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ROBUST QUASI-NEWTON EQUATIONS IN QUASI-NEWTON METHOD FOR SOLVING UNCONSTRAINED OPTIMIZATION PROBLEMS

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ABSTRACT

Quasi-Newton methods are among the most widely used and effective general-purpose algorithms for unconstrained optimization. These methods traditionally rely on the quasi-Newton equation, which serves as the foundation for updating approximations of the Hessian matrix at each iteration. The goal is to construct accurate second-order curvature information to accelerate convergence toward the optimum. In this paper, we derive a novel quasi-Newton equation based on an enhanced quadratic model. A key feature of this new formulation is that it incorporates both gradient information and objective function values, enabling higher-order accuracy in approximating the second-order curvature of the objective function. This new equation stands out for its ability to provide a more precise representation of the function's curvature, which in turn improves the overall efficiency and performance of the optimization method. Theoretical analysis shows that the proposed method is globally convergent under certain reasonable assumptions. To validate the effectiveness of the approach, we conducted a series of numerical experiments using standard benchmark problems. The results demonstrate that the modified Broyden, Fletcher, Goldfarb, and Shanno (BFGS) method, which integrates the new quasi-Newton equation, outperforms existing BFGS-type methods in terms of numerical efficiency and solution accuracy.

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1. INTRODUCTION

Optimization techniques are essential tools in applied mathematics, engineering, machine learning, and operations research. As real-world problems grow in complexity and dimensionality, the need for efficient and reliable optimization algorithms becomes increasingly critical [1], [2]. Among the various optimization methods, second-order techniques—particularly Newton's method-are recognized for their rapid convergence properties. However, these methods require explicit computation of the Hessian matrix, which can be computationally expensive or even impractical for large-scale problems. To address this limitation, quasi-Newton methods were developed as a class of optimization algorithms that approximate the Hessian matrix using only gradient information [2]. These methods preserve many of the desirable convergence characteristics of Newton's method while significantly reducing computational costs, making them highly suitable for practical applications [3].

Quasi-Newton methods are iterative techniques designed to find local minima of differentiable functions without requiring the direct computation of second derivatives. Instead, they build an approximation of the inverse Hessian matrix through successive gradient evaluations. One of the most prominent quasi-Newton algorithms is the Broyden–Fletcher–Goldfarb–Shanno (BFGS) method, which updates the inverse Hessian approximation at each iteration using rank-two updates [4]. For high-dimensional problems, the limited-memory version, known as L-BFGS, has been developed. This variant stores only a limited number of vectors that represent the approximation, making it particularly effective in machine learning and large-scale data modeling tasks [5].

The motivation behind this study stems from the increasing demand for optimization algorithms that are computationally efficient and capable of robust convergence across a wide range of problems. While first-order methods, such as stochastic gradient descent (SGD), are widely used due to their simplicity and scalability, they often converge slowly, especially in ill-conditioned problems [6]. On the other hand, second-order methods achieve faster convergence but at a significantly higher computational cost. Quasi-Newton methods offer a balanced alternative, especially in scenarios where the objective function is smooth and gradients can be computed efficiently. The central problem this research addresses is how to improve optimization performance with respect to convergence speed and computational efficiency, particularly for high-dimensional or ill-conditioned problems. The study aims to explore the mathematical structure, implementation challenges, and empirical performance of quasi-Newton methods. It focuses on comparing quasi-Newton methods with other optimization techniques in terms of convergence rate and computational cost, identifying practical considerations in their implementation, and understanding the conditions under which these methods may underperform or fail.

The objectives of this research are threefold. First, to provide a comprehensive analysis of the theoretical foundations of quasi-Newton methods, including their convergence behavior and numerical stability. Second, to implement and evaluate these methods on a set of benchmark optimization problems to assess their practical effectiveness. Third, to identify possible enhancements or hybrid strategies that may further improve their performance. In the context of numerical optimization, the study also investigates how quasi-Newton updates can efficiently approximate the Hessian matrix while reducing computational cost. Furthermore, particular focus is placed on the algorithms' adaptability to large-scale numerical problems and their stability in high-dimensional optimization landscapes. This investigation aims to contribute to the broader understanding of optimization algorithms and guide the selection of appropriate methods for different classes of problems, [4].

The anticipated outcomes of this research include a clearer understanding of when and how quasi-Newton methods should be applied, practical guidelines for selecting and tuning optimization algorithms, and recommendations for improving existing quasi-Newton frameworks. This work also seeks to bridge the gap between theoretical development and real-world implementation, especially in domains where optimization plays a critical role in system performance. In particular, the study emphasizes the role of numerical precision and computational efficiency in determining the effectiveness of quasi-Newton updates. Additionally, comparative analyses are conducted to evaluate their robustness and scalability across different numerical optimization scenarios. This introduction is grounded in foundational literature on numerical optimization, particularly the work of Nocedal and Wright [2], as well as more recent studies that highlight the effectiveness of quasi-Newton methods across various application domains [5], [6].

2. RESEARCH METHODS

The quasi-Newton method is highly useful for minimiz-ing a smooth function of n variables:

$$Minf(x)\ ,\ \mathrm{x} \in R^n, \tag{1}$$

where f is continuously differentiable [3]. As opposed to utilizing the real value of the Hessian or the inverse Hessian, quasi-Newton methods use a symmetric positive definite estimate of the Hessian or the inverse Hessian. The following is its form:

$$x_{k+1} = x_k + \alpha_k d_k, \tag{2}$$

where the search direction (d_k) and the step length (α_k) are determined bya using Wolfe conditions such that as:

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^T d_k, \tag{3}$$

$$d_k^T g(x_k + \alpha_k d_k) \ge \sigma\, d_k^T g_k, \tag{4}$$

where $0 < \delta < \sigma < 1$, are usually used. For more details see [7]. The parameter α_k was computed using an exact line-search:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T Q d_k}. \tag{5}$$

For more details see [8]. Its direction is computed by solving system:

$$B_k d_k + g_k = 0. \tag{6}$$

2.1 Related Work

For each iteration, B_k is an updated estimate of the Hessian. The BFGS approach, which was separately proposed by Broyden, Fletcher, Goldfarb, and Shanno, is now one of the most effective quasi-Newton methods. By using the following formula, the matrix B_{k+1} in the BFGS technique may be updated:

$$B_{k+1}^{BFGS} = B_k - \frac{B_k s_k s_k^T {B_k}^T}{s_k^T B_k s_k} + \frac{y_k y_k^T}{s_k^T y_k}, \tag{7}$$

where $s_k = x_{k+1} - x_k = \alpha_k d_k$ and $y_k = g_{k+1} - g_k$, see [2]. Let H_k be the inverse of B_k. Undoubtedly, the BFGS update (8) is publicly known as:

$$H_{k+1}^{BFGS} = H_k - \frac{H_k y_k s_k^T + s_k y_k^T H_k}{s_k^T y_k} + \left[1 + \frac{y_k^T H_k y_k}{s_k^T y_k}\right] \frac{s_k s_k^T}{s_k^T y_k}, \tag{8}$$

see, [9] for details. For quasi-Newton processes, the quasi-Newton equation's definiteness is essential. In general, quasi-Newton methods fulfill the Quasi-Newton equation:

$$B_{k+1} s_k = y_k. \tag{9}$$

Numerical experiments indicate that the BFGS technique outperforms other quasi-Newton methods in many scenarios. Convex minimization using the BFGS approach has been extensively studied; for examples, see [10], [11], [12]. However, to illustrate that the BFGS method with the Wolfe line search may fail for non-convex functions, Dai constructed an example containing six cycling points [13]. To address this limitation, numerous improvements have been proposed. Some modifications have been applied to the standard BFGS technique, and a modified BFGS algorithm (MBFGS) has been developed to enhance its global convergence properties [15], [20]. These studies demonstrate that MBFGS achieves global convergence even for non-convex optimization problems. Further investigations have been conducted to assess whether novel quasi-Newton methods exhibit global convergence and can outperform BFGS computationally [21]. In practice, the modified BFGS method is generally preferred for efficiently approximating the Hessian matrix (or its inverse) using a symmetric positive definite matrix. While the standard secant relation relies solely on gradient values, the modified quasi-Newton secant equation utilizes both gradient and function information. Importantly, global convergence can be established without assuming convexity of the objective function [22].

Extensive research has focused on designing quasi-Newton methods that guarantee global convergence while simultaneously improving numerical performance, particularly those based on the BFGS update. In the following section, we review the key modifications made to the quasi-Newton equations and their impact on algorithmic efficiency.

Table 1. Overview of Quasi-Newton Conditions Reported in the Literature

Authors	QN – conditions	References
Li and Fukushima	$B_{k+1}s_k = \tilde{y}_k = y_k + t_k s_k\ ,\quad t_k \le 10^{-6}$	[26]
Wei, Li and Qi	$B_{k+1}s_k = \tilde{y}_k = y_k + \frac{2(f_k - f_{k+1}) + (g_{k+1} + g_k)^T s_k}{\lVert s_k \rVert^2} s_k$	[27]
Zhang, Deng, and Chen	$B_{k+1}s_k = \tilde{y}_k = y_k + \frac{6(f_k - f_{k+1}) + 3(g_{k+1} + g_k)^T s_k}{\lVert s_k \rVert^2} s_k$	[19]
Basim	$B_{k+1}s_k = \tilde{y}_k = \frac{1}{2} y_k + \frac{(f_k - f_{k+1}) - 1/2(g_{k+1}^T s_k)}{s_k^T y_k} y_k$	[14]
Basim and Mohamed	$B_{k+1}s_k = \tilde{y}_k = y_k - \frac{g_{k+1}^T s_k}{\lVert s_k \rVert^2} s_k$	[16]
Basim and Ghada	$B_{k+1}s_k = \tilde{y}_k = y_k + \frac{2(f_k - f_{k+1}) - s_k^T y_k}{\lVert s_k \rVert^2} s_k$	[28]
Basim and Ranen	$B_{k+1}s_k = \tilde{y}_k = \frac{3}{2} y_k + \frac{(f_{k+1} - f_k) - 3/2 g_{k+1}^T s_k}{s_k^T y_k} y_k$	[17]
Basim and Moghrabi	$B_{k+1}s_k = \tilde{y}_k = 2 y_k + \frac{2(f_k - f_{k+1})}{\lVert s_k \rVert^2} s_k$	[29]
Basim and Abdulrahman	$B_{k+1}s_k = \tilde{y}_k = \frac{1}{2} y_k + \frac{(f_{k+1} - f_k)}{\lVert s_k \rVert^2} s_k$	[18]
Basim and Abdulrahman	$B_{k+1}s_k = \tilde{y}_k = \frac{5}{6} y_k + \frac{(f_k - f_{k+1}) - 1/3 g_k^T s_k}{\lVert s_k \rVert^2} s_k$	[30]

The quasi-Newton equation underlies all quasi-Newton methods. At each iteration, updating the initial quasi-Newton equation requires evaluating the second-order derivative Hessian matrix based on the quadratic model. Line search-based methods have been shown to achieve global convergence. Considerable research has focused on using modified quasi-Newton equations to improve the quality of Hessian updates, thereby enhancing the numerical performance of quasi-Newton techniques.

2.2 Derivation of the Robust QN-Equation

The new additional quasi-Newton equation is derived using a quadratic model of the goal function. As a result, the objective function's quadratic model gives us:

$$f_{k+1} = f_k + s_k^T g_k + \frac{1}{2} s_k^T Q(x_k) s_k, \tag{10}$$

where $Q(x_k)$ is the Hessian matrix. The first derivative of above equation may be written as:

$$\nabla f_{k+1} = g_k + Q(x_k) s_k. \tag{11}$$

Thus, the curvature information Based on two Eqs. (9) and (10) by new approximated:

$$s_k^T Q(x_k) s_k = \frac{2}{3}(f_k - f_{k+1}) + \frac{2}{3} s_k^T Q(x_k) s_k. \tag{12}$$

Since the updated B_{k+1} is supposed to approximate $Q(x_k)$, it is reasonable to having:

$$s_k^T B_{k+1} s_k = \frac{2}{3}(f_k - f_{k+1}) + \frac{2}{3} s_k^T Q(x_k) s_k. \tag{13}$$

By using Eq. (11) in Eq. (13) we have:

$$s_k^T B_{k+1} s_k = \frac{2}{3} s_k^T y_k + \frac{2}{3}(f_k - f_{k+1}). \tag{14}$$

The new QN- equation is given by:

$$s_k^T \tilde{y}_k = \frac{2}{3} s_k^T y_k + \frac{2}{3}(f_k - f_{k+1}). \tag{15}$$

From the above equation, the different gradient can be written as:

$$B_{k+1} s_k = \tilde{y}_k, \tilde{y}_k = \frac{2}{3} y_k + \frac{2/3(f_k - f_{k+1})}{s_k^T u_k} u_k. \tag{16}$$

where u_k is any vector such that $s_k^T u_k \neq 0$. Based on the revised quasi-Newton equation, the BFGS update was modified. Alternatively, the vector u_k choices in Eq. (16) might be expressed as:

In the case $u_k = y_k$ Eq. (16) becomes:

$$\tilde{y}_k = \frac{2}{3} y_k + \frac{2/3(f_k - f_{k+1})}{s_k^T y_k} y_k. \tag{17}$$

In the case $u_k = g_k$ Eq. (16) becomes:

$$\tilde{y}_k = \frac{2}{3} y_k + \frac{2/3(f_k - f_{k+1})}{s_k^T g_k} g_k. \tag{18}$$

In the case $u_k = g_{k+1}$ Eq. (16) becomes:

$$\tilde{y}_k = \frac{2}{3} y_k + \frac{2/3(f_k - f_{k+1})}{s_k^T g_{k+`1}} g_{k+`1}. \tag{19}$$

The algorithm that results from the explanation above is as follows:

Algorithm: New algorithms.

Input: Initialization, set $x_0 \in R^n$, $H_0 = I$, $k = 0$. Compute $g_0 = \nabla f(x_0)$.
Output: Optimal point x* and Function value f(x*).
Repeat until convergence:
Stage 1: Check for optimality:If $\|g_k\| = 0$ then x* Optimal point.
Stage 2: Search direction: $d_k = -H_k g_k$.
Stage 3: Step-size determination: compute α_k using Eq. (3) and (4).
Stage 4: Update: $x_{k+1} = x_k + \alpha_k d_k$. If $s_k^T \tilde{y}_k > 0$ then Update H_{k+1} using Eq. (8) (with parameters defined in Eqs. (17)–(19)). Else $H_{k+1} = H_k$, End If.
Stage 5: Iteration: k = k + 1 Continue to Stage 1. End Repeat.

The comparison between the BFGS and proposed update formulas revealed that the method relies on computing differences of vector derivatives, with BFGS defined based on y_k and proposed on $\tilde{y}_k$. It was also found that the BFGS algorithm does not achieve global convergence, whereas the proposed MBFGS formula demonstrates a clear ability to attain global convergence.

3. RESULTS AND DISCUSSION

3.1 Convergence Analysis

We provide the global convergence of innovative approaches in circumstances that are comparatively understated.

Assumption 1

1. Level is set to $L_0 = \{x \in R^n : f(x) \le f(x_0)\}$ be convex.
2. Since the gradient is Lipschitz continuous, there exists a positive constant $L > 0$:

$$\left(\nabla f(\bar{x}) - \nabla f(x^+)\right) \le L\|\bar{x} - x^+\| \ , \forall \bar{x}, x^+ \in L_0. \tag{20}$$

The series $\{x_k\}$ produced by a new algorithm is evidently found in S since $\{f_k\}$ is a decreasing series, and there is a constant f^* that results in:

$$\lim_{k\to\infty} f_k = f^*. \tag{21}$$

3. Let G be the matrix of second derivatives of f. Then, there exist constants r and R such that:

$$r\|z\|^2 \le z^T Qz \quad \le R\|z\|^2. \tag{22}$$

for all $z \in R^n$, see [12], [13], [14].

Theorem 1. *Let $\boldsymbol{B_{k+1}}$ be generated by formula Eq. (6). Then $\boldsymbol{B_{k+1}}$ is positive definite.*

Proof. From the different gradient definition, we have:

$$s_k^T \tilde{y}_k = \frac{2}{3} y_k + \frac{2}{3}(f_k - f_{k+1}). \tag{23}$$

By applying Wolfe's condition to the previous equation, we obtain:

$$s_k^T \tilde{y}_k \ge \frac{2}{3}\left(s_k^T y_k - \delta g_k^T s_k\right). \tag{24}$$

Since $s_k^T y_k > 0$ and $-\delta g_k^T s_k > 0$, thus Eq. (24), we have that:

$$s_k^T \tilde{y}_k \ge 0, \tag{25}$$

therefore, B_{k+1} is positive-definite.

Theorem 2. *Let $\{x_k\}$ be generated by the new algorithm. Then we have :*

$$r\|s_k\|^2 \le s_k^T \tilde{y}_k \le R\|s_k\|^2 \tag{26}$$

and

$$\|\tilde{y}_k\| \le (L+R)\|s_k\|. \tag{27}$$

Proof. By different gradient definition $\tilde{y}_k$ and combining Eq. (10) with Eq. (16) we get:

$$\begin{aligned} s_k^T \tilde{y}_k = s_k^T Q(x_k) s_k &= \frac{2}{3} s_k^T y_k + \frac{2}{3}(f_k - f_{k+1}) \\ &= 2(f_{k+1} - f_k) - 2 s_k^T g_k. \end{aligned} \tag{28}$$

Utilizing the mean value theorem and Taylor's series, it became:

$$f_{k+1} = f_k + s_k^T g_k + \frac{1}{2} s_k^T Q(\eta_k) s_k \tag{29}$$

where $\eta_k = x_k + \xi(x_{k+1} - x_k)$ and $\xi \in (0\,,1)$. As such by Eq. (28) and (29) that:

$$\begin{aligned} s_k^T \tilde{y}_k &= 2\left(s_k^T g_k + \frac{1}{2} s_k^T Q(\eta_k) s_k\right) - 2 s_k^T g_k \\ &= 2 s_k^T g_k + s_k^T Q(\eta_k) s_k - 2 s_k^T g_k \\ &= s_k^T Q(\eta_k) s_k. \end{aligned} \tag{30}$$

Meeting with Assumption 3, it is simple to stumble on:

$$r\|s_k\|^2 \le s_k^T \tilde{y}_k \le R\|s_k\|^2. \tag{31}$$

Then we have different gradient definition $\tilde{y_k}$ by direct calculations:

$$\begin{aligned}\|\tilde{y_k}\| &= \left\|\frac{2}{3}y_k + \frac{[2/3(f_k - f_{k+1})]}{s_k^T u_k}u_k\right\| \\ &\leq \frac{2}{3}\|y_k\| + \frac{|[s_k^T Q(\eta_k)s_k - 2/3(s_k^T y_k)]|}{\|\delta_k\|\|u_k\|}\|u_k\| \\ &\leq \frac{4}{3}\|y_k\| + \frac{|[s_k^T Q(\eta_k)s_k]|}{\|s_k\|} \leq 4/3L\|s_k\| + R\|s_k\| \\ &\leq \left(\frac{4}{3L} + R\right)\|s_k\|.\end{aligned} \tag{32}$$

The proof is thus complete.

Theorem 3. *Then constants* $a_1 > 0$ *and* $a_2 > 0$ *exists, the following inequality holds:*

$$\|B_k s_k\| \leq a_1\|s_k\| \text{ and } \quad s_k^T B_k s_2 \geq a_2\|s_k\|^2 \tag{33}$$

for any k. *The sequence* $\{x_k\}$ *is obtained by new Algorithm, the we have:*

$$\liminf_{k\to\infty} \|g_k\| = 0 . \tag{34}$$

Proof: In a manner similar to the proof of Theorem 3 in [17], the argument proceeds straightforwardly.

In this study, we prove a global convergence theorem for non-convex problems and suggest a cautious updating strategy that is comparable to the one mentioned before. We state a Powell-related lemma for motivational purposes [20].

Lemma 1. *In [23], a smooth function f that is limited below can be treated using the BFGS technique if a constant* $M > 0$ *exists that makes the inequality:*

$$\|\tilde{y_k}\|^2 / s_k^T \tilde{y_k} \leq M. \tag{35}$$

and

$$\liminf_{k\to\infty} \|g_k\| = 0 . \tag{36}$$

Theorem 4. *Let Assumptions holds and* $\{\boldsymbol{x_k}\}$ *be generated by new Algorithm. Then Eq. (36) holds.*

Proof. If Eq. (36) fails to hold, a constant $\varepsilon > 0$ exists, such that:

$$\|g_{k+1}\| \geq \varepsilon. \tag{37}$$

Therefore, there is a constant $r > 0$ such that:

$$r\|s_k\|^2 \leq s_k^T \tilde{y_k}. \tag{38}$$

So, combining Eqs. (32) and (38) implies:

$$\|\tilde{y_k}\|^2 / s_k^T \tilde{y_k} \leq M. \tag{39}$$

This completes the proof.

3.2. Numerical Results

We give a numerical comparison of QN-techniques for minimization of 30 test functions with various variables derived from [24], [31] by we use the MATLAB programming. The total number of iterations and the total number of function evaluations are denoted by NI and NF, respectively. The findings for approaches applying Wolfe conditions (3)-(4) with 2 and 3 are shown in the table. We stopped utilizing the algorithms by employing the law of Himmeblau [24], [32]: "If $|f(x_k)| > 10^{-5}$, let $stop1 = |f(.x_k) - f(.x_{k+1})| / |f(x_k)|$; Otherwise, let $stop1 = \big||f(.x_k) - f(.x_{k+1})|\big|$. For every problem, if $\|g_k\| < \varepsilon$ or $stop1 <$

10^{-5} is satisfied, the program will be stopped. There are new resources like in the optimization field, [22], [23], [24], [25].

Table 2. Table of Problems, Starting Points and Dimensions.

Problem	Dim	Function No.	Starting points
Rosenbrock function	2	1	$x_0 = (-1.2,1.0)$
Freudenstein and Roth function	2	2	$x_0 = (0.5,-2.0)$
Powell badly scaled function	2	3	$x_0 = (1.0,1.0)$
Brown badly scaled function	2	4	$x_0 = (1.0,1.0)$
Beale function	2	5	$x_0 = (1.0,1.0)$
Jennrich and Sampson function	2	6	$x_0 = (0.3,0.4)$
Helical valley function	3	7	$x_0 = (-1.0,0,0)$
Bard function	3	8	$x_0 = (1.0,1.0,1.0)$
Gaussian function	3	9	$x_0 = (0.4,1.1)$
Gulf function	3	10	$x_0 = (5.0,2.5,1.5)$
Powell singular function	4	11	$x_0 = (3,-1,0,1)$
Wood function	4	12	$x_0 = (-3,-1,-3,-1)$
Kowalik function	4	13	$x_0 = (.25,.39,.415,.39)$
Brown function	4	14	$x_0 = (25,5,-5,-1)$
Osborne 1 function	5	15	$x_0 = (0.5,1.5,-1,0.01,0.02)$
Biggs function	6	16	$x_0 = (1,2,1,1,1,1)$
Osborne 2 function	11	17	$x_0 = (1.3,0.65,0.65,0.7,0.7,0.6,3,5,7,2,4.5,5.5)$
Watson function	20	18	$x_0 = (0,\ldots\ldots\ldots,0)$
Penalty 1 function	400	19	$x_0 = (1/2,..,1/2)$
Penalty 2 function	200	20	$x_0 = (1/2,..,1/2)$
Vardim function	100	21	$x_0 = (1-(j/n))$
Trigonometric function	500	22	$x_0 = (1/n,\ldots.,1/n)$
Boundary value function	500	23	$x_0 = (t_i(t_i-1))$
Discrete integral function	500	24	$x_0 = (t_i(t_i-1))$
Broyden banded function	500	25	$x_0 = (-1,\ldots.,-1))$
Linear function - full rank	500	26	$x_0 = (-1,\ldots.,-1)$
Linear function - rank 1	500	27	$x_0 = (1,\ldots\ldots..,1)$
Trigonometric function	1000	28	$x_0 = (1/n,\ldots.,1/n)$
Broyden banded function	1000	29	$x_0 = (-1,\ldots.,-1)$

Quasi-Newton approaches can perform better when an appropriate quasi-Newton equation is used. The new update with $u_k = g_{k+1}$ performs the best on average out of the three ways; the BFGS technique performs somewhat better than the new update with $u_k = y_k$ and $u_k = g_k$. Consequently, the most structured of the QN-procedures for unbound problems is the new update with $u_k = g_{k+1}$.

Table 3. Numerical Results of the BFGS Method and New Methods: Number of Iterations (NI), and Function Evaluations (NF) for Test Problems

Problem	BFGS		$u_k = y_k$		$u_k = g_k$		$u_k = g_{k+1}$	
	NI	NF	NI	NF	NI	NF	NI	NF
1.	35	140	28	92	29	96	26	175
2.	9	26	8	23	8	23	8	23
3.	43	166	33	116	37	132	3	31
4.	3	30	3	30	3	30	3	30
5.	15	50	12	41	13	39	6	21
6.	2	27	2	27	2	27	2	27
7.	34	113	30	94	34	107	8	23
8.	16	54	16	53	16	48	10	30
9.	2	4	2	4	2	4	2	4
10.	2	27	2	27	2	27	2	27
11.	20	60	20	65	16	49	5	17
12.	19	61	19	59	16	50	5	17
13.	21	65	23	97	23	79	5	13
14.	17	54	16	49	16	50	5	17
15.	2	27	2	27	2	27	2	27
16.	25	72	7	42	5	38	2	12
17.	3	31	3	31	3	31	3	31
18.	31	102	32	102	34	111	4	13

Problem	BFGS		$u_k = y_k$		$u_k = g_k$		$u_k = g_{k+1}$	
	NI	NF	NI	NF	NI	NF	NI	NF
19.	2	27	2	27	2	27	2	27
20.	2	5	2	5	2	5	2	5
21.	2	27	2	27	2	27	2	27
22.	9	33	9	34	9	29	2	29
23.	2	4	2	4	2	4	2	4
24.	6	16	7	19	7	19	5	14
25.	57	281	14	82	10	37	5	17
26.	2	4	2	4	2	4	2	4
27.	3	7	3	7	3	7	3	7
28.	7	25	6	22	6	20	6	20
29.	14	114	9	73	10	37	6	22
Total	**405**	**1399**	**316**	**1283**	**316**	**1184**	**138**	**714**

To conduct a more rigorous evaluation of the proposed methods, the results were further examined using the performance profile methodology developed by Dolan and Moré [25]. This methodology utilizes a cumulative distribution function to quantify the probability that a given algorithm solves a particular problem within a factor of the best observed performance across all methods. In the resulting plots, the x-axis represents the performance ratio, while the y-axis reflects the cumulative fraction of problems for which the algorithm achieves that ratio or better. An algorithm whose curve lies above those of other methods across a wide range of performance ratios is considered more robust and efficient, as it successfully solves a higher proportion of problems with relatively superior performance.

Fig. 1 illustrates the performance profile comparing the proposed algorithms with the classical BFGS method, using the number of iterations (NI) as the performance metric. Comment: This figure highlights the convergence speed advantage of the proposed algorithm, showing it reaches solutions with fewer iterations across most test problems. The curve indicates that the proposed algorithm consistently achieves higher cumulative performance at lower iteration counts, reflecting faster convergence across most test problems.

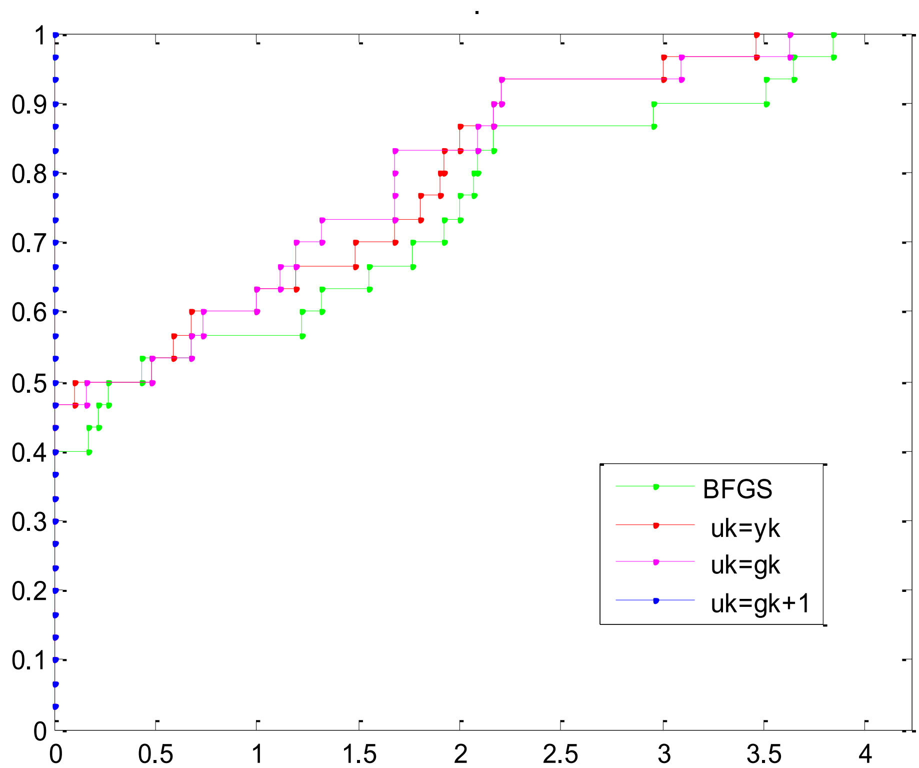

Figure 1. Performance on the number of iterations

Similarly, Fig. 2 presents the performance profile based on the number of function evaluations (NF). Comment: This demonstrates the computational efficiency of the proposed algorithms, as they require fewer function evaluations to solve a larger proportion of problems compared to classical BFGS. The proposed algorithms again outperform the classical BFGS, solving a greater proportion of problems with fewer evaluations. This highlights the improved computational efficiency and effectiveness of the proposed method in practical optimization tasks.

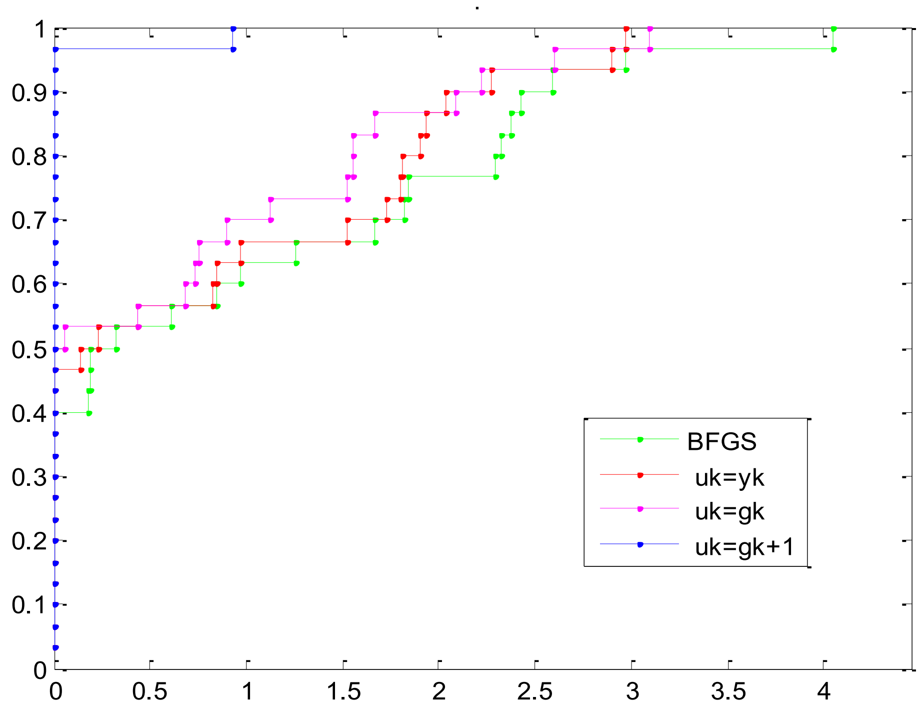

Figure 2. Function evaluations performance

Overall, these results demonstrate that the proposed method is more efficient than the classical BFGS algorithm, achieving faster convergence with fewer iterations and reduced computational effort. The combined insights from Figs. 1 and 2 emphasize that the proposed method is not only faster in convergence but also more resource-efficient, making it suitable for large-scale or computationally expensive optimization problems.

4. CONCLUSION

Many of the quasi-Newton techniques are based on the quasi-Newton equation itself. Hence, we get the improved BFGS quasi-Newton updating formulae using the robust QN-equation that has been described. The second order information from the objective function's Hessian is used in this research to develop a novel quasi-Newton equation. Assign this algorithm to a line search rule and demonstrate its global convergence. Relative Efficiency of the New Algorithms presents a comparative analysis of the performance of the BFGS algorithm against a set of newly proposed algorithms, using two key performance indicators: the number of iterations (NI) and the number of function evaluations (NF). In this comparison, BFGS serves as the reference point, with its efficiency standardized at 100% for both criteria. The other algorithms demonstrate varying degrees of relative performance. In terms of iterations (NI), the new algorithms achieved efficiencies of 78.02%, 78.02%, and 34.07%, indicating that while some approaches approach the speed of BFGS in reaching a solution, others require significantly more iterations. Regarding function evaluations (NF), the relative efficiencies were 91.70%, 84.63%, and 51.03%. This suggests that some of the new algorithms are quite effective in reducing computational effort, while others fall noticeably behind in this aspect. For futuristic work, use the derivation idea to derive self-scaling QN to improve the numerical performance of BFGS.

Author Contributions

Basim A. Hassan: Conceptualization, Writing – Review, Formal Analysis, Methodology. Software, Validation, Resources. Manal I. Mohammed: Data Curation, Visualization, Writing - Original Draft, Supervision. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare no competing interest.

Declaration of Generative AI and AI-assisted Technologies

The authors declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, including for writing, editing, data analysis, or the creation of tables and figures.

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