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NUMERICAL SOLUTIONS OF HERMITE DIFFERENTIAL EQUATIONS USING LEGENDRE MULTIWAVELETS

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ABSTRACT

This paper presents a numerical method for solving Hermite differential equations (HDEs) using operational integration matrices derived from Legendre multiwavelets of linear, quadratic, and cubic orders. The proposed technique transforms HDEs into algebraic systems, enabling efficient and accurate numerical solutions. Through several illustrative examples, the method's effectiveness is demonstrated, with Cubic Legendre Multiwavelets (CLMW) exhibiting superior accuracy in approximating exact solutions.

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1. INTRODUCTION

Wavelets have revolutionized numerical analysis and computational mathematics by providing a robust framework for approximating and solving complex equations with high accuracy and efficiency. Unlike traditional methods, wavelets allow the representation of functions with both time and frequency localization, making them particularly suitable for analyzing problems with varying levels of detail or abrupt changes. Wavelets are characterized by their compact support, orthogonality, and multiresolution properties, which enable efficient representation and manipulation of functions across different scales. Legendre multiwavelets, derived from Legendre polynomials, are widely regarded as one of the most effective types of multiwavelets due to their compact support, orthogonality, and ease of implementation [1]. By leveraging the polynomial structure of Legendre functions, these multiwavelets provide a powerful basis for approximating functions and solving differential equations with improved computational efficiency [2].

Khellat [3] developed the linear Legendre mother wavelets operational matrix of integration and its application. While Babolian [4] discussed the Numerical solution of differential equations by using the Chebyshev wavelet operational matrix of integration. Shiralashetti [5] explained the Hermite wavelets operational matrix of integration for the numerical solution of nonlinear singular initial value problems. Khan [6] formulated the Haar wavelet operational matrix method for pantograph fractional differential equations. Ravichandran [7] studied a Hermite-type radial basis function-based differential quadrature approach that allows for free vibration beams for higher-order equations. Alpert [8] explained a class of bases for the sparse representation of integral operators. Zhang [9] discussed a generalized collocation method in reproducing kernel space for solving a weakly singular Fredholm integro-differential equation. Du [10] developed the least residue method for solving a nonlinear fractional integro-differential equation with a weakly singular kernel. Later, Khan [11] developed a new method for finding numerical solutions to integro-differential equations based on Legendre multiwavelets collocation. Matoog [12] formulated a hybrid numerical technique for solving fractional Fredholm-Volterra integro-differential equations using Ramadan group integral transform and Hermite polynomials.

Jarczewska [13] introduced multiwavelets and multiwavelet packets of Legendre functions in the direct method for solving variational problems. Daneshkhah [14] presented an approximation multivariate distribution with pair copula using the orthonormal polynomial and Legendre multiwavelets basis functions. Further, Geronimo [15] obtained Alpert multiwavelets and Legendre-Angelesco multiple orthogonal polynomials. Paul [16] introduced Legendre multiwavelets to solve Carleman-type singular integral equations. Ashpazzadeh [17] formulated Hermite multiwavelets representation for the sparse solution of the nonlinear Abel’s integral equation.

Devi [18] developed numerical solutions of a system of linear differential equations using the Haar wavelet approach. Further, Devi [19] established numerical solutions of integral equations using linear Legendre multiwavelets. Singh [20] derived a solution of linear differential equations using the operational matrix of Bernoulli orthonormal polynomials. Furthermore, Kheirdeh [21] studied s-elementary wavelets in R and their applications in solving integral equations. While Sahani [22] analyzed the numerical solution of the non-linear Lienard equation using the Haar wavelet method.

Liu [23] discussed the bivariate Hermite polynomials. Kumbinarasaiah [24] discussed applications of the Hermite wavelet method to nonlinear differential equations arising in heat transfer. Faheem [25] studied a high-resolution Hermite wavelet technique for solving space-time-fractional partial differential equations. Further, Pourfattah [26] formulated an algorithm based on the Pseudospectral method for solving Abel’s integral equation using Hermite cubic spline scaling bases.

Furthermore, Ali [27] developed an adaptive algorithm for numerically solving fractional partial differential equations using Hermite wavelet artificial neural networks. While Wani [28] explained generalized 1-Parameter 3-Variable Hermite–Frobenius–Euler Polynomials. Santra [29] determined a simultaneous space–time Hermite wavelet method for time-fractional nonlinear weakly singular integro-partial differential equations.

The integration properties of CLMW are specifically derived and utilized to enhance the computational efficiency and accuracy of the solutions. This paper focuses on employing LLMW, QLMW, and CLMW to solve HDE of orders zero through five. It is observed that the approximate solution using CLMW closely matches the exact solution and outperforms the results from LLMW and QLMW. Hermite differential equations, which play a fundamental role in mathematical physics, quantum mechanics, and signal processing, are solved by transforming them into algebraic systems using the operational matrix of integration of LLMW, QLMW, and CLMW. Through illustrative examples and comparative analyses, this study demonstrates that the proposed multiwavelet-based approach not only provides highly accurate solutions but also reduces computational overhead, offering a versatile and efficient method for solving HDE.

2. RESEARCH METHODS

2.1 Hermite Differential Equations (HDE)

The differential equation is of the form

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0, \tag{1}$$

and is called HDE. The solution of Eq.(1) is known as the Hermite polynomial.

The Hermite polynomials, denoted by $H_n(x)$, is the set of orthogonal polynomials over the domain $(-\infty, \infty)$ with a weighting function e^{-x^2}. Hermite polynomials $H_n(x)$ are defined as

$$H_n(x) = \frac{n!}{2\pi i}\oint e^{-x^2+2xz}x^{-n-1}dx, \tag{2}$$

where the contour encloses the origin and is traversed in a counterclockwise direction.

2.2 Wavelet

A wavelet is a family of functions based on well well-localized oscillating function $\psi(x)$ of real variable x. The function $\psi(x)$ is called "Mother Wavelet" because all other wavelet functions within the family are generated from translation and dilation of $\psi(x)$. The following expression is the family of functions generated from $\psi(x)$ by translation and dilation

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right), \quad a,b \in R, \quad a \neq 0, \tag{3}$$

where, b is translation parameter and a is dilation parameter.

2.3 Linear Legendre Multiwavelets (LLMW)

The scaling functions for LLMW are defined as

$$\phi_0(t) = 1, \phi_1(t) = \sqrt{3}(2t-1), 0 \leq t < 1$$

Now, by the definition of multiresolution analysis (MRA), the corresponding mother wavelets $\psi^0(t)$ and $\psi^1(t)$ are given as

$$\psi^0(t) = \begin{cases} -\sqrt{3}(4t-1); & 0 \leq t < \frac{1}{2} \\ \sqrt{3}(4t-3); & \frac{1}{2} \leq t < 1 \end{cases}, \psi^1(t) = \begin{cases} (6t-1); & 0 \leq t < \frac{1}{2} \\ (6t-5); & \frac{1}{2} \leq t < 1 \end{cases}.$$

Further, by translation and dilation of the mother wavelets $\psi^0(t)$ and $\psi^1(t)$, the LLMW are constructed and are in the following form

$$\psi_{k,n}^j(t) = 2^{\frac{k}{2}}\psi^j(2^k t - n), \qquad k,n,j \in \mathbb{Z}.$$

For $k = 0,1$; $j = 0,1$ and $n = 0,1$,

$$\psi_{0,0}^0(t) = \begin{cases} -\sqrt{3}(4t-1); & 0 \leq t < \frac{1}{2} \\ \sqrt{3}(4t-3); & \frac{1}{2} \leq t < 1 \end{cases}, \psi_{0,0}^1(t) = \begin{cases} (6t-1); & 0 \leq t < \frac{1}{2} \\ (6t-5); & \frac{1}{2} \leq t < 1 \end{cases},$$

$$\psi_{1,0}^{0}(t)=\begin{cases}-\sqrt{6}(8t-1); & 0\le t<\frac{1}{4}\\ \sqrt{6}(8t-3); & \frac{1}{4}\le t<\frac{1}{2}\\ 0; & \frac{1}{2}\le t<1\end{cases},\ \psi_{1,0}^{1}(t)=\begin{cases}\sqrt{2}(12t-1); & 0\le t<\frac{1}{4}\\ \sqrt{2}(12t-5); & \frac{1}{4}\le t<\frac{1}{2}\\ 0; & \frac{1}{2}\le t<1\end{cases},$$

$$\psi_{1,1}^{0}(t)=\begin{cases}-\sqrt{6}(8t-5); & \frac{1}{2}\le t<\frac{3}{4}\\ \sqrt{6}(8t-7); & \frac{3}{4}\le t<1\\ 0; & 0\le t<\frac{1}{2}\end{cases},\ \psi_{1,1}^{1}(t)=\begin{cases}\sqrt{2}(12t-7); & \frac{1}{2}\le t<\frac{3}{4}\\ \sqrt{2}(12t-11); & \frac{3}{4}\le t<1.\\ 0; & 0\le t<\frac{1}{2}\end{cases}$$

The scaling functions and subfamily members $\{\phi_0(t),\phi_1(t),\psi_{0,0}^{0}(t),\psi_{0,0}^{1}(t),\psi_{1,0}^{0}(t),\psi_{1,0}^{1}(t),\psi_{1,1}^{0}(t),\psi_{1,1}^{1}(t)\}$ constitute the LLMW.

2.4 Quadratic Legendre Multiwavelets (QLMW)

The scaling functions for QLMW are defined

$$\phi_0(t)=1,\phi_1(t)=\sqrt{3}(2t-1),\phi_2(t)=\sqrt{5}(6t^2-6t+1),0\le t\le 1.$$

By the definition of MRA,

$$\psi^{0}(t)=\begin{cases}\frac{1}{3}(-7+72t-120t^2); & 0\le t<\frac{1}{2}\\ \frac{1}{3}(55-168t+120t^2); & \frac{1}{2}\le t<1\\ 0; & Otherwise\end{cases},$$

$$\psi^{1}(t)=\begin{cases}\sqrt{3}(1-14t+30t^2); & 0\le t<\frac{1}{2}\\ \sqrt{3}(17-46t+30t^2); & \frac{1}{2}\le t<1\\ 0; & Otherwise\end{cases},$$

$$\psi^{2}(t)=\begin{cases}-\frac{1}{3}\sqrt{5}(1-18t+48t^2); & 0\le t<\frac{1}{2}\\ \frac{1}{3}\sqrt{5}(31-78t+48t^2); & \frac{1}{2}\le t<1\\ 0; & Otherwise\end{cases}.$$

The QLMW is constructed by translating and dilating of mother wavelet $\psi(t)$ and is given by

$$\psi^{j}{}_{k,n}(t)=2^{\frac{k}{2}}\psi^{j}\left(2^{k}t-n\right),\quad k,n,j\in\mathbb{Z}$$

For $k=0,1,2$; $j=0,1$ and $n=0,1$,

$$\psi_{0,0}{}^{0}(t)=\begin{cases}\frac{1}{3}(-7+72t-120t^2); & 0\le t<\frac{1}{2}\\ \frac{1}{3}(55-168t+120t^2); & \frac{1}{2}\le t<1\\ 0; & Otherwise\end{cases},$$

$$\psi_{0,0}{}^{1}(t)=\begin{cases}\sqrt{3}(1-14t+30t^2); & 0\le t<\frac{1}{2}\\ \sqrt{3}(17-46t+30t^2); & \frac{1}{2}\le t<1\\ 0; & Otherwise\end{cases},$$

$$\psi_{0,0}{}^{2}(t)=\begin{cases}-\frac{1}{3}\sqrt{5}(1-18t+48t^2); & 0\le t<\frac{1}{2}\\ \frac{1}{3}\sqrt{5}(31-78t+48t^2); & \frac{1}{2}\le t<1\\ 0; & Otherwise\end{cases},$$

$$\psi_{1,0}{}^{0}(t)=\begin{cases}-\frac{1}{3}\sqrt{2}(7-144t+480t^2); & 0\le t<\frac{1}{4}\\ \frac{1}{3}\sqrt{2}(55-336t+480t^2); & \frac{1}{4}\le t<\frac{1}{2}\\ 0; & Otherwise\end{cases},$$

$$\psi_{1,0}{}^{1}(t)=\begin{cases}\sqrt{6}(1-28t+120t^2); & 0\le t<\frac{1}{4}\\ \sqrt{6}(17-92t+120t^2); & \frac{1}{4}\le t<\frac{1}{2}\\ 0; & Otherwise\end{cases},$$

$$\psi_{1,0}{}^{2}(t)=\begin{cases}-\frac{1}{3}\sqrt{10}(1-36t+192t^2); & 0\le t<\frac{1}{4}\\ \frac{1}{3}\sqrt{10}(31-156t+192t^2); & \frac{1}{4}\le t<\frac{1}{2}\\ 0; & Otherwise\end{cases},$$

$$\psi_{1,1}{}^{0}(t)=\begin{cases}-\frac{1}{3}\sqrt{2(7-72(-1+2t)+120(-1+2t)^2)}; & \frac{1}{2}\le t<\frac{3}{4}\\ \frac{1}{3}\sqrt{2(55-168(-1+2t)+120(-1+2t)^2)}; & \frac{3}{4}\le t<1\\ 0; & Otherwise\end{cases},$$

$$\psi_{1,1}{}^{1}(t)=\begin{cases}\sqrt{6(1-14(-1+2t)+30(-1+2t)^2)}; & \frac{1}{2}\le t<\frac{3}{4}\\ \sqrt{6(17-46(-1+2t)+30(-1+2t)^2)}; & \frac{3}{4}\le t<1\\ 0; & Otherwise\end{cases},$$

$$\psi_{1,1}{}^{2}(t)=\begin{cases}-\frac{1}{3}\sqrt{10(1-18(-1+2t)+48(-1+2t)^2)}; & \frac{1}{2}\le t<\frac{3}{4}\\ \frac{1}{3}\sqrt{10(31-78(-1+2t)+48(-1+2t)^2)}; & \frac{3}{4}\le t<1\\ 0; & Otherwise\end{cases}.$$

2.5 Cubic Legendre Multiwavelets (CLMW)

The scaling functions for CLMW are defined as

$\phi_0(t)=1, \phi_1(t)=\sqrt{3}(2t-1), \phi_2(t)=\sqrt{5}(6t^2-6t+1), \phi_3(t)=\sqrt{7}(-1+12t-30t^2+20t^3)$, $0\le t\le 1$.

By the definition of MRA,

$$\psi^{0}(t)=\begin{cases}-\sqrt{\frac{15}{17}}(-3+56t-216t^2+224t^3); & 0\le t<\frac{1}{2}\\ \sqrt{\frac{15}{17}}(-61+296t-456t^2+224t^3); & \frac{1}{2}\le t<1\\ 0; & Otherwise\end{cases},$$

$$\psi^{1}(t)=\begin{cases}\frac{-11+270t-1320t^2+1680t^3}{\sqrt{21}}; & 0\le t<\frac{1}{2}\\ \frac{-619+2670t-3720t^2+1680t^3}{\sqrt{21}}; & \frac{1}{2}\le t<1\\ 0; & Otherwise\end{cases},$$

$$\psi^{2}(t)=\begin{cases}-\sqrt{\frac{35}{17}}(-1+30t-174t^2+256t^3); & 0\le t<\frac{1}{2}\\ \sqrt{\frac{35}{17}}(-111+450t-594t^2+256t^3); & \frac{1}{2}\le t<1\\ 0; & Otherwise\end{cases},$$

$$\psi^3(t) = \begin{cases} \sqrt{\frac{5}{42}}(-1 + 36t - 246t^2 + 420t^3); & 0 \le t < \frac{1}{2} \\ \sqrt{\frac{5}{42}}(-209 + 804t - 1014t^2 + 420t^3); & \frac{1}{2} \le t < 1 \\ 0; & Otherwise \end{cases}.$$

The CLMW is constructed by translating and dilating of mother wavelet $\psi(t)$ and is given by

$$\psi^j{}_{k,n}(t) = 2^{\frac{k}{2}}\psi^j(2^k t - n), \quad k, n, j \in \mathbb{Z}$$

For $k = 0,1,2,3$; $j = 0,1$ and $n = 0,1$,

$$\psi^0{}_{0,0}(t) = \begin{cases} -\sqrt{\frac{15}{17}}(-3 + 56t - 216t^2 + 224t^3); & 0 \le t < \frac{1}{2} \\ \sqrt{\frac{15}{17}}(-61 + 296t - 456t^2 + 224t^3); & \frac{1}{2} \le t < 1 \\ 0; & Otherwise \end{cases},$$

$$\psi^1{}_{0,0}(t) = \begin{cases} \frac{-11 + 270t - 1320t^2 + 1680t^3}{\sqrt{21}}; & 0 \le t < \frac{1}{2} \\ \frac{-619 + 2670t - 3720t^2 + 1680t^3}{\sqrt{21}}; & \frac{1}{2} \le t < 1 \\ 0; & Otherwise \end{cases},$$

$$\psi^2{}_{0,0}(t) = \begin{cases} -\sqrt{\frac{35}{17}}(-1 + 30t - 174t^2 + 256t^3); & 0 \le t < \frac{1}{2} \\ \sqrt{\frac{35}{17}}(-111 + 450t - 594t^2 + 256t^3); & \frac{1}{2} \le t < 1 \\ 0; & Otherwise \end{cases},$$

$$\psi^3{}_{0,0}(t) = \begin{cases} \sqrt{\frac{5}{42}}(-1 + 36t - 246t^2 + 420t^3); & 0 \le t < \frac{1}{2} \\ \sqrt{\frac{5}{42}}(-209 + 804t - 1014t^2 + 420t^3); & \frac{1}{2} \le t < 1 \\ 0; & Otherwise \end{cases},$$

$$\psi^0{}_{1,0}(t) = \begin{cases} \sqrt{\frac{30}{17}}(3 - 112t + 864t^2 - 1792t^3); & 0 \le t < \frac{1}{4} \\ \sqrt{\frac{30}{17}}(-61 + 592t - 1824t^2 + 1792t^3); & \frac{1}{4} \le t < \frac{1}{2} \\ 0; & Otherwise \end{cases},$$

$$\psi^1{}_{1,0}(t) = \begin{cases} \sqrt{\frac{2}{21}}(-11 + 540t - 5280t^2 + 13440t^3); & 0 \le t < \frac{1}{4} \\ \sqrt{\frac{2}{21}}(-619 + 5340t - 14880t^2 + 13440t^3); & \frac{1}{4} \le t < \frac{1}{2} \\ 0; & Otherwise \end{cases},$$

$$\psi^2{}_{1,0}(t) = \begin{cases} \sqrt{\frac{70}{17}}(1 - 60t + 696t^2 - 2048t^3); & 0 \le t < \frac{1}{4} \\ \sqrt{\frac{70}{17}}(-111 + 900t - 2376t^2 + 2048t^3); & \frac{1}{4} \le t < \frac{1}{2} \\ 0; & Otherwise \end{cases},$$

$$\psi^3{}_{1,0}(t) = \begin{cases} \sqrt{\frac{5}{21}}(-1 + 72t - 984t^2 + 3360t^3); & 0 \le t < \frac{1}{4} \\ \sqrt{\frac{5}{21}}(-209 + 1608t - 4056t^2 + 3360t^3); & \frac{1}{4} \le t < \frac{1}{2} \\ 0; & Otherwise \end{cases},$$

$$\psi^0{}_{1,1}(t) = \begin{cases} \sqrt{\frac{30}{17}}(499 - 2320t + 3552t^2 - 1792t^3); & \frac{1}{2} \le t < \frac{3}{4} \\ \sqrt{\frac{30}{17}}(-1037 + 3760t - 4512t^2 + 1792t^3); & \frac{3}{4} \le t < 1 \\ 0; & Otherwise \end{cases},$$

$$\psi^1{}_{1,1}(t) = \begin{cases} \sqrt{\frac{2}{21}}(-3281 + 15900t - 25440t^2 + 13440t^3); & \frac{1}{2} \le t < \frac{3}{4} \\ \sqrt{\frac{2}{21}}(-8689 + 30300t - 35040t^2 + 13440t^3); & \frac{3}{4} \le t < 1 \\ 0; & Otherwise \end{cases},$$

$$\psi^2{}_{1,1}(t) = \begin{cases} \sqrt{\frac{70}{17}}(461 - 2292t + 3768t^2 - 2048t^3); & \frac{1}{2} \le t < \frac{3}{4} \\ \sqrt{\frac{70}{17}}(-1411 + 4812t - 5448t^2 + 2048t^3); & \frac{3}{4} \le t < 1 \\ 0; & Otherwise \end{cases},$$

$$\psi^3{}_{1,1}(t) = \begin{cases} \sqrt{\frac{5}{21}}(-703 + 3576t - 6024t^2 + 3360t^3); & \frac{1}{2} \le t < \frac{3}{4} \\ \sqrt{\frac{5}{21}}(-2447 + 8184t - 9096t^2 + 3360t^3); & \frac{3}{4} \le t < 1 \\ 0; & Otherwise \end{cases}.$$

2.6 Function Approximations

The square integrable function $f(t)$ defined on [0, 1] can be expanded in terms of LLMW, QLMW, and CLMW, respectively, as follows:

$$f(t) = \sum_{i=0}^{1} c_i\phi_i(t) + \sum_{k=0}^{\infty}\sum_{j=0}^{1}\sum_{n=0}^{2^{k-1}} c_{k,n}^{j}\psi_{k,n}^{j}(t), \tag{4}$$

$$f(t) = \sum_{i=0}^{2} c_i\phi_i(t) + \sum_{k=0}^{\infty}\sum_{j=0}^{2}\sum_{n=0}^{2^{k-1}} c_{k,n}^{j}\psi_{k,n}^{j}(t), \tag{5}$$

$$f(t) = \sum_{i=0}^{3} c_i\phi_i(t) + \sum_{k=0}^{\infty}\sum_{j=0}^{3}\sum_{n=0}^{2^{k-1}} c_{k,n}^{j}\psi_{k,n}^{j}(t), \tag{6}$$

where, $c_i = \langle f(t), \phi_i(t)\rangle$ and $c_{k,n}^{j} = \left\langle f(t), \psi_{k,n}^{j}(t)\right\rangle$.

After truncating the infinite series, Eqs. (4), (5), and (6) take the following form:

$$f(t)=\sum_{i=0}^{1} c_i\phi_i(t)+\sum_{k=0}^{M}\sum_{j=0}^{1}\sum_{n=0}^{2^{k-1}} c_{k,n}^{j}\psi_{k,n}^{j}(t)=C_1{}^{T}\Psi_1(t), \tag{7}$$

$$f(t)=\sum_{i=0}^{2} c_i\phi_i(t)+\sum_{k=0}^{M}\sum_{j=0}^{2}\sum_{n=0}^{2^{k-1}} c_{k,n}^{j}\psi_{k,n}^{j}(t)=C_2{}^{T}\Psi_2(t), \tag{8}$$

$$f(t)=\sum_{i=0}^{3} c_i\phi_i(t)+\sum_{k=0}^{M}\sum_{j=0}^{3}\sum_{n=0}^{2^{k-1}} c_{k,n}^{j}\psi_{k,n}^{j}(t)=C_3{}^{T}\Psi_3(t), \tag{9}$$

where,

$$C_1=\left[c_0,c_1,c_{0,0}^{0},c_{0,0}^{1},\dots,c_{M,0}^{0},c_{M,1}^{0},\dots,c_{M,(2^{M-1})}^{0},c_{M,0}^{1},c_{M,1}^{1},\dots,c_{M,(2^{M-1})}^{1}\right], \tag{10}$$

$$C_2=\begin{bmatrix}c_0,c_1,c_2,c_{0,0}^{0},c_{0,0}^{1},\dots,c_{M,0}^{0},c_{M,1}^{0},\dots,c_{M,(2^{M-1})}^{0},c_{M,0}^{1},c_{M,1}^{1},\dots,c_{M,(2^{M-1})}^{1},\\ c_{M,0}^{2},c_{M,1}^{2},\dots,c_{M,(2^{M-1})}^{2}\end{bmatrix}, \tag{11}$$

$$C_3=\begin{bmatrix}c_0,c_1,c_2,c_3,c_{0,0}^{0},c_{0,0}^{1},\dots,c_{M,0}^{0},c_{M,1}^{0},\dots,c_{M,(2^{M-1})}^{0},c_{M,0}^{1},c_{M,1}^{1},\dots,c_{M,(2^{M-1})}^{1},\\ c_{M,0}^{2},c_{M,1}^{2},\dots,c_{M,(2^{M-1})}^{2},c_{M,0}^{3},c_{M,1}^{3},\dots,c_{M,(2^{M-1})}^{3}\end{bmatrix}, \tag{12}$$

and

$$\psi_1(t)=\begin{bmatrix}\phi_0(t),\phi_1(t),\psi_{0,0}^{0}(t),\psi_{0,0}^{1}(t),\dots,\psi_{M,0}^{0}(t),\psi_{M,1}^{0}(t),\dots,\psi_{M,(2^{M-1})}^{0}(t),\\ \psi_{M,0}^{1}(t),\psi_{M,1}^{1}(t),\dots,\psi_{M,(2^{M-1})}^{1}(t)\end{bmatrix}, \tag{13}$$

$$\psi_2(t)=\begin{bmatrix}\phi_0(t),\phi_1(t),\phi_2(t),\psi_{0,0}^{0}(t),\psi_{0,0}^{1}(t),\dots,\psi_{M,0}^{0}(t),\psi_{M,1}^{0}(t),\dots,\psi_{M,(2^{M-1})}^{0}(t),\\ \psi_{M,0}^{1}(t),\psi_{M,1}^{1}(t),\dots,\psi_{M,(2^{M-1})}^{1}(t),\psi_{M,0}^{2}(t),\psi_{M,1}^{2}(t),\dots,\psi_{M,(2^{M-1})}^{2}(t)\end{bmatrix}, \tag{14}$$

$$\psi_3(t)=\begin{bmatrix}\phi_0(t),\phi_1(t),\phi_2(t),\psi_{0,0}^{0}(t),\psi_{0,0}^{1}(t),\dots,\psi_{M,0}^{0}(t),\psi_{M,1}^{0}(t),\dots,\psi_{M,(2^{M-1})}^{0}(t),\\ \psi_{M,0}^{1}(t),\psi_{M,1}^{1}(t),\dots,\psi_{M,(2^{M-1})}^{1}(t),\psi_{M,0}^{2}(t),\psi_{M,1}^{2}(t),\dots,\psi_{M,(2^{M-1})}^{2}(t),\\ \psi_{M,0}^{3}(t),\psi_{M,1}^{3}(t),\dots,\psi_{M,(2^{M-1})}^{3}(t)\end{bmatrix}. \tag{15}$$

Theorem 1. [8] *Let the function* $f:[0,1]\to\mathbb{R}$ *and* $f\in C^3[0,1]$ *then* $G^T\psi$ *approximates f with mean error bound as follows:*

$$\| f-G^T\psi \| \le \frac{1}{3!2^{3k}} \sup_{x\in[0,1]} |f'''(x)|, \tag{16}$$

where, $\|.\|$ *denotes the norm in* $L_2(\mathbb{R})$ *space.*

Proof. We see that the factor $\frac{1}{3!2^{3k}}$ shows the best approximation of the function because if k increases, then the error decreases rapidly. ■

2.7 Operational Matrix of Integration of LLMW, QLMW, and CLMW

The integration of LLMW can be obtained as

$$\int_0^t \psi(t)dt = P\ \psi(t), \tag{17}$$

where, P is an operational matrix of integration of order (8×8) [3], i.e.,

$$P=\begin{bmatrix}
\frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac{1}{2\sqrt{3}} & 0 & \frac{1}{8} & 0 & \frac{1}{32\sqrt{2}} & \frac{1}{32\sqrt{2}} & 0 & 0\\
0 & -\frac{1}{8} & 0 & \frac{1}{8\sqrt{3}} & -\frac{1}{16\sqrt{2}} & \frac{1}{16\sqrt{2}} & 0 & 0\\
0 & 0 & -\frac{1}{8\sqrt{3}} & 0 & \frac{\sqrt{\frac{3}{2}}}{32} & \frac{\sqrt{\frac{3}{2}}}{32} & 0 & 0\\
0 & -\frac{1}{32\sqrt{2}} & \frac{1}{16\sqrt{2}} & -\frac{1}{32\sqrt{2}} & 0 & 0 & \frac{1}{16\sqrt{3}} & 0\\
0 & -\frac{1}{32\sqrt{2}} & -\frac{1}{16\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{32} & 0 & 0 & 0 & \frac{1}{16\sqrt{3}}\\
0 & 0 & 0 & 0 & -\frac{1}{16\sqrt{3}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{3}} & 0 & 0
\end{bmatrix}. \tag{18}$$

The integration of QLMW can be obtained as

$$\int_0^t \psi(t)dt = Q\ \psi(t), \tag{19}$$

where, Q is an operational matrix of integration of order (12×12) [3].

$$Q=\begin{bmatrix}
\frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{15}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{1}{2\sqrt{15}} & 0 & \frac{3}{16\sqrt{5}} & 0 & 0 & \frac{3}{128\sqrt{10}} & \frac{3}{128\sqrt{10}} & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{3}{16\sqrt{5}} & 0 & \frac{5}{48\sqrt{3}} & 0 & -\frac{1}{32\sqrt{2}} & \frac{1}{32\sqrt{2}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\frac{5}{48\sqrt{3}} & 0 & \frac{1}{6\sqrt{15}} & \frac{3\sqrt{\frac{3}{2}}}{128} & \frac{3\sqrt{\frac{3}{2}}}{128} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\frac{1}{6\sqrt{15}} & 0 & -\frac{1}{16\sqrt{10}} & \frac{1}{16\sqrt{10}} & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{3}{128\sqrt{10}} & \frac{1}{32\sqrt{2}} & -\frac{3\sqrt{\frac{3}{2}}}{128} & \frac{1}{16\sqrt{10}} & 0 & 0 & \frac{5}{96\sqrt{3}} & 0 & 0 & 0\\
0 & 0 & -\frac{3}{128\sqrt{10}} & -\frac{1}{32\sqrt{2}} & -\frac{3\sqrt{\frac{3}{2}}}{128} & -\frac{1}{16\sqrt{10}} & 0 & 0 & 0 & \frac{5}{96\sqrt{3}} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{96\sqrt{3}} & 0 & 0 & 0 & \frac{1}{12\sqrt{15}} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{96\sqrt{3}} & 0 & 0 & 0 & \frac{1}{12\sqrt{15}}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{12\sqrt{15}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{12\sqrt{15}} & 0 & 0
\end{bmatrix}. \tag{20}$$

The integration of CLMW can be obtained as

$$\int_0^t \psi(t)dt = C\ \psi(t), \tag{21}$$

where, C is an operational matrix of integration of order (16×16).

$$C = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{15}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2\sqrt{15}} & 0 & \frac{1}{2\sqrt{35}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2\sqrt{35}} & 0 & \frac{\sqrt{\frac{85}{21}}}{32} & 0 & 0 & 0 & \frac{\sqrt{\frac{85}{42}}}{512} & \frac{\sqrt{\frac{85}{42}}}{512} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\sqrt{\frac{85}{21}}}{32} & 0 & \frac{29}{24\sqrt{595}} & 0 & \frac{1}{96\sqrt{238}} & -\frac{1}{64\sqrt{2}} & \frac{1}{64\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{29}{24\sqrt{595}} & 0 & \frac{3\sqrt{\frac{3}{85}}}{14} & 0 & \frac{\sqrt{\frac{85}{14}}}{128} & \frac{\sqrt{\frac{85}{14}}}{128} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{3\sqrt{\frac{3}{85}}}{14} & 0 & \frac{\sqrt{\frac{3}{34}}}{14} & -\frac{1}{8\sqrt{42}} & \frac{1}{8\sqrt{42}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{96\sqrt{238}} & 0 & -\frac{\sqrt{\frac{3}{34}}}{14} & 0 & \frac{5\sqrt{\frac{17}{7}}}{1024} & \frac{5\sqrt{\frac{17}{7}}}{1024} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\sqrt{\frac{85}{42}}}{512} & \frac{1}{64\sqrt{2}} & -\frac{\sqrt{\frac{85}{14}}}{128} & \frac{1}{8\sqrt{42}} & -\frac{5\sqrt{\frac{17}{7}}}{1024} & 0 & 0 & \frac{29}{48\sqrt{595}} & 0 & 0 & 0 & \frac{1}{192\sqrt{238}} & 0 \\
0 & 0 & 0 & -\frac{\sqrt{\frac{85}{42}}}{512} & -\frac{1}{64\sqrt{2}} & -\frac{\sqrt{\frac{85}{14}}}{128} & -\frac{1}{8\sqrt{42}} & -\frac{5\sqrt{\frac{17}{7}}}{1024} & 0 & 0 & 0 & \frac{29}{48\sqrt{595}} & 0 & 0 & 0 & \frac{1}{192\sqrt{238}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{29}{48\sqrt{595}} & 0 & 0 & 0 & \frac{3\sqrt{\frac{3}{85}}}{28} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{29}{48\sqrt{595}} & 0 & 0 & 0 & \frac{3\sqrt{\frac{3}{85}}}{28} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3\sqrt{\frac{3}{85}}}{28} & 0 & 0 & 0 & \frac{\sqrt{\frac{3}{34}}}{28} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3\sqrt{\frac{3}{85}}}{28} & 0 & 0 & 0 & \frac{\sqrt{\frac{3}{34}}}{28} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{192\sqrt{238}} & 0 & 0 & 0 & -\frac{\sqrt{\frac{3}{34}}}{28} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{192\sqrt{238}} & 0 & 0 & 0 & -\frac{\sqrt{\frac{3}{34}}}{28} & 0 & 0
\end{bmatrix}. \tag{22}$$

2.8 Advantages of CLMW over LLMW and QLMW

CLMW offers several notable advantages over LLMW and QLMW counterparts, making them highly effective for solving complex differential equations. One of the primary strengths of CLMW lies in its higher approximation accuracy, which stems from its third-degree polynomial basis. This enhanced degree allows the wavelets to better capture intricate and smoothly varying features of solutions, outperforming the lower-order wavelets in precision. Furthermore, CLMW exhibits superior convergence behaviour, enabling faster attainment of the desired accuracy with fewer basis functions. This results in more efficient computations, especially beneficial for higher-order or stiff differential equations.

In addition to accuracy and efficiency, CLMW provides improved numerical stability. Their higher smoothness and stronger orthogonality minimize numerical oscillations and instabilities, particularly in domains with steep gradients or when iterative solvers are involved. Most notably, the cubic framework of CLMW enhances its capacity to model complex solution behaviours, such as inflection points, sharp transitions, and boundary layer phenomena. Such capabilities make CLMW exceptionally well-suited for tackling higher-order Hermite differential equations, where LLMW and QLMW often fall short in capturing critical solution features.

3. RESULTS AND DISCUSSION

3.1 Method of Solution of HDE

Consider the following HDE of n-th order

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0, \quad with \quad y(0) = a \quad \text{and} \quad y'(0) = b, \tag{23}$$

where, $n = 0,1,2, \ldots$ and $y(x)$ is an unknown function.

First approximate a, b, $\frac{d^2y}{dx^2}, \frac{dy}{dx}$, $y(x)$ and $2x$, with the help of **Subsection 2.6.**

3.1.1 For LLMW

Consider

$$a = A_1^T \psi_1(x), b = B_1^T \psi_1(x), 2x = X_1^T \psi_1(x), 2n = N_1^T \psi_1(x). \tag{24}$$

Now

$$\frac{d^2y}{dx^2} \approx Y^T \psi_1(x), \tag{25}$$

and

$$\frac{dy}{dx} = \int_0^x \frac{d^2y}{dx^2} \, dx + \; y'(0) \approx Y^T \; P\psi_1(x) + B_1^T \psi_1(x), \tag{26}$$

$$y(x) = \int_0^x \frac{dy}{dx} \, dx + \; y(0) \approx Y^T P^2 \psi_1(x) + B_1^T \; P\psi_1(x) + A_1^T \psi_1(x). \tag{27}$$

Substituting Eqs. (24)-(27) into Eq. (23) yields

$$\psi_1^T(x)Y - X_1^T\psi_1(x)\left[\psi_1^T(x)P^TY + \psi_1^T(x)B_1\right] + N_1^T\psi_1(x)\left[\psi_1^T(x)P^{2T}Y + \psi_1^T(x)P^TB_1 + \psi_1^T(x)A_1\right] = 0 \tag{28}$$

Applying the property of matrix multiplication [4]

$$c_i^T \psi_i(x)\psi_i^T(x) \approx \psi_i^T(x)\widetilde{c_i}, \quad \text{i.e.,} \qquad c_1^T \psi_1(x)\psi_1^T(x) \approx \psi_1^T(x)\widetilde{c_1}. \tag{29}$$

Substituting Eq. (29) into Eq. (28) gives

$$Y - \left[\widetilde{X_1}P^TY + \widetilde{X_1}B_1\right] + \left[\widetilde{N_1}P^{2T}Y + \widetilde{N_1}P^TB_1 + \widetilde{N_1}A_1\right] = 0, \tag{30}$$

$$\left[I - \widetilde{X_1}P^T + \widetilde{N_1}P^{2T}\right]Y = \widetilde{X_1}B_1 - \widetilde{N_1}P^TB_1 - \widetilde{N_1}A_1. \tag{31}$$

After solving the system of algebraic equations above, i.e., Eq. (31), we obtain Y. Putting the value of Y in Eq. (27), we can find an approximate solution of the HDE of n-th order.

Algorithm 1. LLMW Algorithm

Input: i, j, n, k, M; $i = 0,1$; $j = 0,1$; $n = 0,1,2, \ldots, 2^{k-1}, k = 0,1,2, \ldots, M$; $M = 0,1,2, \ldots$.
Output: $y(t) \approx Y^T P^2 \psi_1(t) + B_1^T P \psi_1(t) + A_1^T \psi_1(t)$.

1: Computei, j, n, k, M
2: Define LLMW $\psi_1(t)$ from Eq. (13).
3: Introduce the unknown vector C_1 from Eq. (10).
4: Compute the operational matrix of integration of LLMW "P" from Eq. (17).
5: Approximate the terms $a, b, 2x$ and $2n$ from Eq. (24) in term of LLMW by using Eq. (4).
6: Convert Eq. (23) into approximation form by using Steps 1-5.
7: Extract the set of algebraic equations as in Eq. (27).
8: Solve the system of equations obtained in Step 7.
9: Evaluate Y.
10: Increase k and n for a better approximation of $y(t)$.
11: Calculate $y(t) \approx Y^T P^2 \psi_1(t) + B_1^T P \psi_1(t) + A_1^T \psi_1(t)$.

3.1.2 For QLMW

Consider

$$a = A_2^T \psi_2(x), \quad b = B_2^T \psi_2(x), 2x = X_2^T \psi_2(x), 2n = N_2^T \psi_2(x). \tag{32}$$

Now

$$\frac{d^2y}{dx^2} \approx Y^T \psi_2(x), \tag{33}$$

and

$$\frac{dy}{dx} = \int_0^x \frac{d^2y}{dx^2} \, dx + \; y'(0) \approx Y^T \; Q\psi_2(x) + B_2^T \psi_2(x), \tag{34}$$

$$y(x) = \int_0^x \frac{dy}{dx} \, dx + \; y(0) \approx Y^T Q^2 \psi_2(x) + B_2^T \; Q\psi_2(x) + A_2^T \psi_2(x). \tag{35}$$

Substituting Eqs. (32)-(35) into Eq. (23) yields

$$\psi_2^T(x)Y - X_2^T\psi_2(x)\left[\psi_2^T(x)Q^TY + \psi_2^T(x)B_2\right] + N_2^T\psi_2(x)\left[\psi_2^T(x)Q^{2T}Y + \psi_2^T(x)Q^TB_2 + \psi_2^T(x)A_2\right] = 0 \quad (36)$$

Applying the property of matrix multiplication [4]

$$c_2^T\psi_2(x)\psi_2^T(x) \approx \psi_2^T(x)\widetilde{c_2}. \quad (37)$$

Substituting Eq. (37) into Eq. (36) gives

$$Y - \left[\widetilde{X_2}P^TY + \widetilde{X_2}B_2\right] + \left[\widetilde{N_2}P^{2T}Y + \widetilde{N_2}P^TB_2 + \widetilde{N_2}A_2\right] = 0, \quad (38)$$

$$\left[I - \widetilde{X_2}P^T + \widetilde{N_2}P^{2T}\right]Y = \widetilde{X_2}B_2 - \widetilde{N_2}P^TB_2 - \widetilde{N_2}A_2. \quad (39)$$

After solving the system of algebraic equations above, i.e., Eq. (39), we obtain Y. Putting the value of Y in Eq. (35), we can find an approximate solution of the HDE of n-th order.

Algorithm 2. QLMW Algorithm

Input: i, j, n, k, M; $i = 0,1$; $j = 0,1$; $n = 0,1,2,\ldots,2^{k-1}$, $k = 0,1,2,\ldots,M$; $M = 0,1,2,\ldots$.
Output: $y(t) \approx Y^TQ^2\psi_2(t) + B_2^TQ\psi_2(t) + A_2^T\psi_2(t)$.

1: Computei, j, n, k, M
2: Define QLMW $\psi_2(t)$ from Eq. (14).
3: Introduce the unknown vector C_2 from Eq. (11).
4: Compute the operational matrix of integration of QLMW "Q" from Eq. (19).
5: Approximate the terms $a, b, 2x$ and $2n$ from Eq. (32) in term of QLMW by using Eq. (5).
6: Convert Eq. (23) into approximation form by using Steps 1-5.
7: Extract the set of algebraic equations as in Eq. (35).
8: Solve the system of equations obtained in Step 7.
9: Evaluate Y.
10: Increase k and n for a better approximation of $y(t)$.
11: Calculate $y(t) \approx Y^TQ^2\psi_2(t) + B_2^TQ\psi_2(t) + A_2^T\psi_2(t)$.

3.1.3 For CLMW

Consider

$$a = A_3^T\psi_3(x), \quad b = B_3^T\psi_3(x), 2x = X_3^T\psi_3(x), 2n = N_3^T\psi_3(x). \quad (40)$$

Now

$$\frac{d^2y}{dx^2} \approx Y^T\psi_3(x), \quad (41)$$

and

$$\frac{dy}{dx} = \int_0^x \frac{d^2y}{dx^2}\,dx + y'(0) \approx Y^T C\psi_3(x) + B_3^T\psi_3(x), \quad (42)$$

$$y(x) = \int_0^x \frac{dy}{dx}\,dx + y(0) \approx Y^TC^2\psi_3(x) + B_3^T C\psi_3(x) + A_3^T\psi_3(x). \quad (43)$$

Substituting Eqs. (40)-(43) into Eq. (23) yields

$$\psi_3^T(x)Y - X_3^T\psi_3(x)\left[\psi_3^T(x)C^TY + \psi_3^T(x)B_3\right] + N_3^T\psi_3(x)\left[\psi_3^T(x)C^{2T}Y + \psi_3^T(x)C^TB_3 + \psi_3^T(x)A_3\right] = 0 \quad (44)$$

Applying the property of matrix multiplication [4]

$$c_3^T\psi_3(x)\psi_3^T(x) \approx \psi_3^T(x)\widetilde{c_3}. \quad (45)$$

Substituting Eq. (45) into Eq. (44) gives

$$Y - \left[\widetilde{X_3}C^TY + \widetilde{X_3}B_3\right] + \left[\widetilde{N_3}C^{2T}Y + \widetilde{N_3}P^TB_3 + \widetilde{N_3}A_3\right] = 0, \quad (46)$$

$$\left[I - \widetilde{X_3}C^T + \widetilde{N_3}P^{2T}\right]Y = \widetilde{X_3}B_3 - \widetilde{N_3}C^TB_3 - \widetilde{N_3}A_3. \quad (47)$$

After solving the system of algebraic equations above, i.e., Eq. (47), we obtain Y. Putting the value of Y in Eq. (43), we can find an approximate solution of the HDE of n-th order.

Algorithm 3. CLMW Algorithm

Input: $i, j, n, k, M;\ i = 0,1;\ j = 0,1;\ n = 0,1,2, \ldots, 2^{k-1}, k = 0,1,2, \ldots, M;\ M = 0,1,2, \ldots$
Output: $y(t) \approx Y^T C^2 \psi_3(t) + B_3^T C \psi_3(t) + A_3^T \psi_3(t)$.

1: Compute i, j, n, k, M
2: Define CLMW $\psi_3(t)$ from Eq. (15).
3: Introduce the unknown vector C_3 from Eq. (12).
4: Compute the operational matrix of integration of QLMW "C" Eq. (21).
5: Approximate the terms $a, b, 2x$ and $2n$ from equation (40) in term of CLMW by using Eq. (6).
6: Convert Eq. (23) into approximation form by using Steps 1-5.
7: Extract the set of algebraic equations as in Eq. (43).
8: Solve the system of equations obtained in Step 7.
9: Evaluate Y.
10: Increase k and n for a better approximation of $y(t)$.
11: Calculate $y(t) \approx Y^T C^2 \psi_3(t) + B_3^T C \psi_3(t) + A_3^T \psi_3(t)$.

Example 1. Consider the HDE for $n = 0$, i.e., order zero

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = 0, \quad y(0) = 1, \quad \text{and} \quad y'(0) = 0. \tag{48}$$

The exact solution is

$$y(x) = 1. \tag{49}$$

Employing LLMW, QLMW, and CLMW bases along with their operational matrices, the approximate solution is given as

$$\tilde{y}(x) = \begin{cases} 1; & 0 \le x < 1 \\ 0; & Otherwise \end{cases}. \tag{50}$$

It is observed that for the problem in this example, the approximate solution is the same as the exact solution.

Example 2. Consider the HDE for $n = 1$, i.e., order one

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0, \quad y(0) = 0, \quad \text{and} \quad y'(0) = 2. \tag{51}$$

The exact solution is

$$y(x) = 2x. \tag{52}$$

Employing LLMW, QLMW, and CLMW bases along with their operational matrices, the approximate solution is given as

$$\tilde{y}(x) = \begin{cases} 2x; & 0 \le x < 1 \\ 0; & Otherwise \end{cases}. \tag{53}$$

Again, it is observed that for the problem in this example, the approximate solution is the same as the exact solution.

Example 3. Consider the HDE for $n = 2$, i.e., order two

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 4y = 0, \quad y(0) = -2, \quad \text{and} \quad y'(0) = 0. \tag{54}$$

The exact solution is

$$y(x) = 4x^2 - 2. \tag{55}$$

Using LLMW, QLMW, and CLMW bases along with their corresponding operational matrices, the approximate solutions are expressed as follows:

$$\tilde{y_1}(x) = \begin{cases} -\frac{49}{24} + x; & 0 \leq x < \frac{1}{4} \\ -\frac{61}{24} + 3x; & \frac{1}{4} \leq x < \frac{1}{2} \\ -\frac{85}{24} + 5x; & \frac{1}{2} \leq x < \frac{3}{4} \\ -\frac{121}{24} + 7x; & \frac{3}{4} \leq x < 1 \\ 0; & Otherwise \end{cases},$$

$$\tilde{y_2}(x) = \begin{cases} -2 + 4x^2; & 0 \leq x < 1 \\ 0; & Otherwise \end{cases},$$

$$\tilde{y_3}(x) = \begin{cases} -2 + 4x^2; & 0 \leq x < 1 \\ 0; & Otherwise \end{cases}.$$

The exact and approximate solutions of Eq. (54) are illustrated in Fig. 1. Table 1 presents a comparison of the exact and approximate solutions obtained by LLMW, QLMW, and CLMW. Notably, the approximate solution for CLMW matches the exact solution and demonstrates superior accuracy compared to LLMW and QLMW.

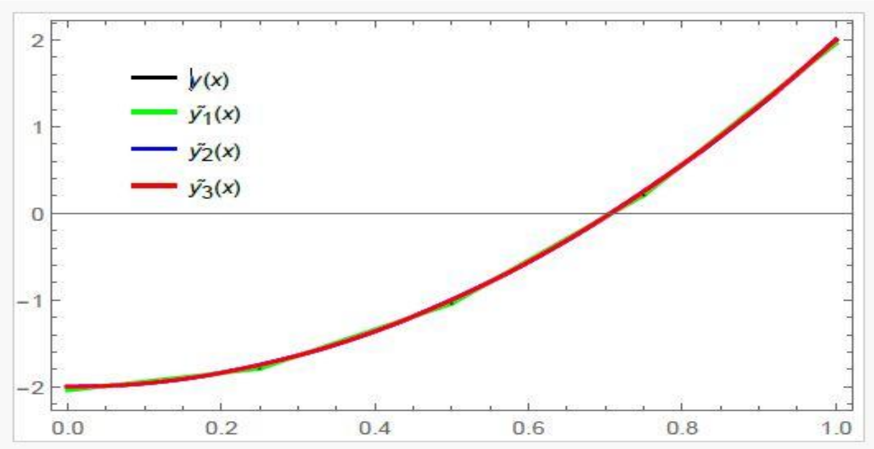


Figure 1. The Graph of Exact and Approximate Solutions for LLMW, QLMW, and CLMW

Table 1. Comparison of Exact and Approximate Solutions for LLMW, QLMW, and CLMW

x	$Exact: y(x)$	$LLMW: \tilde{y}_1(x)$	$QLMW: \tilde{y}_2(x)$	$CLMW: \tilde{y}_3(x)$
0.1	-1.96	-1.94167	-1.96	-1.96
0.2	-1.84	-1.84167	-1.84	-1.84
0.3	-1.64	-1.64167	-1.64	-1.64
0.4	-1.36	-1.34167	-1.36	-1.36
0.5	-1	-1.04167	-1	-1
0.6	-0.56	-0.541667	-0.56	-0.56
0.7	-0.04	-0.041666	-0.04	-0.04
0.8	0.56	0.558333	0.56	0.56
0.9	1.24	1.25833	1.24	1.24

Example 4. Consider the HDE for $n = 3$, i.e., order three

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y = 0, \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = -12. \tag{56}$$

The exact solution is

$$y(x) = 8x^3 - 12x. \tag{57}$$

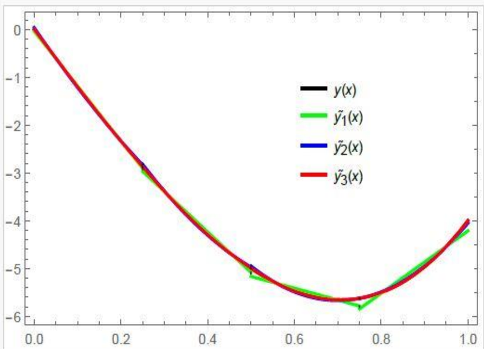
Using LLMW, QLMW, and CLMW bases along with their corresponding operational matrices, the approximate solutions are expressed as follows:

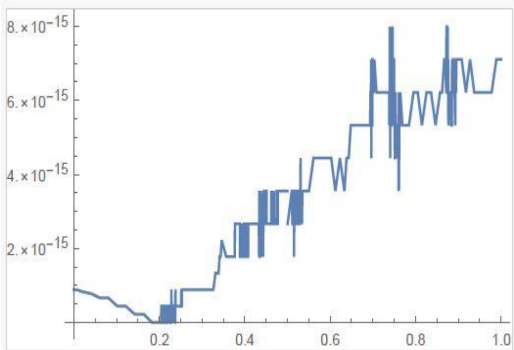
$$\tilde{y_1}(x) = \begin{cases} -3.9290533083765706 - 2.472998103453275t; & 0 \leq t < \frac{1}{4} \\ -0.8472184254306346 - 8.471019823081729t; & \frac{1}{4} \leq t < \frac{1}{2} \\ -0.03621027477645411 - 11.46611079696396t; & \frac{1}{2} \leq t < \frac{3}{4} \\ -10.74349981063365 + 6.530324978402067t; & \frac{3}{4} \leq t < 1 \\ 0; & Otherwise \end{cases}, \tag{58}$$

$$\tilde{y_2}(x) = \begin{cases} 0.03937440702747928 - 13.223872590567915t + 6.95174072943174t^2; & 0 \leq t < \frac{1}{4} \\ 0.9528742641745869 - 18.32816276654199t + 12.952885080321302t^2; & \frac{1}{4} \leq t < \frac{1}{2} \\ 3.1726952665293453 - 25.2655869478412t + 18.046996292760326t^2; & \frac{1}{2} \leq t < \frac{3}{4} \\ 3.133816642737621 - 25.163403688621727t + 17.97888708834267t^2; & \frac{3}{4} \leq t < 1 \\ 0; & Otherwise \end{cases}, \tag{59}$$

$$\tilde{y_3}(x) = \begin{cases} -8.881784197001252 \times 10^{-16} - 11.999999999999998t + 1.953992523340275 \times 10^{-14}t^2 + 7.999999999999988t^3; & 0 \leq t < \frac{1}{4} \\ -8.553925320480403 \times 10^{-16} - 11.999999999999998t + 2.05489288103852 \times 10^{-14}t^2 + 7.999999999999986t^3; & \frac{1}{4} \leq t < \frac{1}{2} \\ -6.472666656430343 \times 10^{-16} - 12.t + 2.379991685974288 \times 10^{-14}t^2 + 7.999999999999984t^3; & \frac{1}{2} \leq t < \frac{3}{4} \\ 3.432026936776186 \times 10^{-16} - 12.000000000000004t + 2.783683677425238 \times 10^{-14}t^2 + 7.999999999999982t^3; & \frac{3}{4} \leq t < 1 \\ 0; & Otherwise \end{cases}. \tag{60}$$

The exact and approximate solutions of Eq. (56) are illustrated in Fig. 2 (a), while Fig. 2 (b) shows the absolute error of the exact and approximate solutions obtained by CLMW. Table 2 provides a comparison of the exact and approximate solutions obtained by LLMW, QLMW, and CLMW. The results indicate that CLMW achieves superior numerical accuracy compared to LLMW and QLMW, with its approximate solution matching the exact solution.





(a) **(b)**

Figure 2. The Graphs of Exact and Approximate Solutions and Absolute Error
(a) The Graph of Exact and Approximate Solutions for LLMW, QLMW, and CLMW, (b) The Graph of the Absolute Error of Exact and Approximate Solutions Obtained by CLMW

Table 2. Comparison of Exact and Approximate Solutions for LLMW, QLMW, and CLMW

x	$Exact: y(x)$	$LLMW: \tilde{y}_1(x)$	$QLMW: \tilde{y}_2(x)$	$CLMW: \tilde{y}_3(x)$
0.1	-1.192	-1.18282	-1.21135	-1.192
0.2	-2.336	-2.32942	-2.32733	-2.336
0.3	-3.384	-3.38852	-3.37981	-3.384
0.4	-4.288	-4.23463	-4.30593	-4.288
0.5	-5	-5.16555	–4.94835	-5
0.6	-5.472	-5.41285	-5.48974	–5.472
0.7	-5.656	-5.66015	–5.67019	-5.656
0.8	-5.504	-5.51924	-5.49042	-5.504

x	$Exact: y(x)$	$LLMW: \tilde{y}_1(x)$	$QLMW: \tilde{y}_2(x)$	$CLMW: \tilde{y}_3(x)$
0.9	-4.968	-4.86621	-4.95035	-4.968

Example 5. Consider the HDE for $n = 4$, i.e., order four

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 8y = 0, \quad \text{with} \quad y(0) = 12 \quad \text{and} \quad y'(0) = 0. \tag{61}$$

The exact solution is

$$y(x) = 16x^4 - 48x^2 + 12. \tag{62}$$

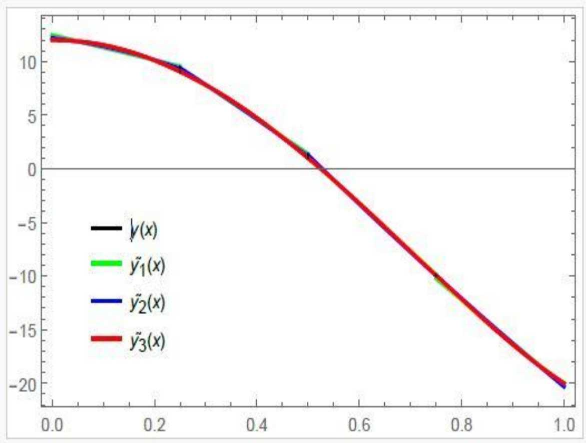
Using LLMW, QLMW, and CLMW bases along with their corresponding operational matrices, the approximate solutions are expressed as follows:

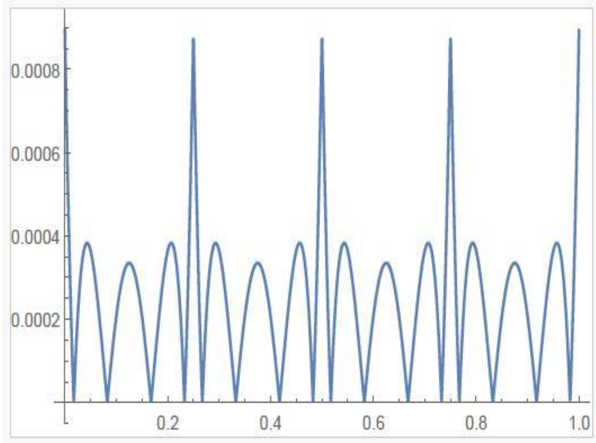
$$\tilde{y}_1(x) = \begin{cases} 12.46781646828935 - 11.689163563369931t; & 0 \le t < \frac{1}{4} \\ 17.453275148039435 - 32.128007436555514t; & \frac{1}{4} \le t < \frac{1}{2} \\ 23.044670085535344 - 43.86250846027291t; & \frac{1}{2} \le t < \frac{3}{4} \\ 19.808054595788704 - 40.0443244803025t; & \frac{3}{4} \le t < 1 \\ 0; & Otherwise \end{cases}, \tag{63}$$

$$\tilde{y}_2(x) = \begin{cases} 12.194414315389041 - 5.472409821378477t - 23.995991863672657t^2; & 0 \le t < \frac{1}{4} \\ 16.23781745460151 - 24.753241481825686t - 10.736016431327272t^2; & \frac{1}{4} \le t < \frac{1}{2} \\ 26.168764617741413 - 52.990025965625584t + 6.450169869128377t^2; & \frac{1}{2} \le t < \frac{3}{4} \\ 26.744160519376116 - 54.524235386758306t + 7.4714931357687835t^2; & \frac{3}{4} \le t < 1 \\ 0; & Otherwise \end{cases}, \tag{64}$$

$$\tilde{y}_3(x) = \begin{cases} 11.999107142857149 + 0.07142857142855874t - 49.28571428571429t^2 + 8.000000000000016t^3; & 0 \le t < \frac{1}{4} \\ 11.713392857142862 + 3.2142857142857006t - 61.2857142857143t^2 + 24.000000000000014t^3; & \frac{1}{4} \le t < \frac{1}{2} \\ 9.641964285714291 + 15.357142857142849t - 85.28571428571429t^2 + 40.000000000000014t^3; & \frac{1}{2} \le t < \frac{3}{4} \\ 2.784821428571435 + 42.499999999999986t - 121.28571428571429t^2 + 56.000000000000014t^3; & \frac{3}{4} \le t < 1 \\ 0; & Otherwise \end{cases}. \tag{65}$$

The exact and approximate solutions of Eq. (61) are illustrated in Fig.3 (a), while Fig.3 (b) shows the absolute error of the exact and approximate solutions obtained by CLMW. Table 3 provides a comparison of the exact and approximate solutions obtained by LLMW, QLMW, and CLMW. The results show that CLMW provides better numerical accuracy compared to LLMW and QLMW.





(a) **(b)**

Figure 3. The Graphs of Exact and Approximate Solutions and Absolute Error
(a) The Graph of Exact and Approximate Solutions for LLMW, QLMW, and CLMW, (b) The Graph of the Absolute Error of Exact and Approximate Solutions Obtained by CLMW

Table 3. Comparison of Exact and Approximate Solutions for LLMW, QLMW, and CLMW

x	$Exact: y(x)$	$LLMW: \tilde{y}_1(x)$	$QLMW: \tilde{y}_2(x)$	$CLMW: \tilde{y}_3(x)$
0.1	11.5216	11.2989	11.4072	11.5214
0.2	10.1056	10.13	10.1401	10.106
0.3	7.8096	7.81487	7.8456	7.80996
0.4	4.7296	4.60207	4.61876	4.72939
0.5	1	1.11342	1.28629	0.999107
0.6	-3.2064	-3.27283	-3.30319	-3.20661
0.7	-7.6784	-7.65909	-7.76367	-7.67804
0.8	-12.1664	-12.2274	-12.0935	-12.166
0.9	-16.3824	-16.2318	-16.2757	-16.3826

Example 6. Consider the HDE for $n = 5$, i.e., order five

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 10y = 0, \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 120. \tag{66}$$

The exact solution is

$$y(x) = 32x^5 - 160x^3 + 120x. \tag{67}$$

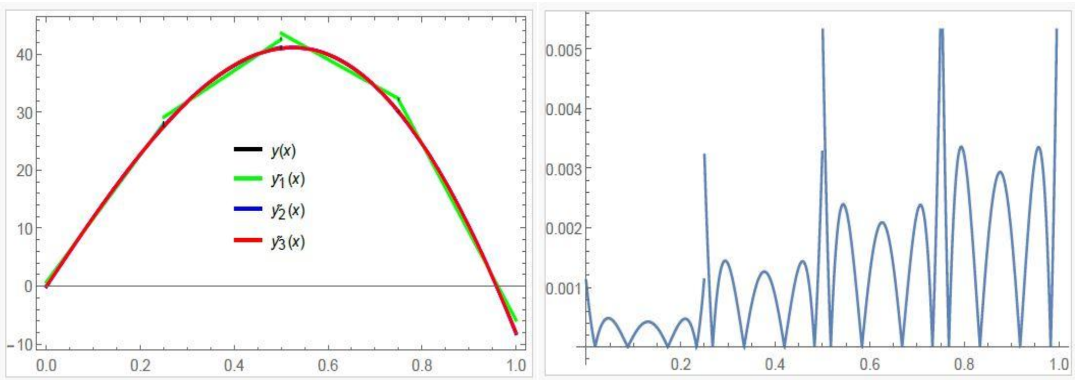
Using LLMW, QLMW, and CLMW bases along with their corresponding operational matrices, the approximate solutions are expressed as follows:

$$\tilde{y_1}(x) = \begin{cases} 0.6929982885864265 + 109.49162535907598t; & 0 \le t < \frac{1}{4} \\ 15.727767304684253 + 53.59213878193939t; & \frac{1}{4} \le t < \frac{1}{2} \\ 66.11256225076067 - 45.0407616273358t; & \frac{1}{2} \le t < \frac{3}{4} \\ 145.99283478458105 - 151.81697489373812t; & \frac{3}{4} \le t < 1 \\ 0; & Otherwise \end{cases}, \tag{68}$$

$$\tilde{y}_2(x) = \begin{cases} -0.04928569770808855 + 123.91928377438413t - 51.14461347551531t^2; & 0 \leq t < \frac{1}{4} \\ -4.94567325113264 + 168.00117260079517t - 151.51138237114097t^2; & \frac{1}{4} \leq t < \frac{1}{2} \\ -20.980999116823664 + 235.67363184565323t - 223.53341447819705t^2; & \frac{1}{2} \leq t < \frac{3}{4}, \\ -14.793938035448159 + 219.35949096293407t - 212.7399101612205t^2; & \frac{3}{4} \leq t < 1 \\ 0; & Otherwise \end{cases} \quad (69)$$

$$\tilde{y}_3(x) = \begin{cases} -0.0011100933046233052 + 120.07998491357321t - 1.2449735485994327t^2 - 154.3030435168597t^3; & 0 \leq t < \frac{1}{4} \\ -0.8728022039437281 + 129.17922941381957t - 33.72920026584301t^2 - 114.30296920810378t^3; & \frac{1}{4} \leq t < \frac{1}{2} \\ -11.854615101374808 + 192.38334289485203t - 156.2190139610411t^2 - 34.302985250229355t^3; & \frac{1}{2} \leq t < \frac{3}{4}. \\ -64.68265005944309 + 399.6809172784151t - 428.7030373842242t^2 + 85.69695159118216t^3; & \frac{3}{4} \leq t < 1 \\ 0; & Otherwise \end{cases} \quad (70)$$

The exact and approximate solutions of Eq. (66) are illustrated in Fig. 4 (a), while Fig. 4 (b) shows the absolute error of the exact and approximate solutions obtained by CLMW. Table 4 provides a comparison of the exact and approximate solutions obtained by LLMW, QLMW, and CLMW. The results show that CLMW provides better numerical accuracy compared to LLMW and QLMW.



(a) (b)

Figure 4. The Graphs of Exact and Approximate Solutions and Absolute Error (a)The Graph of Exact and Approximate Solutions for LLMW, QLMW, and CLMW, (b) The Graph of the Absolute Error of Exact and Approximate Solutions Obtained by CLMW

Table 4. Comparison of Exact and Approximate Solutions for LLMW, QLMW, and CLMW

x	$Exact: y(x)$	$LLMW: \tilde{y}_1(x)$	$QLMW: \tilde{y}_2(x)$	$CLMW: \tilde{y}_3(x)$
0.1	11.8403	11.6422	11.8312	11.8401
0.2	22.7302	22.5913	22.6888	22.7307
0.3	31.0877	31.8054	31.8187	31.7592
0.4	38.0877	37.1646	38.013	38.0868
0.5	41	43.4922	40.9725	40.9944
0.6	39.9283	39.0881	39.9512	39.9271
0.7	34.4982	34.584	34.4592	34.5005
0.8	24.5658	24.5393	24.5401	24.569
0.9	10.2557	9.35756	10.3103	10.2538

4. CONCLUSION

The manuscript presents an effective method for numerically solving HDEs of orders zero through five by leveraging LLMW, QLMW, and CLMW. By employing these methods, the study successfully transforms these equations into algebraic systems, enabling efficient computational solutions. The integration properties are derived and effectively utilized, resulting in enhanced computational accuracy and efficiency. Among the methods, CLMW consistently yields more accurate results, occasionally matching the analytical solutions exactly.

Author Contributions

Meenu Devi: Conceptualization, Data curation, Writing - Original Draft, Writing - Review and Editing; Sunil Rawan: Conceptualization, Data Curation, Formal Analysis, Methodology, Software, Writing - Original Draft, Writing - Review and Editing; Sushil Chandra Rawan: Data Curation, Resources, Software; Vineet Kishore Srivastava: Data Curation, Software, Supervision. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare that they have no potential conflict of interest related to this study.

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