unsafe because there is a possibility that the ship can reach a state that is considered unsafe based on the established criteria.

Based on the previous analysis, we have found that the intersection between the set of unsafe states (U_s) and the union of the set of reachable states (X_0, X_1, and X_2) is not empty. This non-empty intersection indicates that there is a possibility that the scheduling system can reach a state that is included in the "unsafe" category (U_s). In other words, the scenarios considered unsafe by the definition of U_s have the possibility of occuring in the ship's operation. The existence of this intersection reveals the presence of a potential risk in the scheduling system. Although the system may not always end up in an unsafe state, this possibility means that the risk of accidents or incidents cannot be ignored. This result emphasizes the need for action to reduce or eliminate these risks. The scheduling system needs to be evaluated and improved to ensure that the ship operates within safe limits. In the context of KM. Lambelu, this finding has implication including schedule evaluation, risk mitigation, and safety improvement.

Based on these findings, several recommendations can be made to improve KM. Lambelu's schedule: adding buffer time to the departure and arrival schedules to accommodate variations in travel time, improving coordination between ports to ensure smooth ship flow and reduce potential delays, and setting a maximum limit on the difference in departure time between ships to mitigate the risk of traffic congestion.

Despite its limitations, this study contributes to the existing literature by applying the uMPL approach to analyze the schedule security of passenger ships in Indonesia, a topic that has not been previously explored. This research also offers a methodology for verifying schedule security by considering travel time uncertainty, which can be valuable tool for ship operators and port authorities.

4. CONCLUSION

This research has successfully demonstrated the application of the uncertain Max-Plus Linear (uMPL) approach in analyzing the safety ship scheduling, specifically for the KM. Lambelu passenger ship on the Parepare-Balikpapan route. The key finding of this research is that the current scheduling system of KM. Lambelu is not safe, because there is an intersection between the set of unsafe states and the set of states that the system can reach. This implies that there is a possibility for the ship to operate under conditions that are considered unsafe.

The analysis highlighted that the variability in travel times between ports is a critical factor influencing the safety of the schedule. To improve safety, this research recommends adding buffer time, improving coordination between ports, and setting maximum limits on departure time differences. This research contributes to the literature by applying the uMPL approach in the context of Indonesian passenger ships and offering a methodology for verifying schedule safety by considering travel time uncertainty.

Author Contributions

Nurul Fuady Adhalia H: Conceptualization, Methodology, Writing-Original Draft, Software, Validation. Mardhiyyah Rafrin: Data Curation, Resources, Writing-Review. Aditya Putra Pratama: Data Curation, Resources, Draft Preparation, Writing-Review. Rifaldy Atlant Tungga: Formal Analysis, Validation, Writing-Review and Editing. Bayu: Software, Visualization. Syahrul Ramadhan Tahir: Software, Visualization. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare no competing interest.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools were used solely for language refinement (grammar, spelling, and clarity). The scientific content, analysis, interpretation, and conclusions were developed entirely by the authors. The authors reviewed and approved all final text.

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