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AN INVESTOR'S OPTIMAL PLAN IN A DC SCHEME WITH REFUND OF CONTRIBUTIONS FOR MORTGAGE HOUSING SCHEME

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ABSTRACT

In this paper, we investigate an investor's portfolios in a defined contributory (DC) Pension Scheme with return of contributions for a mortgage housing scheme and managerial fees for time-inconsistent utility. A portfolio with a fixed deposit (risk-free asset) and two stocks (risky assets) is taken into consideration, where the stock market prices of the risky assets follow the geometric Brownian motion (GBM) and the instantaneous volatilities form a positive definite matrix. To determine the number of scheme members (SM) interested in the mortgage housing, the Abraham De Moivre function is used. Furthermore, the dynamic programming and game techniques were used to obtain our optimization problem by maximizing the expected utility (mean-variance utility) subject to the SM's wealth. Using the variable change technique, the optimal value function (OVF), investor's optimal plan (IOP), and the efficient frontier were obtained under mean variance utility function. . Furthermore, some numerical results of some sensitive parameters such as risk-free interest rate (RIR), risk averse coefficient (RAC), entry age (EA) of SM, managerial charges (MC), optimal fund size (OFS), instantaneous volatilities (IV) and appreciation rates (AR) of the risky assets were presented to explain their impact on the IOP. It was observed that the IOP, which is the fraction of SM's accumulations invested in the risky assets, is a decreasing function of RIR, RAC, IV, EA, OFS, and MC but an increasing function of the AR. Finally, we observed that since there is truncation in the investment period, mean-variance (MV) utility is most suitable for our problem because it provides us with additional information on how to set up our investment strategies, such as the efficient frontier, which gives the relationship between the expectation and risk. Our result also correlates with the existing literature on the choice of utility for problems involving the return of premium.

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1. INTRODUCTION

The Nigerian pension system, before 2004, was fundamentally marred by disputes and dissatisfaction among members due to its poor administrative structure. Before then, every SM of the scheme was under the defined benefit (DB) pension scheme, where eligible SM of the scheme had a guaranteed income for life after retirement. In this scheme, employers mostly set aside a certain retirement benefit amount for each participant based on factors that include SM's age, number of years in service, SM's salary history, retirement level, and so on. Although the majority of the SMs initially appreciated and applauded the scheme since the employers carry the financial burden alone, it has, for some years, created dissatisfaction among SMs due to untimely and non-payment of their retirement benefits and gratuity by the scheme [1]. This has led to an alternative scheme known as the DC pension scheme, which is known to be both employer- and employee-dependent; in this scheme, employees and employers contribute 8% and 10% of the employee's income, respectively, to the member's personal retirement savings account (RSA). These funds are managed by the Pension Fund Administrator (PFA) of their choice [2]. Unlike the DB scheme, members of the DC scheme are fully involved in the contributions and investment process [3]- [7]. As good and attractive as this scheme may look, it requires SMs and PFAs to have a good investment plan in the financial market for optimal returns, considering the volatile nature of the risky assets such as stocks, bonds, loans, and so on. This has led to the study of IOP [8]- [15].

It is noteworthy that the most common and interesting aspect of pension schemes in Nigeria is the payment of retirement benefits to its SMs. However, after the pension reform act in 2014, there are other aspects of the reform that the Nigerian Pension Commission has adopted. One such is the Residential Mortgage Scheme, as can be seen in Section 89 (2) of the PRA 2014, which provides that: "A PFA may, subject to guidelines issued by the Commission, apply for 25% of the pension assets in the retirement savings account towards payment of equity contribution for payment of residential mortgage by a holder of RSA". Pension is borne out of social security. Access to housing is also borne out of social security, and shelter is a basic human need. Owning a home is a big deal anywhere in the world, but having the means to fulfill the desire is often a challenge, particularly in countries where lenders tend to focus on only high-net-worth individuals [16]. One of the ways employees can access funds for a housing scheme is through a pension scheme, as stated in Section 89 (2) of the PRA 2014 by the Nigerian Pension Commission. As this may sound like good news to the RSA holders, it brings about a new investment challenge for PFAs after returning 25% equity contribution for residential mortgages. Giving out such funds during the accumulation phase and how the PFAs will be able to manage the investments and returns of retirement benefits to their SMs forms the basis of this research. Hence, our aim is to develop an investment plan for a PFA with a return of contribution clause for the purpose of a mortgage housing scheme. By this clause, SMs can request for 25% of their contributions to acquire a residential mortgage. The implication of this study is that PFAs cannot use the investment strategy when the return clause is with mortality rate for investment when the return clause is for mortgage housing. The reason is that under the mortality rate, everything is returned to the next of kin, whereas under a mortgage, only 25% is returned, and the member is still part of the scheme.

Before now, some authors have studied IOP with a return of contribution clause based on the number of dead members during the accumulation phase; they include, but are not limited to [17], who studied the IOP with return of premium when the risky asset follows geometric Brownian motion. They considered one risk-free and one risky asset. [18], [19] studied the same problem in [11] but the risky asset followed the Heston volatility model under different assumptions. [12] considered investment in three assets where the two risky assets followed the constant elasticity of variance model and geometric Brownian motion. [20] solved the same problem similar to [11], but their risky asset followed the jump diffusion model. [21], studied a time-consistent investment model in the presence of inflation and a return of premium clause; in their work, they introduced an inflation index bond into their model and determined explicit solutions for both pre-commitment strategy and time-consistent strategy. In [22], pre-commitment and equilibrium investment plan with return of premium was considered; they achieved this by using a discrete time model instead of a continuous time model. Other authors, including [23]- [27], studied the IOP with a return clause under different assumptions. [28], [29], obtained the IOP using the Weibull force function. Other authors, such as [30] and [31], obtained the OIP with a return clause when the returned contribution is with predetermined interest. The research is therefore structured as follows: in section 1, the Nigerian pension scheme and its policies with respect to the mortgage housing scheme are discussed; section 2 will discuss the model formulations and methodology; section 3 discusses the optimization problem, IOP, efficient frontier, and numerical results; section 4 concludes the research.

2. RESEARCH METHODS

2.1 SM's Portfolio with Managerial Charges

Let us consider a market that is open continuously for a period of time $t \in [0, T]$ such that $T > 0$, is the expiration date of the investment. Furthermore, we shall consider a portfolio which comprise of a fixed deposit and two risky assets (stocks). Let $\{\mathfrak{J}_1(t), \mathfrak{J}_2(t), t \geq 0\}$ be two standard BM defined on a complete probability space $(\Omega, \mathfrak{B}, \mathbb{P})$ where Ω is a real space and $\mathbb{P}$ is a probability measure and $\mathfrak{B}$ is the filtration representing information generated by $\mathfrak{J}_1(t)$ and $\mathfrak{J}_2(t)$.

Suppose $\mathcal{H}_t^0(t)$ is the price process of the fixed deposit at time $t > 0$ and $r > 0$, the predetermined interest. From the work of [4], [8], [11], [32], the price dynamics are given thus

$$\begin{cases} \dfrac{d\mathcal{H}_t^0(t)}{\mathcal{H}_t^0(t)} = r dt, \\ \mathcal{H}_t^0(t) = 0, \end{cases} \tag{1}$$

where $r(t)$ is the predetermined interest such that $r > 0$.

Also, we assume the price process of the two risky assets to be $\mathcal{H}_t^1(t)$ and $\mathcal{H}_t^2(t)$ modelled by the GBM model, and the dynamics of the price processes are given similarly to those in [33], [34] by the stochastic differential equations at $t \geq 0$ as follows

$$\begin{cases} \dfrac{d\mathcal{H}_t^1(t)}{\mathcal{H}_t^1(t)} = \pi_1 dt + m_{11} d\mathfrak{J}_1 + m_{12} d\mathfrak{J}_2, \\ \mathcal{H}_t^1(0) = 1, \end{cases} \tag{2}$$

$$\begin{cases} \dfrac{d\mathcal{H}_t^2(t)}{d\mathcal{H}_t^2(t)} = \pi_2 dt + m_{21} d\mathfrak{J}_1 + m_{22} d\mathfrak{J}_2, \\ \mathcal{H}_t^2(0) = 1\,. \end{cases} \tag{3}$$

Where π_1 and π_2 are instantaneous expected rate of return and $m_{11}, m_{12}, m_{21}, m_{22}$ are the instantaneous volatilities for the two risky assets. The volatilities form a 2×2 positive definite matrix $m = \{m_{ab}\}_{2X2.}$ [33], [34].

Let $\mathfrak{H}$ be the PFA's managerial charges, which is determined based on the value of the stock by the PFAs [23]. Corresponding to the investor's optimal plan, $\kappa = (\kappa_0, \kappa_{1,} \kappa_2)$, where $\kappa_0 = 1 - \kappa_1 - \kappa_2$ and the accumulation phase period $\left[t, t + \frac{1}{t}\right]$, the stochastic differential equation for the wealth function $\mathcal{G}(t)$ similar to [11], [12], [33], is given as:

$$\mathcal{G}\left(t + \frac{1}{t}\right) = \left[\begin{matrix} \mathcal{G}(t) \left(\begin{matrix} (1 - \kappa_1(t) - \kappa_2(t)) \dfrac{\mathcal{H}^0_{t+\frac{1}{t}}}{\mathcal{H}_t^0} + \mathfrak{T}_1(t) \dfrac{\mathcal{H}^1_{t+\frac{1}{t}}}{\mathcal{H}_t^1} + \kappa_2(t) \dfrac{\mathcal{H}^2_{t+\frac{1}{t}}}{\mathcal{H}_t^2} \\ \left(1 - 0.25 t k \emptyset_{\frac{1}{t}, b_0 + t}\right) (1 - \kappa_1(t) - \kappa_2(t)) \dfrac{\mathcal{H}^0_{t+\frac{1}{t}}}{\mathcal{H}_t^0} \end{matrix} \right) \\ -\mathfrak{H} \mathcal{G}(t) \dfrac{1}{t} - 0.25 t k \emptyset_{\frac{1}{t}, b_0 + t} + t k \dfrac{1}{t} \end{matrix}\right]. \tag{4}$$

Simplifying Eq. (4), we have

$$\mathcal{G}\left(t+\frac{1}{t}\right)-\mathcal{G}(t)=\left[\begin{array}{c}\mathcal{G}(t)\left(\begin{array}{c}\left(1-\kappa_1(t)-\kappa_2(t)\right)\left(1-0.25tk\emptyset_{\frac{1}{t},b_0+t}\right)\\+\left(2-0.25tk\emptyset_{\frac{1}{t},b_0+t}\right)\left(\frac{\mathcal{H}^0_{t+\frac{1}{t}}-\mathcal{H}^0_t}{\mathcal{N}^0_t}\right)\\+\kappa_1(t)\left(\frac{\mathcal{H}^1_{t+\frac{1}{t}}-\mathcal{H}^1_t}{\mathcal{N}^1_t}\right)+\kappa_2(t)\left(\frac{\mathcal{H}^2_{t+\frac{1}{t}}-\mathcal{H}^2_t(t)}{\mathcal{N}^2_t(t)}\right)\end{array}\right)\\-\mathfrak{H}\mathcal{G}(t)\frac{1}{t}-0.25tk\emptyset_{\frac{1}{t},b_0+t}+tk\frac{1}{t}\end{array}\right]. \tag{5}$$

From [11], we have

$$\left\{\mathfrak{H}\frac{1}{t}=\mathfrak{H}dt,\ tk\frac{1}{t}\to tkdt,\frac{\mathcal{H}^0_{t+\frac{1}{t}}-\mathcal{H}^0_t}{\mathcal{H}^0_t}\to\frac{d\mathcal{H}^0_t}{\mathcal{H}^0_t},\ \frac{\mathcal{H}^1_{t+\frac{1}{t}}-\mathcal{H}^1_t}{\mathcal{H}^1_t}\to\frac{d\mathcal{H}^1_t}{\mathcal{H}^1_t},\ \frac{\mathcal{H}^2_{t+\frac{1}{t}}-\mathcal{H}^2_t(t)}{\mathcal{H}^2_t(t)}\to\frac{d\mathcal{H}^2_t(t)}{\mathcal{H}^2_t(t)}.\right. \tag{6}$$

2.2 Abraham De Moivre Force Function

In this section, we discuss a pension fund system with a return clause of premium based on a mortgage housing scheme, where the rate of SM involvement is modelled using Abraham De Moivre's force function. Suppose k is the SM monthly contribution at time t, b_0 the entry age of SM during the accumulation period, T the accumulation phase period, such that b_0+T is the retirement age of the SM, $\emptyset_{\frac{1}{t},b_0+t}$ is the fraction of members interested in the mortgage housing scheme for time t to $t+\frac{1}{t}$, tk is the accumulated contributions at time t, $0.25tk\emptyset_{\frac{1}{t},b_0+t}$ is the returned contributions paid toward the mortgage housing scheme.

From the work of [11], [12], [33], we have

$$\begin{cases}\dfrac{\emptyset_{\frac{1}{t},b_0+t}}{1-\emptyset_{\frac{1}{t},b_0+t}}=(b_0+t)dt,\\ \emptyset_{\frac{1}{t},b_0+t}=\mathfrak{D}(b_0+t)dt,\end{cases} \tag{7}$$

where $\emptyset(t)$ is the force function and e is the maximal age of the life table. From [11], [27], the Abraham De Moivre force function formula is given as

$$\mathfrak{D}(t)=\frac{1}{(b-t)}\quad 0\le t<b. \tag{8}$$

This implies that

$$\mathfrak{D}(b_0+t)=\frac{1}{(b-b_0-t)}. \tag{9}$$

Substituting Eqs. (6) and (7) into (5), we have

$$d\mathcal{G}(t)=\left[\begin{array}{c}\mathcal{G}(t)\left(\begin{array}{c}\kappa_1\frac{d\mathcal{H}^1_t}{\mathcal{H}^1_t}+\kappa_2\frac{d\mathcal{H}^2_t(t)}{\mathcal{H}^2_t(t)}-\mathfrak{H}dt\\+(1-\kappa_1-\kappa_2)\left(\begin{array}{c}(1-0.25\emptyset(b_0+t)dt\\+\frac{d\mathcal{H}^0_t}{\mathcal{H}^0_t}(2-0.25\emptyset(b_0+t)dt)\end{array}\right)\end{array}\right)\\+tkdt-0.25tk\emptyset(b_0+t)dt\end{array}\right]. \tag{10}$$

Substituting Eqs. (1), (2), (3), and (9) into (10) and simplifying it, we have

$$d\mathcal{G}(t)=\left[\begin{array}{c}\left(\begin{array}{c}\mathcal{G}(t)\left[\begin{array}{c}\kappa_1\left(\mathfrak{n}_1+\frac{0.25}{\mathscr{b}-\mathscr{b}_0-t}-1\right)+\kappa_2\left(\mathfrak{n}_2+\frac{0.25}{\mathscr{b}-\mathscr{b}_0-t}-1\right)\\+\left(2r-\frac{0.25}{\mathscr{b}-\mathscr{b}_0-t}\right)\end{array}\right]\\-\mathfrak{H}+t\mathscr{k}\left(1-\frac{0.25}{\mathscr{b}-\mathscr{b}_0-t}\right)\end{array}\right)dt\\+\mathcal{G}(t)\left((\kappa_1 m_{11}+\kappa_2 m_{21})d\mathfrak{J}_1+(\kappa_1 m_{12}+\kappa_2 m_{22})d\mathfrak{J}_2\right)\end{array}\right]. \quad (11)$$

3. RESULTS AND DISCUSSION

3.1 SM's Optimization Problem and Value Function

Next, we consider the fact that surviving SMs are usually interested in portfolio distribution with the aim of maximizing the fund size while minimizing the volatility of the wealth accumulated over a period of time. It therefore becomes necessary for the PFAs to develop an optimal problem under the mean-variance condition as follows:

$$\mathcal{K}(t,\mathscr{g}).=sup_{\kappa}\{E_{t,\mathscr{g}}[\mathcal{G}^{\kappa}(T)-Var_{t,\mathscr{g}}\mathcal{G}^{\kappa}(T)]\}, \quad (12)$$

Where κ is the control variable representing the optimal control. Next, we observed that the mean variance control problem in Eq. (12) is similar to the following Markovian time-inconsistent stochastic optimal control problem value function $\mathcal{K}(t,\mathscr{g})$.

$$\begin{cases}\mathcal{L}(t,\mathscr{g},\kappa)=E_{t,\mathscr{g}}[\mathcal{G}^{\kappa}(T)]-\frac{\gamma}{2}Var_{t,\mathscr{g}}[\mathcal{G}^{\kappa}(T)],\\ \mathcal{L}(t,\mathscr{g},\kappa)=E_{t,\mathscr{g}}[\mathcal{G}^{\kappa}(T)]-\frac{\gamma}{2}\left(E_{t,\mathscr{g}}[\mathcal{G}^{\kappa}(T)^2]-(E_{t,\mathscr{g}}[\mathcal{G}^{\kappa}(T)])^2\right),\\ \mathcal{K}(t,\mathscr{g}).=sup_{\kappa}\mathcal{L}(t,\mathscr{g},\kappa),\end{cases} \quad (13)$$

γ is a constant representing the risk aversion coefficient of the SMs.

Following the procedure in [11] and [35] the investor's optimal plan κ^* satisfies:

$$\mathcal{K}(t,\mathscr{g}).=sup_{\kappa}\mathcal{L}(t,\mathscr{g},\kappa). \quad (14)$$

Let $\mathscr{y}^{\kappa}(t,\mathscr{g})=E_{t,\mathscr{g}}[\mathcal{G}^{\kappa}(T)]$, $z^{\kappa}(t,\mathscr{g})=E_{t,\kappa}[\mathcal{G}^{\kappa}(T)^2]$ then

$$\mathcal{K}(t,\mathscr{g}). \quad =\sup_{\kappa} c\left(t,\mathscr{g},\mathscr{y}^{\kappa}(t,\mathscr{g}),z^{\kappa}(t,\mathscr{g})\right),$$

where

$$c(t,\kappa,\mathscr{y},z)=\mathscr{y}-\frac{\gamma}{2}(z-\mathscr{y}^2). \quad (15)$$

Since we are interested in maximizing the members' utility in Eq. (12) subject to their wealth in Eq. (11), from [11], [12], [33], [35], we establish our optimization problem, i.e., the extended HJB, using a game theoretic approach by applying Ito's lemma. This is given by the verification theorem below

Theorem 1. **(Verification Theorem)** *If there exist three real functions* $\mathcal{C},\mathcal{D},\mathcal{E}:[0,T]\times\mathfrak{R}\rightarrow\mathfrak{R}$ *satisfying the following EHJB equations:*

$$sup\left\{\begin{array}{c}\mathcal{C}_t-c_t+(\mathcal{C}_{\mathscr{g}}-c_{\mathscr{g}})\left[\begin{array}{c}\kappa_1\mathscr{g}\left(\mathfrak{n}_1+\frac{0.25}{\mathscr{b}-\mathscr{b}_0-t}-1\right)+\kappa_2\mathscr{g}\left(\mathfrak{n}_2+\frac{0.25}{\mathscr{b}-\mathscr{b}_0-t}-1\right)\\+\mathscr{g}\left(2r-\frac{0.25}{\mathscr{b}-\mathscr{b}_0-t}\right)-\mathfrak{H}+t\mathscr{k}\left(1-\frac{0.25}{\mathscr{b}-\mathscr{b}_0-t}\right)\end{array}\right]\\+\left[\begin{array}{c}\frac{1}{2}\mathfrak{J}\kappa_1^2+\mathfrak{L}\kappa_1\kappa_2\\+\frac{1}{2}\mathfrak{K}\kappa_2^2\end{array}\right]\mathscr{g}^2(\mathcal{C}_{\mathscr{g}\mathscr{g}}-\mathcal{U})\\ \mathcal{C}(T,\mathscr{g})=c(T,\mathscr{g},\mathscr{g},\mathscr{g}^2)\end{array}\right\}=0, \quad (16)$$

where $\mathcal{U}=c_{\mathscr{g}\mathscr{g}}+2c_{\mathscr{g}\mathscr{y}}\mathscr{y}_{\mathscr{g}}+2c_{\mathscr{y}z}\mathscr{y}_z z_{\mathscr{y}}+c_{zz}z_{\mathscr{g}}^2=\gamma\mathscr{y}_{\mathscr{g}}^2$,

$$\begin{cases} \mathcal{D}_t + \mathcal{D}_{\mathscr{g}} \begin{bmatrix} \kappa_1 \mathscr{g}\left(\eta_1 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right) + \kappa_2 \mathscr{g}\left(\eta_2 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right) \\ + \mathscr{g}\left(2\mathscr{r} - \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t}\right) - \mathfrak{H} + t\mathscr{k}\left(1 - \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t}\right) \end{bmatrix} \\ + \begin{bmatrix} \frac{1}{2}\mathfrak{J}\kappa_1^2 + \mathfrak{L}\kappa_1\kappa_2 \\ + \frac{1}{2}\mathfrak{K}\kappa_2^2 \end{bmatrix} \mathscr{g}^2 \mathcal{D}_{\mathscr{g}\mathscr{g}} \\ \mathcal{D}(T, \mathscr{g}) = \mathscr{g} \end{cases} = 0, \tag{17}$$

$$\begin{cases} \mathcal{E}_t + \mathcal{E}_{\mathscr{g}} \begin{bmatrix} \kappa_1 \mathscr{g}\left(\eta_1 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right) + \kappa_2 \mathscr{g}\left(\eta_2 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right) + \\ \mathscr{g}\left(2\mathscr{r} - \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t}\right) - \mathfrak{H} + t\mathscr{k}\left(1 - \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t}\right) \end{bmatrix} \\ + \begin{bmatrix} \frac{1}{2}\mathfrak{J}\kappa_1^2 + \mathfrak{L}\kappa_1\kappa_2 \\ + \frac{1}{2}\mathfrak{K}\kappa_2^2 \end{bmatrix} \mathscr{g}^2 \mathcal{E}_{\mathscr{g}\mathscr{g}} \\ \mathcal{E}(T, \mathscr{g}) = \mathscr{g}^2 \end{cases} = 0, \tag{18}$$

$\mathcal{K}(t, \mathscr{g}) = \mathcal{C}(t, \mathscr{g})$, $y^{\kappa^*} = D(t, \mathscr{g})$, $z^{\kappa^*} = \mathcal{E}(t, \mathscr{g})$ for the investor's optimal plan κ^*.

Proof: The proof is similar to one in [17], [36], [37]

$$\mathcal{C}_y = 1 + \gamma y, c_{yy=}\gamma, c_z = -\frac{\gamma}{2}, c_z = c_{\mathscr{g}} = c_{\mathscr{g}} = c_{\mathscr{g}y} = c_{\mathscr{g}z} = c_{zz} = 0 \tag{19}$$

where $m_{11}^2 + m_{12}^2 = \mathfrak{J}, m_{21}^2 + m_{22}^2 = \mathfrak{K}, m_{11}m_{21} + m_{12}m_{22} = \mathfrak{L}$.

Differentiating with respect to κ_1 and κ_2, we obtain the first order optimizing problem as follows

$$\kappa_1 = \frac{\left[\mathfrak{L}\left(\eta_2 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right) - \mathfrak{K}\left(\eta_1 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right)\right]\mathcal{C}_{\mathscr{g}}}{(\mathfrak{J}\mathfrak{K} - \mathfrak{L}^2)\mathscr{g}\left(\mathcal{C}_{\mathscr{g}\mathscr{g}} - \gamma\mathcal{D}_{\mathscr{g}}^2\right)} \tag{20}$$

$$\kappa_2 = \frac{\left[\mathfrak{L}\left(\eta_1 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right) - \mathfrak{J}\left(\eta_2 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right)\right]\mathcal{C}_{\mathscr{g}}}{(\mathfrak{J}\mathfrak{K} - \mathfrak{L}^2)\mathscr{g}\left(\mathcal{C}_{\mathscr{g}\mathscr{g}} - \gamma\mathcal{D}_{\mathscr{g}}^2\right)} \tag{21}$$

Substituting Eqs. (20) and (21) into Eqs. (16) and (17).

$$\mathcal{C}_t - \frac{1}{2}(\Lambda_1 - \Lambda_2 - \Lambda_3)\frac{\mathcal{C}_{\mathscr{g}}^2}{\mathcal{C}_{\mathscr{g}\mathscr{g}} - \gamma\mathcal{D}_{\mathscr{g}}^2} + \begin{bmatrix} \mathscr{g}\left(2\mathscr{r} - \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t}\right) \\ -\mathfrak{H} + t\mathscr{k}\left(1 - \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t}\right) \end{bmatrix} \mathcal{C}_{\mathscr{g}} = 0, \tag{22}$$

$$\mathcal{D}_t + \frac{1}{2}(\Lambda_1 - \Lambda_2 - \Lambda_3)\frac{\mathcal{C}_{\mathscr{g}}^2}{\mathcal{C}_{\mathscr{g}\mathscr{g}} - \gamma\mathcal{D}_{\mathscr{g}}^2} + \begin{bmatrix} \mathscr{g}\left(2\mathscr{r} - \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t}\right) - \mathfrak{H} + t\mathscr{k}\left(1 - \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t}\right) \\ - \frac{\mathcal{C}_{\mathscr{g}}(\Lambda_1 - \Lambda_2 - \Lambda_3)}{\left(\mathcal{C}_{\mathscr{g}\mathscr{g}} - \gamma\mathcal{D}_{\mathscr{g}}^2\right)} \end{bmatrix} \mathcal{D}_{\mathscr{g}}, \tag{23}$$

where $\Lambda_1 = \frac{2\mathcal{R}\left(\eta_1 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right)\left(\eta_2 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right)}{\mathfrak{J}\mathfrak{K} - \mathfrak{L}^2}, \Lambda_2 = \frac{\mathcal{Q}\left(\eta_1 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right)^2}{\mathfrak{J}\mathfrak{K} - \mathfrak{L}^2}, \Lambda_3 = \frac{\left(\eta_2 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right)^2}{\mathfrak{J}\mathfrak{K} - \mathfrak{L}^2}$.

Suppose we conjecture a solution to $\mathcal{C}(t, \mathscr{g})$ and $\mathcal{D}(t, \mathscr{g})$ according to [11], [31], [33] as

$$\begin{cases} \mathcal{C}(t, \mathscr{g}) = \mathscr{g}\mathfrak{T}_1(t) + \mathfrak{T}_2(t), \\ \mathfrak{T}_1(T) = 1, \ \mathfrak{T}_2(T) = 0, \end{cases} \tag{24}$$

$$\begin{cases} \mathcal{D}(t, \mathscr{g}) = \mathscr{g}\nu_1(t) + \nu_2(t), \\ \nu_1(T) = 1, \ \nu_2(T) = 0. \end{cases} \tag{25}$$

Differentiating $\mathcal{C}(t,g)$ and $\mathcal{D}(t,g)$ with respect to g and t, we have

$$\begin{cases} \mathcal{C}_t(t,g) = g\mathfrak{T}_{1t}(t) + \mathfrak{T}_{2t}(t)\,, \mathcal{C}_g(t,g) = \mathfrak{T}_1(t), \mathcal{C}_{gg}(t,g) = 0, \\ \mathfrak{T}_1(T) = 1,\ \ \mathfrak{T}_2(T) = 0, \end{cases} \tag{26}$$

$$\begin{cases} \mathcal{D}_t(t,g) = g\nu_{1t}(t) + \nu_{2t}(t)\,, \mathcal{D}_g(t,g) = \nu_1(t), \mathcal{D}_{gg}(t,g) = 0, \\ \nu_1(T) = 1,\ \ \nu_2(T) = 0. \end{cases} \tag{27}$$

Substituting Eqs. (26) and (27) into Eqs. (22) and (23), we have

$$\begin{cases} \begin{bmatrix} g\mathfrak{T}_{1t}(t) + \mathfrak{T}_{2t}(t) + \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3)\frac{\mathfrak{T}_1^2}{\gamma\nu_1^2} \\ +g\left[\mathfrak{T}_1(t)\left(2r - \frac{0.25}{b - b_0 - t}\right)\right] + \left[-\mathfrak{H} + tk\left(1 - \frac{0.25}{b - b_0 - t}\right)\right]\mathfrak{T}_1(t) \end{bmatrix} = 0, \\ \mathfrak{T}_1(T) = 1,\ \ \mathfrak{T}_2(T) = 0, \end{cases} \tag{28}$$

$$\begin{cases} \begin{bmatrix} g\nu_{1t}(t) + \nu_{2t}(t) + \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3)\frac{\mathfrak{T}_1^2}{\gamma\nu_1^2} + \frac{\mathfrak{T}_1(t)\nu_1(t)(\lambda_1 - \lambda_2 - \lambda_3)}{\gamma\nu_1^2} \\ +g\left[\nu_1(t)\left(2r - \frac{0.25}{b - b_0 - t}\right)\right] + \left[-\mathfrak{H} + tk\left(1 - \frac{0.25}{b - b_0 - t}\right)\right]\nu_1(t) \end{bmatrix} = 0, \\ \nu_1(T) = 1,\ \ \nu_2(T) = 0. \end{cases} \tag{29}$$

Simplifying Eqs. (28) and (29), we have

$$\begin{cases} \mathfrak{T}_{1t}(t) + \left(2r - \frac{0.25}{b - b_0 - t}\right)\mathfrak{T}_1(t) = 0, \\ \mathfrak{T}_1(T) = 1, \end{cases} \tag{30}$$

$$\begin{cases} \mathfrak{T}_{2t}(t) + \left[-\mathfrak{H} + tk\left(1 - \frac{0.25}{b - b_0 - t}\right)\right]\mathfrak{T}_1(t) + \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3)\frac{\mathfrak{T}_1^2}{\gamma\nu_1^2} = 0, \\ \mathfrak{T}_2(T) = 0, \end{cases} \tag{31}$$

$$\begin{cases} \nu_{1t} + \left(2r - \frac{0.25}{e - e_0 - t}\right)\nu_1(t) = 0, \\ \nu_1(T) = 1, \end{cases} \tag{32}$$

$$\begin{cases} \nu_{2t} + \left[-\mathfrak{H} + tk\left(1 - \frac{0.25}{b - b_0 - t}\right)\right]\nu_1(t) + \frac{\mathfrak{T}_1(t)\nu_1(t)}{\eta\nu_1^2}(\lambda_1 - \lambda_2 - \lambda_3) = 0, \\ \nu_2(T) = 0. \end{cases} \tag{33}$$

Solving Eqs. (30), (31), (32), and (33), we have

$$\mathfrak{T}_1(t) = \left(\frac{b - b_0 - T}{b - b_0 - t}\right)^{0.25} e^{2r(T-t)}, \tag{34}$$

$$\nu_1(t) = \left(\frac{b - b_0 - T}{b - b_0 - t}\right)^{0.25} e^{2r(T-t)}. \tag{35}$$

Since $\mathfrak{T}_1(t) = \nu_1(t)$

$$\begin{gathered} \mathfrak{T}_{2t}(t) + \left[-\mathfrak{H} + tk\left(1 - \frac{0.25}{b - b_0 - t}\right)\right]\left(\frac{b - b_0 - T}{b - b_0 - t}\right)^{0.25} e^{2r(T-t)} + \frac{1}{2\gamma}(\lambda_1 - \lambda_2 - \lambda_3) = 0, \\ \mathfrak{T}_2(T) = 0. \end{gathered} \tag{36}$$

Solving Eq. (36), we have

$$\mathfrak{T}_2(t)=\begin{bmatrix}\frac{1}{2\gamma}(\lambda_1-\lambda_2-\lambda_3)(T-t)\\ -k\int_t^T\tau\left(1-\frac{0.25}{b-b_0-\tau}\right)\left(\frac{b-b_0-T}{b-b_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\\ +\mathfrak{H}\int_t^T\left(1-\frac{0.25}{b-b_0-\tau}\right)\left(\frac{b-b_0-T}{b-b_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\end{bmatrix}. \tag{37}$$

Also,

$$\begin{cases}\nu_{2t}+\left[-\mathfrak{H}+tk\left(1-\frac{0.25}{b-b_0-t}\right)\right]\left(\frac{b-b_0-T}{b-b_0-t}\right)^{0.25}e^{2r(T-t)}+\frac{1}{\gamma}(\lambda_1-\lambda_2-\lambda_3)=0,\\ \nu_2(T)=0.\end{cases} \tag{38}$$

Solving Eq. (38), we have

$$\nu_2(t)=\begin{bmatrix}\frac{1}{\gamma}(\lambda_1-\lambda_2-\lambda_3)(T-t)\\ -k\int_t^T\tau\left(1-\frac{0.25}{b-b_0-\tau}\right)\left(\frac{b-b_0-T}{b-b_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\\ +\mathfrak{H}\int_t^T\left(1-\frac{0.25}{b-b_0-\tau}\right)\left(\frac{b-b_0-T}{b-b_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\end{bmatrix}. \tag{39}$$

Substituting Eqs. (34) and (37) into Eq. (24) and Eqs. (35) and (39) into Eq. (25), we have

$$\mathcal{C}(t,g)=\begin{bmatrix}g\left(\frac{b-b_0-T}{b-b_0-t}\right)^{0.25}e^{2r(T-t)}\\ +\begin{bmatrix}\frac{1}{2\gamma}(\lambda_1-\lambda_2-\lambda_3)(T-t)\\ -k\int_t^T\tau\left(1-\frac{0.25}{b-b_0-\tau}\right)\left(\frac{b-b_0-T}{b-b_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\\ +\mathfrak{H}\int_t^T\left(1-\frac{0.25}{b-b_0-\tau}\right)\left(\frac{b-b_0-T}{b-b_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\end{bmatrix}\end{bmatrix}, \tag{40}$$

$$\mathcal{D}(t,g)=\begin{bmatrix}g\left(\frac{b-b_0-T}{b-b_0-t}\right)^{0.25}e^{2r(T-t)}\\ +\begin{bmatrix}\frac{1}{\gamma}(\lambda_1-\lambda_2-\lambda_3)(T-t)\\ -k\int_t^T\tau\left(1-\frac{0.25}{b-b_0-\tau}\right)\left(\frac{b-b_0-T}{b-b_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\\ +\mathfrak{H}\int_t^T\left(1-\frac{0.25}{b-b_0-\tau}\right)\left(\frac{b-b_0-T}{b-b_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\end{bmatrix}\end{bmatrix}. \tag{41}$$

3.2 Efficient Frontier and Investor's Optimal Plan

Proposition 1. *The Efficient frontier of the SM is given as*

$$E\left(\mathcal{G}^{\kappa^*}(T)\right)=\begin{bmatrix}g\left(\frac{b-b_0-T}{b-b_0-t}\right)^{0.25}e^{2r(T-t)}+\sqrt{Var[\mathcal{G}^{\kappa^*}(T)][\lambda_1-\lambda_2-\lambda_3](T-t)}\\ +\begin{bmatrix}-k\int_t^T & \tau\left(1-\frac{0.25}{b-b_0-\tau}\right)\left(\frac{b-b_0-T}{b-b_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\\ & +\mathfrak{H}\int_t^T\left(1-\frac{0.25}{b-b_0-\tau}\right)\left(\frac{b-b_0-T}{b-b_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\end{bmatrix}\end{bmatrix} \tag{42}$$

Proof. Recall from Eqs. (13) and (14),

$$E\left(\mathcal{G}^{\kappa^*}(T)\right) = \mathcal{D}(t, g) \text{ and } \mathcal{V}ar[\mathcal{G}^{\kappa^*}(T)] = E[\mathcal{G}^{\kappa^*}(T)]^2 - \left[E\left(\mathcal{G}^{\kappa^*}(T)\right)\right]^2,$$

then

$$\mathcal{V}ar[\mathcal{G}^{\kappa^*}(T)] = E[\mathcal{G}^{\kappa^*}(T)]^2 - \left[E\left(\mathcal{G}^{\kappa^*}(T)\right)\right] = \frac{2}{\gamma}\left(\mathcal{D}(t, g) - \mathcal{C}(t, g)\right)^2. \tag{43}$$

Substituting Eqs. (40) and (41) into Eq. (43), we have

$$\mathcal{V}ar[\mathcal{G}^{\kappa^*}(T)] = \frac{2}{\gamma}\left[\begin{array}{c}\left[\begin{array}{c} g\left(\frac{\mathcal{b}-\mathcal{b}_0-T}{\mathcal{b}-\mathcal{b}_0-t}\right)^{0.25} e^{2r(T-t)} \\ +\left[\begin{array}{c}\frac{1}{\gamma}(\lambda_1-\lambda_2-\lambda_3)(T-t) \\ -\mathcal{k}\int_t^T \tau\left(1-\frac{0.25}{\mathcal{b}-\mathcal{b}_0-\tau}\right)\left(\frac{\mathcal{b}-\mathcal{b}_0-T}{\mathcal{b}-\mathcal{b}_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau \\ +\mathfrak{H}\int_t^T \left(1-\frac{0.25}{\mathcal{b}-\mathcal{b}_0-\tau}\right)\left(\frac{\mathcal{b}-\mathcal{b}_0-T}{\mathcal{b}-\mathcal{b}_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\end{array}\right]\end{array}\right] \\ -\left[\begin{array}{c} g\left(\frac{\mathcal{b}-\mathcal{b}_0-T}{\mathcal{b}-\mathcal{b}_0-t}\right)^{0.25} e^{2r(T-t)} \\ +\left[\begin{array}{c}\frac{1}{2m}(\lambda_1-\lambda_2-\lambda_3)(T-t) \\ -\mathcal{k}\int_t^T \tau\left(1-\frac{0.25}{\mathcal{b}-\mathcal{b}_0-\tau}\right)\left(\frac{\mathcal{b}-\mathcal{b}_0-T}{\mathcal{b}-\mathcal{b}_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau \\ +\mathfrak{H}\int_t^T \left(1-\frac{0.25}{\mathcal{b}-\mathcal{b}_0-\tau}\right)\left(\frac{\mathcal{b}-\mathcal{b}_0-T}{\mathcal{b}-\mathcal{b}_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\end{array}\right]\end{array}\right]\end{array}\right]$$

$$= \frac{2}{\gamma}\left[\left(\frac{1}{\gamma}-\frac{1}{2\gamma}\right)(\lambda_1-\lambda_2-\lambda_3)(T-t)\right] = \frac{2}{\gamma}\left[\frac{1}{\gamma}(\lambda_1-\lambda_2-\lambda_3)(T-t)\right]$$

$$\mathcal{V}ar[\mathcal{G}^{\kappa^*}(T)] = \frac{1}{\gamma^2}[\lambda_1-\lambda_2-\lambda_3](T-t),$$

$$\frac{1}{\gamma} = \sqrt{\frac{\mathcal{V}ar[\mathcal{G}^{\kappa^*}(T)]}{[\lambda_1-\lambda_2-\lambda_3](T-t)}}. \tag{44}$$

Also,

$$E\left(\mathcal{G}^{\kappa^*}(T)\right) = \mathcal{D}(t, g). \tag{45}$$

Substituting Eq. (41) into Eq. (45), we have

$$E\left(\mathcal{G}^{\kappa^*}(T)\right) = \left[\begin{array}{c} g\left(\frac{\mathcal{b}-\mathcal{b}_0-T}{\mathcal{b}-\mathcal{b}_0-t}\right)^{0.25} e^{2r(T-t)} \\ +\left[\begin{array}{c}\frac{1}{\gamma}(\lambda_1-\lambda_2-\lambda_3)(T-t) \\ -\mathcal{k}\int_t^T \tau\left(1-\frac{0.25}{\mathcal{b}-\mathcal{b}_0-\tau}\right)\left(\frac{\mathcal{b}-\mathcal{b}_0-T}{\mathcal{b}-\mathcal{b}_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau \\ +\mathfrak{H}\int_t^T \left(1-\frac{0.25}{\mathcal{b}-\mathcal{b}_0-\tau}\right)\left(\frac{\mathcal{b}-\mathcal{b}_0-T}{\mathcal{b}-\mathcal{b}_0-\tau}\right)^{0.25}\tau e^{2r(T-\tau)}d\tau\end{array}\right]\end{array}\right]. \tag{46}$$

Substituting Eq. (43) into Eq. (46) and simplifying it, we obtain Eq. (42), which completes the proof. ■

Proposition 2. *The IOP is given as*

$$\kappa_1 = \left(\frac{\mathcal{b}-\mathcal{b}_0-t}{\mathcal{b}-\mathcal{b}_0-T}\right)^{0.25} e^{2r(t-T)} \times \frac{\left[\mathfrak{L}\left(\mathfrak{n}_2+\frac{0.25}{\mathcal{b}-\mathcal{b}_0-t}-1\right)-\mathfrak{K}\left(\mathfrak{n}_1+\frac{0.25}{\mathcal{b}-\mathcal{b}_0-t}-1\right)\right]}{(\mathfrak{J}\mathfrak{K}-\mathfrak{L}^2)g}, \tag{47}$$

$$\kappa_2 = \left(\frac{\mathscr{b} - \mathscr{b}_0 - t}{\mathscr{b} - \mathscr{b}_0 - T}\right)^{0.25} e^{2r(t-T)} \times \frac{\left[\mathfrak{L}\left(\mathfrak{n}_1 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right) - \mathfrak{J}\left(\mathfrak{n}_2 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right)\right]}{(\mathfrak{J}\mathfrak{K} - \mathfrak{L}^2)\mathscr{g}}. \quad (48)$$

Proof. Also, substituting Eqs. (26) and (27) into Eqs. (20) and (21), we have

$$\kappa_1 = \frac{\left[\mathfrak{L}\left(\mathfrak{n}_2 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right) - \mathfrak{K}\left(\mathfrak{n}_1 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right)\right]}{(\mathfrak{J}\mathfrak{K} - \mathfrak{L}^2)\mathscr{g}} \times \frac{1}{\mathfrak{T}_1(t)}, \quad (49)$$

$$\kappa_2 = \frac{\left[\mathfrak{L}\left(\mathfrak{n}_1 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right) - \mathfrak{J}\left(\mathfrak{n}_2 + \frac{0.25}{\mathscr{b} - \mathscr{b}_0 - t} - 1\right)\right]}{(\mathfrak{J}\mathfrak{K} - \mathfrak{L}^2)\mathscr{g}} \times \frac{1}{\mathfrak{T}_1(t)}. \quad (50)$$

Now, substituting Eq. (34) into Eqs. (49) and (50) and simplify it, we obtained Eqs. (47) and (48), which completes the proof. ■

3.3 Numerical Simulations

In this section, some results of our work showing the relationship between the IOP and some sensitive parameters will be presented and discussed using Eqs. (44), (49) and (50). Also, data similar to those in [11] and [32], this is accomplished, unless otherwise stated: $\mathfrak{n}_1 = 1.6$, $\mathfrak{n}_2 = 0.05$, $\mathscr{r} = 0.1$, $\gamma = 0.01$, $\mathscr{g}_0 = 1$, $\mathscr{m}_{11} = 0.12$, $\mathscr{m}_{12} = 0.10$, $\mathscr{m}_{21} = 1.12$, $\mathscr{m}_{22} = 1.1$, $t = 0{:}\,0.0005{:}\,20$, $\mathscr{b} = 100$, $\mathscr{b}_0 = 20$, $T = 40$.

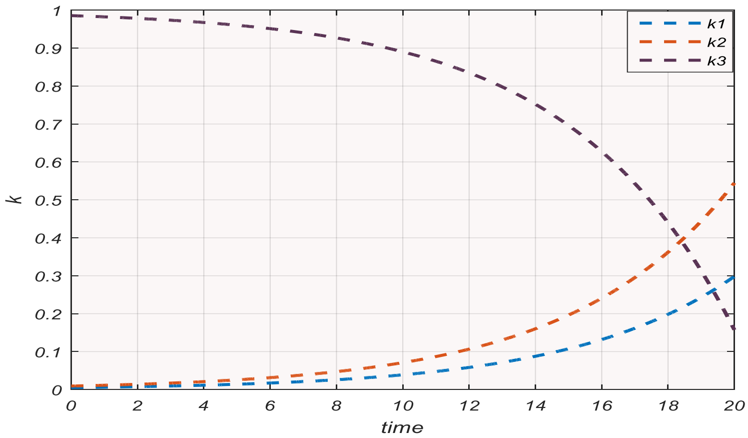

Figure 1. Graph of IOP Against Time

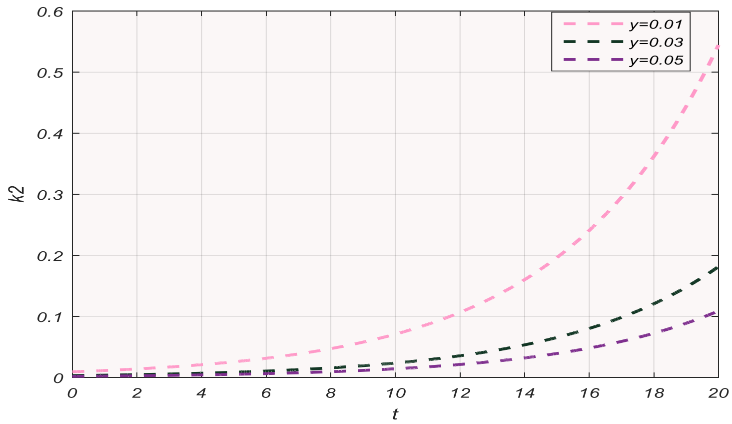

Figure 2. Graph of the IOP of Stock 2 Against Time with Different RAC

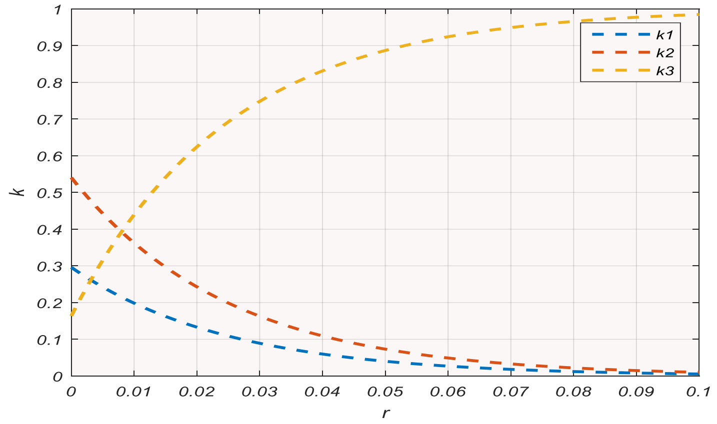

Figure 3. Graph of IOP Against Predetermined Interest

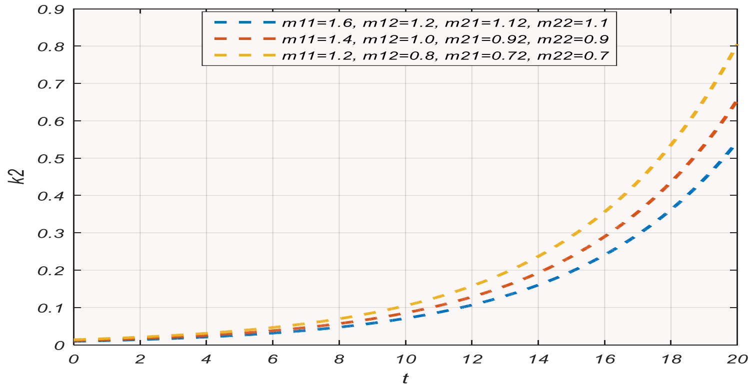

Figure 4. Graph of IOP of Stock 2 against Time with Different Instantaneous Volatilities

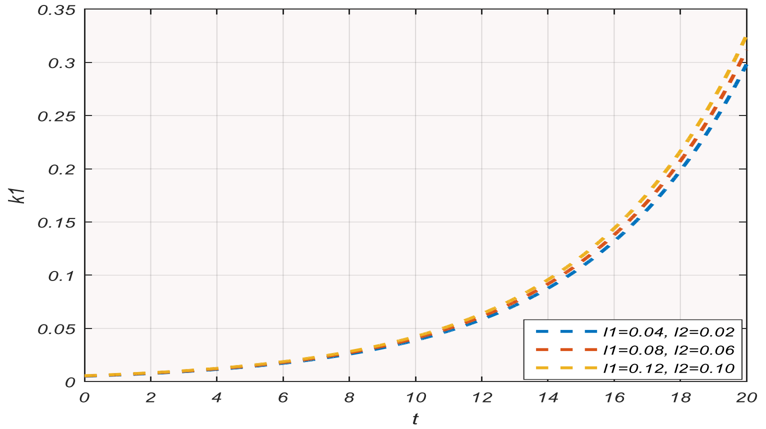

Figure 5. Graph of IOP of Stock 1 against Time with Different Appreciation Rates of the Risky Asset

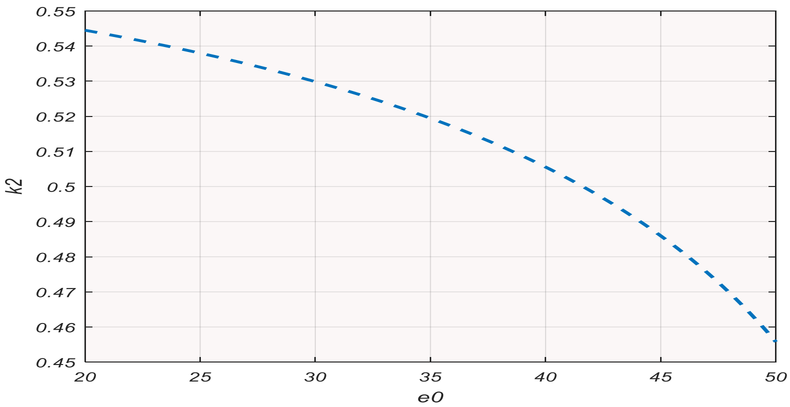

Figure 6. Graph of IOP of Stock 2 against Member's Entry Age

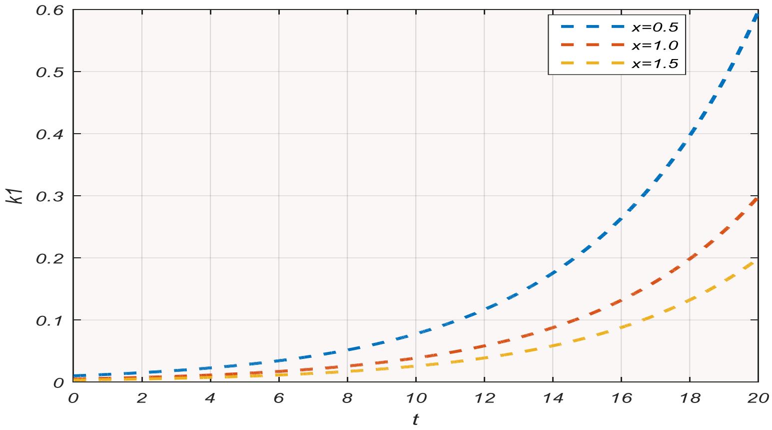

Figure 7. **Graph of IOP of Stock 1 against Time with Different Initial Wealth**

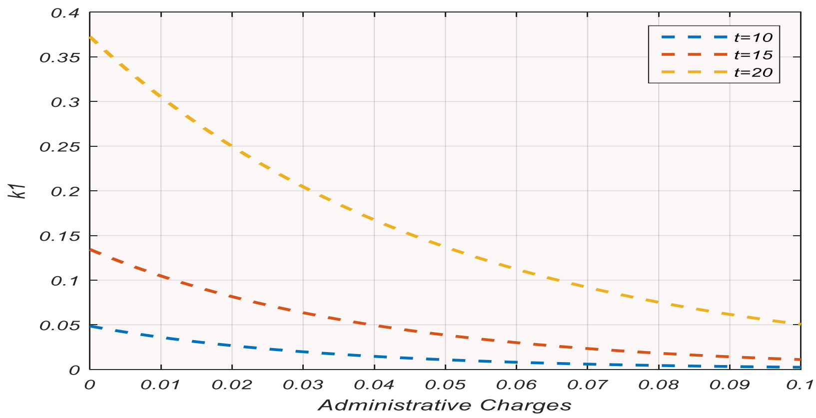

Figure 8. **Graph of the IOP of Stock 1 against Administrative Charges at Different Times**

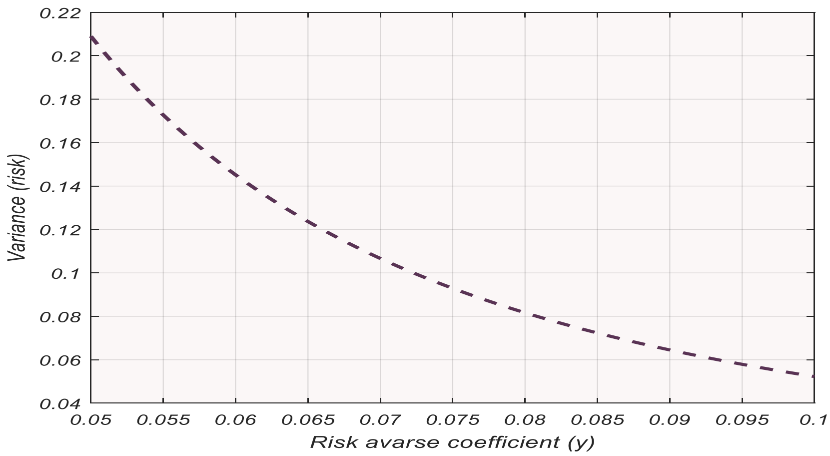

Figure 9. **Relationship between Variance and Risk**

3.4 Summary of Results Discussion

In Fig.1, the graph of IOP for the two stocks and the fixed deposit is presented with respect to time. It is observed that, as SMs get closer to the age of retirement, the proportion of their wealth invested in the two risky assets grows with time and gets significantly higher, while that of fixed deposit gets progressively smaller. This is significant because the initial fund size for investment in stock was used at the early stage of investment, hence giving room for the investor to increase their investment in the risky asset as retirement age gets closer. In Fig. 2, the graph of IOP was plotted against the RAC; it was observed that the IOP is a decreasing function of the RAC. This means that the more dreadful the SM is to investment in the stock market, the more likely that he will gradually switch to invest in a less risky asset. In Fig. 3, the graph of IOP against predetermined interest is presented. We observed that the IOP is a decreasing function of the predetermined. This shows that SMs of the scheme will most likely want to invest less in fixed deposits because it is less risky if the interest rate is attractive. However, if the interest rate in the fixed deposit is lucrative, the SM may decide to increase his investment in stock with the advice of the fund managers.

In Fig. 4, the graph of IOP against time was simulated with different instantaneous volatilities; it was observed that IOP is also a decreasing function of the instantaneous volatilities of the stock. This implies that the higher the instantaneous volatilities, the higher the risk in the investment, which leads to a decrease in IOP of such an asset, especially for risk-averse SMs. Fig. 5 shows the graph of IOP against the appreciation rate of the stock; the graph shows that the IOP is directly proportional to the appreciation rate of the stock. This implies that any asset with a higher appreciation rate will be appealing to the investor; hence, the willingness on the side of the investor to commit more funds into such an asset with the expectation of more returns. However, the shift in the investment plan is that obvious, because the choice of investment plan may be influenced by other sensitive parameters or factors such as the predetermined interest, RAC, and stock market volatilities. In Fig. 6, the graph shows the effect of SM's entry age on IOP. It is observed that the IOP decreases as the entry age increases. This implies that SM with late entry into the pension scheme may increase their wealth to be invested in stocks. This is so because the initial fund size of the SM is indirectly proportional to the entry age of the SM. Also, from the available fund types under the pension scheme, we can see that investment in risky assets decreases as SMs' age increases.

Fig. 7 presents the graph of IOP for stock against time with different SM's wealth. It is observed that the IOP is a decreasing function of the wealth function. This implies that the SM will invest more in the risky assets if his/her wealth is small and vice versa. In Fig. 8, the relationship between the IOP and the managerial charges is presented. It is observed that the IOP is a decreasing function of the managerial charges. The implication in Fig. 8 is that SMs in the scheme are more likely to avoid investing a large amount of their wealth in more risky assets if the cost of managing such asset relatively high. Fig. 9 presents a graph of variance against the RAC; it was observed that the variance is a decreasing function of the RAC. The implication of this is that SMs with lower RAC will avoid taking risks as much as they can, and vice versa. This can also be confirmed from Fig. 3, where the IOP for the stock is a decreasing function of RAC. However, from proposition 2, we observed that the SMs' expectation is directly proportional to variance (risk); this implies that more investment in risky assets may increase the chances of the SM increasing his or her returns on investment. Furthermore, we can deduce that the mean-variance utility is best suited to problems involving return of premium since there is truncation in the investment process.

4. CONCLUSION

In conclusion, the optimal portfolio with refund of contributions for the mortgage housing scheme, managerial charges, a risk-free asset, and two risky assets was considered. Our major aim was to develop an effective and efficient investment model for the PFAs when there is a return of contributions for the mortgage housing scheme, which was obtained in Eqs. (47) and (48). The IOP obtained in Eqs. (47) and (48) gave a shift from the investment model used when the return clause is based on the mortality rate, which makes it more effective in application than if the already existing model were used when the return clause was based on mortgage housing. This is so because our model results handle investment when only 25% is withdrawn during the accumulation phase, while the models in the literature are used when all the accumulations of a deceased SM are returned to his or her next of kin. It was observed that for the PFAs to make an informed investment decision, the following parameters must be taken into consideration as they strongly affect the development of the IOP; these include RIR, initial fund size, instantaneous volatility, managerial charges, RAC, and appreciation rate of the stock price. The impacts of these parameters were clearly discussed in the discussion section. Also, the efficient frontier was obtained in Eq. (46). Finally, we observed that our investment model under this assumption is different from the existing literature.

Author Contributions

Edikan Edem Akpanibah: Conceptualization, Methodology, Data Curation, Project Administration, Supervision, Writing-Original Draft. Sylvanus Kupongoh Samaila: Formal Analysis, Investigation, Software, Validation, Writing-Review and Editing. Ase Matthias Esabai: Data Curation, Resources, Visualization, Writing-Review and Editing. All authors reviewed and approved the final manuscript.

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Declarations

The authors declare no competing interests.

Declaration of Generative AI and AI-assisted Technologies

The authors declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, including writing, editing, data analysis, or the creation of tables and figures.

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