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THE METRIC DIMENSION OF CYCLE BOOK GRAPHS $B_{C_{m,n}}$ FORMED BY A COMMON PATH P_2

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ABSTRACT

This paper investigates the metric dimension of a class of graphs known as cycle books, denoted $B_{C_m,n}$, which feature a shared path P_2 across multiple cycles. We focus on characterizing the minimum number of vertex subsets required so that each vertex in the graph can be uniquely identified by its distances to those subsets. To support our analysis, we present two propositions and a general theorem that establish the metric dimension for various configurations of cycle book graphs. Specifically, we prove that $dim(B_{C_3,n}) = n$ for $n \geq 2$, and $dim(B_{C_4,n}) = n$ for $n = 2,3,4$, while $dim(B_{C_4,n}) = n - 1$ for $n \geq 5$. Furthermore, we provide a general result for $m \geq 5$: the metric dimension is n when m is odd and $m \in \{2,3\}$, or when m is even and $n \geq 2$; and $n - 1$ when m is odd and $n \geq 4$. These findings contribute to the growing body of knowledge on metric properties in graph theory, particularly in structured and cyclic graph families.

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1. INTRODUCTION

The concept of metric dimension has attracted considerable attention among researchers in discrete mathematics, particularly in graph theory, due to its broad range of practical uses such as in robotic navigation [1], network topology identification [2], and computational chemistry [3]. Consider a connected graph G and an ordered set of vertices $\Psi = \{\psi_1, \psi_2, \psi_3, \dots, \psi_n\}$. For any vertex $v \in V(G)$, the representation of v with respect to Ψ is given by the vector $r(v|\Psi) = (d(v,\psi_1), d(v,\psi_2), d(v,\psi_3), \dots, d(v,\psi_n))$, where $d(u,v)$ denotes the shortest path length between vertices u and v. The set Ψ is termed a *resolving set* if every distinct pair of vertices $u, v \in V(G)$ yields different representations. A resolving set with the smallest possible number of elements is called a *minimum resolving set*, and the number of elements it contains defines the *metric dimension* of G, symbolized as $dim(G)$.

Metric dimension remains a key topic of interest in graph theory due to its theoretical importance and wide-ranging applications across various fields. Numerous researchers have explored the concept in different graph classes. The notion of the metric dimension was first introduced by Chartrand *et al.* [4]. In their research, they have shown several metric dimensions in connected graphs such as trees, unicyclic graphs and graphs having metric dimensions 1, $n-1$, and $n-2$.

Several foundational studies have contributed to the development of metric dimension theory across different graph classes. According to Chartrand et al. [4], a connected graph G with n vertices will have a metric dimension of one if it forms a path P_n, and a metric dimension of $n-1$ if it is the complete graph K_n. Further, Buczkowski et al. [5] explored k-dimensional graphs along with their respective bases. In another direction, Bensmail et al. [6] extended the concept of metric dimension to oriented graphs. Moreno et al. [7] investigated the k-metric dimension specifically in the context of lexicographic product graphs. Beaudou *et al.* [8] have shown that upper bounds on graph order via diameter and metric dimension. Li and Pilipczuk [9] introduced the concepts of hardness of metric dimension in graphs of constant treewidth, Hoffmann and Wanke [10] has shown that metric dimension of gabriel unit disk graphs is NP-Complete, Feng *et al.* [11] determined the metric dimension of line graphs. Belmonte *et al.* [12] studied the metric dimension in graphs whose width is bounded, Susilowati *et al.* [13] introduced the concept of dominant metric dimension in graph theory. Hajibarat et al. [14] have shown that metric in higher dimensions, Mashkaria et al. [15] explored metric dimension robustness in grid graphs under edge insertion. Metric dimensions of zero-divisor graphs, certain connected networks, certain carbon nanocone networks, path-related graphs and four-dimensional klein bottle have been discussed in [16], [17], [18] and [19].

Several recent works have addressed the metric dimension in various graph contexts. The metric dimension of the join of paths and cycles was studied in [20], revealing how such operations influence the resolving sets of the resulting graphs. In addition, the concept of mixed metric dimension was investigated for graphs containing edge-disjoint cycles, highlighting the complexity introduced by these disjoint structures [21]. Research on the metric dimension related to the shackle operation applied to the C_3 cycle graph further explored the effects of graph transformations on metric properties [22]. Finally, bounds on the metric dimension for graphs with edge-disjoint cycles have been established, providing important constraints within this class of graphs [23].

The graph referred to as a cycle book, denoted $B_{C_m,n}$, was originally proposed by Santoso and Darmaji [24],[25]. This structure is formed by connecting n identical instances of the cycle C_m through a common subpath. That shared segment consists of two vertices, forming a path isomorphic to P_2. Various investigations have examined the structural and labeling aspects of this graph class. For example, the study by Simanihuruk and colleagues [26] investigated a hypothesis concerning the super edge-magic total labeling applied to cycle books of order four**.** In parallel, the work of Santoso and Darmaji [24] also included an analysis of the partition dimension for $B_{C_m,n}$, focusing on the specific cases where $m = 3,4$ and 5. In their subsequent findings, Santoso and Darmaji [27] generalized the partition dimension of the graph $B_{C_m,n}$. They showed that when the number of vertices in each cycle is at least six, the partition dimension takes the value three for $n = 2$ or 3; it equals n when $n = 4$; and for all other combinations of m and n, it increases linearly as three plus a natural number k. Recent research on graph cycle by Santoso [28] focuses on the chromatic number of cycle book graphs.

Building upon previous research, this paper focuses on determining the metric dimension of the cycle book graph $B_{C_m,n}$, contributing to the deeper analysis of structural characteristics within particular graph classes. Despite being limited to the specific family of cycle books $B_{C_m,n}$, this study presents meaningful findings that support the continuing investigation of metric-based graph parameters. This paper contributes by addressing the metric dimension of cycle book graphs $B_{C_m,n}$, a problem that has not been explicitly studied in previous works.

The main contributions of this study are: providing exact values of the metric dimension for several specific configurations of cycle book graphs; extending and complementing the earlier results of Santoso and Darmaji [24],[27] which focused on the partition dimension; and offering a general characterization of the metric dimension for $B_{C_m,n}$ based on the parity of m and n. These results not only deepen the understanding of metric-based parameters in structured graph families but also open potential directions for further research on generalized cycle book graphs.

2. RESEARCH METHODS

This research employs a literature-based theoretical approach grounded in existing studies on the metric dimension of graphs. To investigate the metric dimension of the cycle book graph $B_{C_m,n}$ with a common path P_2, the study focuses on structural graph analysis. The methodology involves constructing the graph, applying core concepts of metric dimension, analyzing vertex distances, and determining upper and lower bounds through mathematical reasoning.

2.1 Conceptual Foundation and Graph Construction

The first stage involved a comprehensive study of theoretical foundations relevant to the metric dimension, resolving sets, and the structural characteristics of cycle book graphs. This included reviewing scholarly articles, graph theory textbooks, and prior research to understand key concepts and existing results.

The graph $B_{C_m,n}$ referred to as a cycle book graph with a common path P_2 , is constructed by joining n copies of the cycle graph C_m such that all cycles share a common path P_2. This graph model is adapted from a previous construction introduced in [24], where a similar structure was used to study on the partition dimension of $B_{C_m,n}$. In the present study, the same graph is analyzed from the perspective of metric dimension.

2.2 Metric Dimension Framework

The next phase focused on constructing minimal resolving sets for $B_{C_m,n}$. A combinatorial approach was used to analyze vertex distances and demonstrate that certain subsets of vertices could uniquely identify all other vertices via distance vectors. The goal was to determine the smallest such set and thus the metric dimension of the graph.

2.3 Theorem Formulation and Proof

Through an examination of resolving sets, several general theorems have been developed to characterize the metric dimension of $B_{C_m,n}$. These results are supported by detailed and rigorous mathematical proofs. In the context of this research, the theorem below serves to provide a lower estimate of the metric dimension for the graph examined in this work.

Theorem 1.[29] *Let G be a graph consisting of n vertices, where $n \geq 2$. Then*

i) $dim(G) = 1$ *if and only if* $G = P_n$.

ii) $dim(G) = n - 1$ *if and only if* $G = K_n$.

iii) For $n \geq 3$, $dim(C_m) = 2$.

In proving the results, two main strategies are employed. First, a case analysis approach is used, where the values of m (the size of each cycle) and n (the number of cycle copies) are considered separately based on their parity (odd or even). This allows us to establish exact values of the metric dimension in different structural configurations of the cycle book graph. Second, symmetry arguments are applied to reduce redundancy in the analysis, since many vertices in cycle book graphs exhibit equivalent distance patterns due to their repetitive structure. By combining these approaches, minimal resolving sets are identified systematically, ensuring that every distinct pair of vertices has unique distance representations.

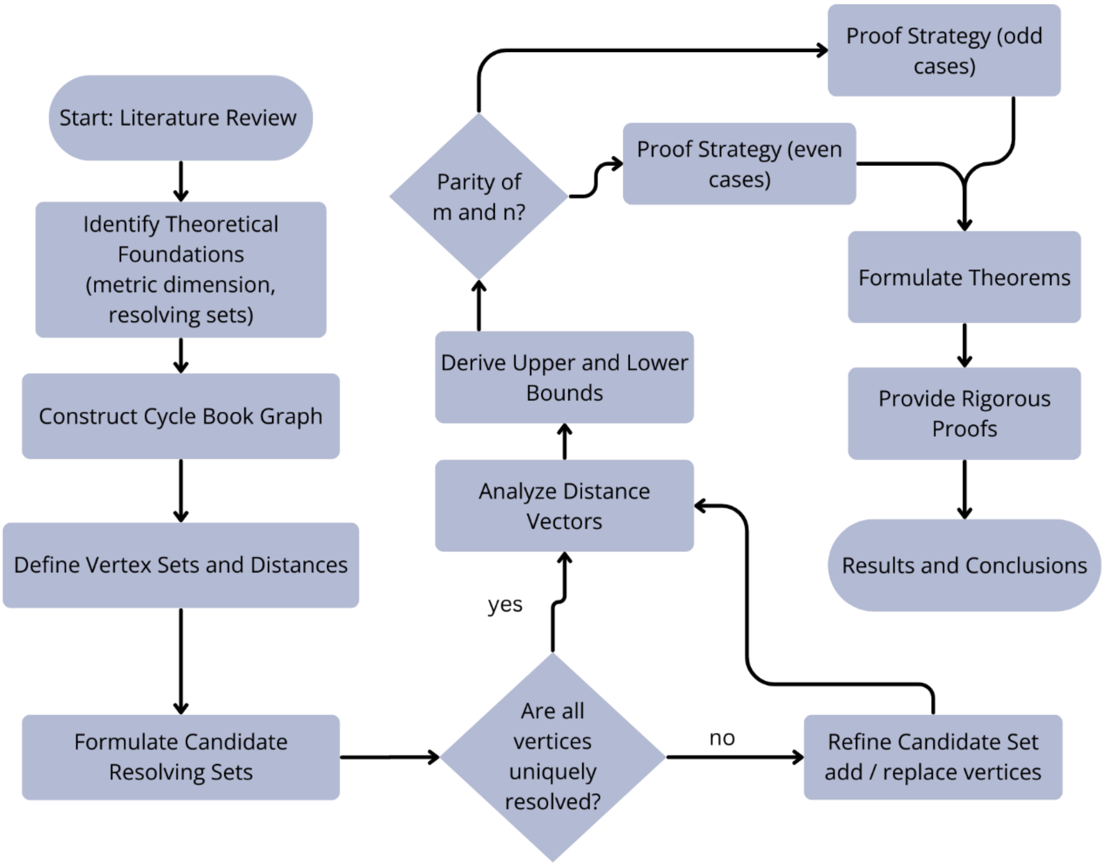

Figure 1. Flowchart of the Research Method for Determining the Metric Dimension

3. RESULTS AND DISCUSSION

In this section, we present the main findings of our study regarding the metric dimension of cycle book graphs $B_{C_m,n}$. The results are organized into two propositions and one theorem, each addressing specific cases based on the parameters m and n. These propositions and theorem collectively characterize the behavior of the metric dimension for different configurations of the cycle book graphs, providing explicit formulas and bounds. The detailed statements are as follows.

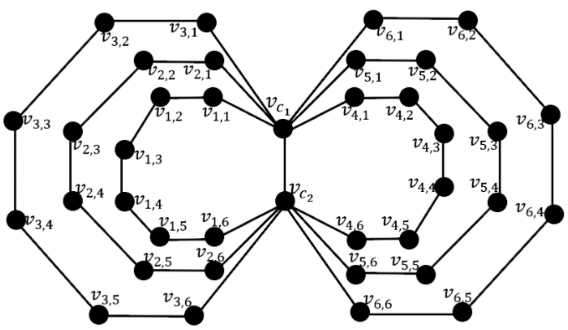

Figure 2. $B_{C_8,6}$ cycle books graph

Theorem 2. *Let* $B_{C_m,n}$ *denote a cycle book graph for* $m = 3$ *and* $n \geq 2$. *Then the metric dimension of* $B_{C_m,n}$ *is given by* $\dim(B_{C_m,n}) = n$.

Proof. Consider the graph $B_{C_3,n}$, whose vertex set is defined as $V(G) = \{v_1, v_2, v_3, \dots, v_n\} \cup \{v_{c_1}, v_{c_2}\}$. To establish the upper bound, we construct a resolving set $\Psi = \{v_{c_1}, v_1, v_2, \dots, v_{n-1}\}$. Each vertex is represented by the following distance vectors based on the set Ψ.

$$r(v_{c_1}|\Psi) = \left(0, \underbrace{1, \dots, 1}_{n-1}\right),$$
$$r(v_{c_2}|\Psi) = \left(\underbrace{1, \dots, 1}_{n}\right),$$
$$r(v_p|\Psi) = \left(1, \underbrace{2, \dots, 2}_{p-1}, 0, \underbrace{2, \dots, 2}_{n-p-1}\right); 1 \le p \le n-1,$$
$$r(v_p|\Psi) = \left(1, \underbrace{2, \dots, 2}_{n-1}\right); p = n.$$

Since all vertices have distinct representations, Ψ is a resolving set. Therefore, $\dim(G) \le n$.

To prove the lower bound, suppose there is a subset Ψ' that serves as a resolving set, with cardinality $|\Psi'| = n-1$. Three cases are considered:

Case 1. $\Psi' = \{v_{c_1}, v_{c_2}, v_1, v_2, \dots, v_{n-3}\}$. Vertices v_{n-2}, v_{n-1}, and v_n are not uniquely resolved, as their metric representations with respect to the resolving set Ψ' are identical. The distance vectors of the vertices in $V(G)$ relative to the set Ψ' are given below.

$$r(v_{c_1}|\Psi') = \left(0, \underbrace{1, \dots, 1}_{n-2}\right),$$
$$r(v_{c_2}|\Psi') = \left(1,0, \underbrace{1, \dots, 1}_{n-3}\right),$$
$$r(v_p|\Psi') = \left(1,1, \underbrace{2, \dots, 2}_{n-1}, 0, \underbrace{2, \dots, 2}_{n-i-2}\right) \; if \; 1 \le p \le n-3,$$
$$r(v_p|\Psi') = \left(1,1, \underbrace{2, \dots, 2}_{n-3}\right) \; if \; n-2 \le p \le n.$$

Case 2. $\Psi' = \{v_{c_1}, v_1, v_2, \dots, v_{n-2}\}$. Vertices v_{n-1} and v_n have identical representations, thus Ψ' fails to resolve all vertices. Each vertex $V(G)$ can be represented by its corresponding distance vector with respect to Ψ', as detailed below.

$$r(v_{c_1}|\Psi') = \left(0, \underbrace{1, \dots, 1}_{n-2}\right),$$
$$r(v_{c_2}|\Psi') = \left(\underbrace{1, \dots, 1}_{n-1}\right),$$
$$r(v_p|\Psi') = \left(1, \underbrace{2, \dots, 2}_{p-1}, 0, \underbrace{2, \dots, 2}_{n-p-2}\right); 1 \le p \le n-2,$$
$$r(v_p|\Psi') = \left(1, \underbrace{2, \dots, 2}_{n-2}\right); n-1 \le p \le n.$$

Case 3. $\Psi' = \{v_1, v_2, \dots, v_{n-1}\}$. Vertices v_{c_1} and v_{c_2} produce identical distance vectors relative to Ψ', the set fails to resolve all vertices. Below are representations of the vertices in $V(G)$ based on the reference set Ψ'.

$$r(v_{c_1}|\Psi') = \left(\underbrace{1, \dots, 1}_{n-1}\right),$$
$$r(v_{c_2}|\Psi') = \left(\underbrace{1, \dots, 1}_{n-1}\right),$$
$$r(v_p|\Psi') = \left(\underbrace{2, \dots, 2}_{p-1}, 0, \underbrace{2, \dots, 2}_{n-p-1}\right); 1 \le p \le n-1,$$
$$r(v_p|\Psi') = \left(\underbrace{2, \dots, 2}_{n-2}\right); p = n.$$

Since no subset of cardinality $n-1$ is a resolving set, it follows that $\dim(G) \geq n$. Thus, combining both bounds, we obtain $\dim(B_{C_3,n}) = n$. ■

Theorem 3. *Let $B_{C_m,n}$ denote a cycle books graph with $m = 4$ and $n \geq 2$. Then the metric dimension of $B_{C_m,n}$ is given by*

$$dim(B_{C_m,n}) = \begin{cases} n, & if\ n = 2,3,4; \\ n-1, & if\ n \geq 5. \end{cases}$$

Proof. The cycle books graph $B_{C_4,n}$ has $V(G) = \{v_{p,q} | 1 \leq p \leq m,\ q = 1,2\} \cup \{v_{c_1}, v_{c_2}\}$. To prove this proposition, we consider the proof in three distinct cases based on the value of n.

Case 1. For $n \in \{2,3,4\}$. Let $\Psi = \{v_{p,1} | 1 \leq p \leq n\}$. We claim that Ψ is resolving set for G. For each vertex $v \in V(G)$, we compute the metric representation $r(v|\Psi)$, which is the vector that contains the shortest path lengths between v and all vertices in Ψ. It can be verified that all such representations are distinct. For any $u, v \in V(G)$ if $u \neq v$, then $r(u|\Psi) \neq r(v|\Psi)$. Hence, Ψ is a resolving set, and thus $\dim(G) \leq |\Psi| = n$. We now present the distance-based representations of the elements of $V(G)$ in relation to the set Ψ.

$$r(v_{c_1}|\Psi) = (\underbrace{1, \dots, 1}_{n-1}),$$
$$r(v_{c_2}|\Psi) = (\underbrace{2, \dots, 2}_{n}),$$
$$r(v_{p,1}|\Psi) = \left(\underbrace{2, \dots, 2}_{p-1}, 0, \underbrace{2, \dots, 2}_{n-1}\right);\ 1 \leq p \leq n,$$
$$r(v_{p,2}|\Psi) = \left(\underbrace{3, \dots, 3}_{p-1}, 1, \underbrace{3, \dots, 3}_{n-p}\right);\ 1 \leq p \leq n,$$

To estimate the lower bound, assume there exists a resolving set $\Psi' \subset V(G)$ with $|\Psi'| = n-1$. Due to the symmetry and repetitive structure of the cycle books graph, it is possible to find a pair of vertices whose distances to Ψ' coincide. For instance, if one omits a vertex $v_{p,1}$ from Ψ, the vertices $v_{p,2}$ and v_{c_1} may share exhibit identical distance representations relative to Ψ'. Therefore, Ψ' does not qualify as a resolving set, and thus $\dim(G) = n$.

Case 2. For $n \geq 5$ and n is odd. Define the set $\Psi = \{v_{p,1} | 1 \leq p \leq n-1,\ p\ odd\} \cup \{v_{p,2} | 1 \leq p \leq n-1, p\ even\}$. Then $|\Psi| = n-1$. One can verify that Ψ is a resolving set of G because the metric representations $r(v|\Psi)$ are distinct for all $v \in V(G)$. Hence, $\dim(G) \leq n-1$. The distance vectors of the vertices in $V(G)$ relative to the set Ψ are given below.

$$r(v_{c_1}|\Psi) = \left(\underbrace{1, \dots, 1}_{\frac{n-1}{2}}, \underbrace{2, \dots, 2}_{\frac{n-1}{2}}\right) \text{ and } r(v_{c_2}|\Psi = \underbrace{\left(\underbrace{2, \dots, 2}_{\frac{n-1}{2}}, \underbrace{1, \dots, 1}_{\frac{n-1}{2}}\right)}_{n},$$

$$r(v_{p,1}|\Psi) = \left(\underbrace{2, \dots, 2}_{\frac{p-1}{2}}, 0, \underbrace{2, \dots, 2}_{\frac{n-1}{2}-1-\frac{p-1}{2}}, \underbrace{3, \dots, 3}_{\frac{n-1}{2}}\right);\ \text{for odd } p,\ 1 \leq p \leq n-1,$$

$$r(v_{p,1}|\Psi) = \left(\underbrace{2, \dots, 2}_{\frac{n-1}{2}}, \underbrace{3, \dots, 3}_{\frac{p-1}{2}}, 1, \underbrace{3, \dots, 3}_{\frac{n-1}{2}-1-\frac{p-2}{2}}\right);\ \text{for even } p,\ 1 \leq p \leq n-1,$$

$$r(v_{p,2}|\Psi) = \left(\underbrace{3, \dots, 3}_{\frac{p-1}{2}}, 1, \underbrace{3, \dots, 3}_{\frac{n-1}{2}-1-\frac{p-1}{2}}, \underbrace{2, \dots, 2}_{\frac{n-1}{2}}\right);\ \text{for odd } p,\ 1 \leq p \leq n-1,$$

$$r(v_{p,2}|\Psi) = \left(\underbrace{3, \dots, 3}_{\frac{n-1}{2}}, \underbrace{2, \dots, 2}_{\frac{p-2}{2}}, 0, \underbrace{2, \dots, 2}_{\frac{n-1}{2}-1-\frac{p-2}{2}}\right);\ \text{for even } p,\ 1 \leq p \leq n-1,$$

$$r(v_{m,1}|\Psi) = \left(\underbrace{2,\dots,2}_{\frac{n-1}{2}},\underbrace{3,\dots,3}_{\frac{n-1}{2}}\right) \text{ and } r(v_{m,2}|\Psi) = \left(\underbrace{3,\dots,3}_{\frac{n-1}{2}},\underbrace{2,\dots,2}_{\frac{n-1}{2}}\right).$$

Suppose there exists a resolving set Ψ' with $|\Psi'| < n-1$. Then by the structure of G, there exists at least one pair of non-adjacent vertices with identical representations with respect to Ψ', which contradicts the definition of a resolving set.

Case 3. For $n \geq 5$ and n is even. The argument proceeds similarly to Case 2. Define the resolving set $\Psi = \{v_{p,1}|1 \leq p \leq n-1,\ p\ odd\} \cup \{v_{p,2}|1 \leq p \leq n-1, p\ even\}$. $|\Psi| = n-1$, one can confirm that each metric representations $r(v|\Psi)$ is unique for all $v \epsilon V(G)$. Therefore, $\dim(G) = n-1$. This completes the proof by considering all three cases. ■

Theorem 4. *Consider the graph $B_{C_m,n}$, which represents a cycle book with $m \geq 5$ and $n \geq 2$. Under these conditions, the metric dimension of $B_{C_m,n}$ can be expressed as follows:*

$$dim(B_{C_m,n}) = \begin{cases} n & if\ m\ is\ odd\ and\ n = 2\ or\ 3; \\ n & if\ m\ is\ even\ and\ n \geq 2; \\ n-1 & if\ m\ is\ odd\ and\ n \geq 4. \end{cases}$$

Proof. To establish this result, the analysis will be partitioned into three scenarios based on the values of m and n. $m \geq 5$ is odd with $n = 2$ or 3; $m \geq 5$ is odd with $n \geq 4$; and when m is even and $n \geq 2$.

Case 1. $m \geq 5$ is odd and $n = 2\ or\ 3$. Let $\Psi = \left\{v_{p,q}|1 \leq p \leq n, 1 \leq q \leq \frac{n-1}{2}\right\}$. We show that Ψ is a resolving set. The metric representations of each vertex in $V(G)$ with respect to Ψ are as follows.

$$r(v_{c_1}|\Psi) = \left(\underbrace{\tfrac{1}{2}(m-3), \tfrac{1}{2}(m-3), \dots, \tfrac{1}{2}(m-3)}_{n}\right),\ r(v_{c_2}|\Psi) = \left(\underbrace{\tfrac{1}{2}(m-3)+1, \tfrac{1}{2}(m-3)+1, \dots, \tfrac{1}{2}(m-3)+1}_{n}\right),$$

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{m-1}{2}+(q-1), \dots, \tfrac{m-1}{2}+(q-1)}_{p-1}, \tfrac{m-1}{2}-(q+1), \underbrace{\tfrac{m-1}{2}+(q-1), \dots, \tfrac{m-1}{2}+(q-1)}_{n-1}\right);\ 1 \leq p \leq n, 1 \leq q \leq \frac{n-1}{2}-1,$$

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{m-1}{2}+(q-1), \dots, \tfrac{m-1}{2}+(q-1)}_{p-1}, 1, \underbrace{\tfrac{m-1}{2}+(q-1), \dots, \tfrac{m-1}{2}+(q-1)}_{n-p}\right);\ 1 \leq p \leq n, q = \frac{n-1}{2},$$

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{m-1}{2}+m-(q+1), \dots, \tfrac{m-1}{2}+m-(q+1)}_{p-1}, (q+1), (q+1) - \tfrac{m-1}{2}, \underbrace{\tfrac{m-1}{2}+m-(q+1), \dots, \tfrac{m-1}{2}+m-(q+1)}_{n-p}\right);\ 1 \leq p \leq n, \frac{n-1}{2}+1 \leq q \leq m-2.$$

Since all vertices have distinct representations, Ψ is a resolving set. Thus, the upper bound is $\dim(G) \leq n$. For the lower bound, suppose there exists a resolving set Ψ' with $|\Psi'| < n$. Then, by the pigeonhole principle, there exist at least two vertices $u \neq v$ such that $r(u|\Psi') = r(v|\Psi')$, which violates the condition that Ψ functions as a resolving set. As a result, we obtain $\dim(G) \geq n$ and therefore $\dim(G) = n$.

Case 2. $m \geq 5$ is odd and $n \geq 4$. Let $\Psi = \left\{v_{p,q}|1 \leq p \leq n-2\ and\ q = \frac{m-3}{2}\right\} \cup \left\{v_{p,q}|p = n-1\ and\ q = \frac{m-3}{2}+1\right\}$. The distance vectors of the vertices in $\{v_{c_1}, v_{c_2}\} \in V(G)$ relative to the set Ψ are given below.

$$r(v_{c_1}|\Psi) = \left(\underbrace{\tfrac{1}{2}(m-3), \dots, \tfrac{1}{2}(m-3)}_{n-2}, \tfrac{1}{2}(m-3)+1\right) \text{ and } r(v_{c_2}|\Psi) = \left(\underbrace{\tfrac{1}{2}(m-3)+1, \dots, \tfrac{1}{2}(m-3)+1}_{n-1}\right).$$

Listed below are the metric representations of all vertices $v_{p,q} \in V(G)$, where $1 \leq p \leq n-2$, measured relative to the set Ψ.

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{m-1}{2}+(q-1),\ldots,\tfrac{m-1}{2}+(q-1)}_{p-1},\tfrac{m-1}{2}-(q+1),\underbrace{\tfrac{m-1}{2}+(q-1),\ldots,\tfrac{m-1}{2}+(q-1)}_{n-p-2},\tfrac{m-1}{2}+q\right); \quad 1 \le q \le \tfrac{m-1}{2}-1,$$

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{m-1}{2}+(q-1),\ldots,\tfrac{m-1}{2}+(q-1)}_{p-1},1,\underbrace{\tfrac{m-1}{2}+(q-1),\ldots,\tfrac{m-1}{2}+(q-1)}_{n-p-2},\tfrac{m-1}{2}+q\right); q = \tfrac{m-1}{2},$$

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{m-1}{2}+m-(q+1),\ldots,\tfrac{m-1}{2}+m-(q+1)}_{i-1},(q+1)-\tfrac{m-1}{2},\underbrace{\tfrac{m-1}{2}+m-(q+1),\ldots,\tfrac{m-1}{2}+m-(q+1)}_{p-1}\right); \tfrac{m-1}{2}+1 \le q \le m-1.$$

For all vertices $v_{p,q} \in V(G)$ with $p = n-1$, their metric representations relative to Ψ are given below.

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{m-1}{2}+m-(q+1),\ldots,\tfrac{m-1}{2}+m-(q+1)}_{n-2},q-\tfrac{m-1}{2}\right); \tfrac{m-1}{2}+1 \le q \le m-2.$$

For all vertices $v_{p,q} \in V(G)$ with $p = n$, their metric representations relative to Ψ are given below.

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{m-1}{2}+(q-1),\ldots,\tfrac{m-1}{2}+(q-1)}_{n-2},\tfrac{m-1}{2}+q\right); 1 \le q \le \tfrac{m-1}{2},$$

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{m-1}{2}+m-(q+1),\ldots,\tfrac{m-1}{2}+m-(q+1)}_{n-1}\right); \tfrac{m-1}{2}+1 \le q \le m-2.$$

Hence, the set Ψ qualifies as a resolving set for the graph G, as the representation $r(v|\Psi)$ is unique for every vertex $v \in V(G)$. This implies that the metric dimension of G satisfies the upper bound $\dim(G) \le n-1$. To establish a lower bound, assume there exists another resolving set Ψ' whose cardinality is strictly smaller than that of Ψ. Under this assumption, there must be at least two distinct vertices in the set $V(G)-\Psi'$ sharing identical representations relative to Ψ'. Consequently, it follows that the metric dimension is bounded below by$\dim(G) \ge n-1$. Combining these results yields the exact value of the metric dimension $\dim(G) = n-1$.

Case 3. Let $m \ge 5$ be an even integer and $n \ge 2$. The upper bound on the metric dimension can be obtained by constructing a resolving set. Consider the set $\Psi = \left\{v_{p,q} | 1 \le p \le n; q = \frac{m-2}{2}\right\}$. The metric representations of the vertices in $V(G)$ with respect to Ψ are described as follows.

$$r(v_{c_1}|\Psi) = \left(\underbrace{\tfrac{(m-2)}{2},\ldots,\tfrac{(m-2)}{2}}_{n}\right) \text{ and } r(v_{c_2}|\Psi) = \left(\underbrace{\tfrac{(m-2)}{2}+1,\ldots,\tfrac{(m-2)}{2}+1}_{n}\right),$$

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{(m-2)}{2}+q,\ldots,\tfrac{(m-2)}{2}+q}_{p-1},\tfrac{(m-2)}{2}-q,\underbrace{\tfrac{(m-2)}{2}+q,\ldots,\tfrac{(m-2)}{2}+q}_{n-p}\right); 1 \le q \le \tfrac{m-2}{2},$$

$$r(v_{p,q}|\Psi) = \left(\underbrace{\tfrac{(m-2)}{2}+m-q,\ldots,\tfrac{(m-2)}{2}+m-q}_{p-1},q-\tfrac{(m-2)}{2},\underbrace{\tfrac{(m-2)}{2}+m-q,\ldots,\tfrac{(m-2)}{2}+m-q}_{n-p}\right); \tfrac{m-2}{2}+1 \le q \le m-2.$$

Therefore, the set Ψ constitutes a resolving set for the graph G, as the representation $r(v|\Psi)$ is unique for every vertex $v \in V(G)$. This implies that the metric dimension of G satisfies the upper bound $\dim(G) \le n$. To determine the corresponding lower bound, assume there exists a resolving set Ψ' whose cardinality is smaller than that of Ψ. Under this assumption, there must be at least two distinct vertices in $V(G)-\Psi'$ that share identical representations relative to Ψ'. Hence, Ψ' cannot serve as a resolving set, leading to the conclusion that $\dim(G) \geq n$. Combining these inequalities yields the exact metric dimension $\dim(G) = n$. This completes the proof of Theorem 4. ■

Example 1. Given a graph $B_{C_5,5}$ shown in Figure 3, can be determined by choosing the resolving set $\Psi = \{v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1}\}$. The representations of each vertex in $V(G)$ with respect to Ψ are as follows.

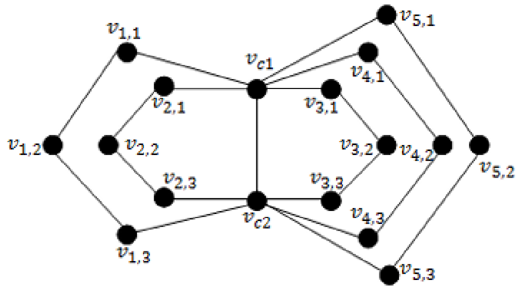

Figure 3. $B_{C_5,5}$ cycle books graph

$r(v_{c_1}\|\Psi) = (1,1,1,2)$;	$r(v_{2,2}\|\Psi) = (3,1,3,4)$;	$r(v_{4,2}\|\Psi) = (3,3,3,0)$;
$r(v_{c_2}\|\Psi) = (2,2,2,2)$;	$r(v_{2,3}\|\Psi) = (3,2,3,3)$;	$r(v_{4,3}\|\Psi) = (3,3,3,1)$;
$r(v_{1,1}\|\Psi) = (0,2,2,3)$;	$r(v_{3,1}\|\Psi) = (2,2,0,3)$;	$r(v_{5,1}\|\Psi) = (2,2,2,3)$;
$r(v_{1,2}\|\Psi) = (1,3,3,4)$;	$r(v_{3,2}\|\Psi) = (3,3,1,4)$;	$r(v_{5,2}\|\Psi) = (3,3,3,4)$;
$r(v_{1,3}\|\Psi) = (2,3,3,3)$;	$r(v_{3,3}\|\Psi) = (3,3,2,3)$;	$r(v_{5,3}\|\Psi) = (3,3,3,3)$;
$r(v_{2,1}\|\Psi) = (2,0,2,3)$;	$r(v_{4,1}\|\Psi) = (2,2,2,1)$.	

The set Ψ forms a resolving set for the graph $B_{C_5,5}$, as each vertex in $V(B_{C_5,5})$ possesses a unique representation $r(v|\Psi)$. Consequently, the metric dimension of $B_{C_5,5}$ is bounded above by $\dim(B_{C_5,5}) \leq 4$. To establish the lower bound, assume the existence of another resolving set Ψ' with cardinality less than that of Ψ, specifically $|\Psi'| = 3 < |\Psi|$. Under this assumption, there must be at least two distinct vertices in the set $V(B_{C_5,5}) - \Psi'$ whose representations relative to Ψ' coincide; notably, these vertices correspond to $v_{p,q}$ for $p = 1,2,3,4,5$ and $q = 1,2,3$. This contradicts the property of a resolving set, thereby implying Ψ' is not a valid resolving set. Hence, it follows that the metric dimension of $B_{C_5,5}$ is exactly $\dim(B_{C_5,5}) = 4$.

4. CONCLUSION

This paper investigates the metric dimension of cycle book graphs $B_{C_m,n}$, focusing on specific structural configurations where multiple cycles share a common path P_2. Through a series of detailed propositions and a general theorem, the following conclusions were obtained. Theorem 2 establishes that the metric dimension of $B_{C_3,n}$ is exactly n for all $n \geq 2$. Theorem 3 shows that the metric dimension of $B_{C_4,n}$ equals n for $n = 2,3,4$ and equals $n - 1$ for $n \geq 5$. Theorem 4 generalizes these findings for $B_{C_m,n}$ with $m \geq 5$, where the metric dimension satisfies: $\dim(B_{C_m,n}) = n$ if m is odd and $n \in \{2,3\}$, or m is even and $n \geq 2$. $\dim(B_{C_m,n}) = n - 1$ if m is odd and $n \geq 4$.

These results reveal that the metric dimension of cycle book graphs is highly dependent on both the number of cycles m and the size n of the constituent cycles, especially their parity. The study not only extends known results in the field but also provides a foundational understanding for further exploration of metric properties in more complex or generalized graph families. Future research may extend the current results by exploring the metric dimension of generalized cycle book graphs with varying common substructures, such as longer shared paths or intersecting cycles. Additionally, the application of metric dimension analysis to practical domains such as network verification, robotics navigation, or chemical graph theory, could open new avenues for interdisciplinary research.

Author Contributions

Jaya Santoso: Conceptualization, Formal Analysis, Methodology, Data Curation, Visualization, Writing–Original Draft, Validation. Darmaji: Supervision, Validation. Ana Muliyana: Data Curation, Resources, Draft Preparation. Asido

Saragih: Visualization, Writing–Review and Editing. All authors discussed the results and contributed to the final manuscript.

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Declarations

Competing interest. The authors declare no competing interest.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools (e.g., ChatGPT) were used solely for language refinement (grammar, spelling, and clarity). The scientific content, analysis, interpretation, and conclusions were developed entirely by the authors. The authors reviewed and approved all final text.

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