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LAPLACE TRANSFORMATION AND MITTAG-LEFFLER FUNCTION FOR THE SOLUTION OF DAMPED OSCILLATOR EQUATION WITH FRACTIONAL ORDER

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ABSTRACT

Fractional calculus has emerged as an active area of research due to its ability to model complex dynamical systems with memory effects and anomalous diffusion. In particular, the Mittag–Leffler function plays a fundamental role in solving fractional differential equations. This study aims to derive the analytical solution of the Linear Fractionally Damped Oscillator using the Laplace transform and the Mittag–Leffler function, where the derivative is of Caputo type with order $0<\alpha<1$. We further extend the analysis to both homogeneous and nonhomogeneous models, the latter corresponding to the presence of an external forcing term. The results indicate that the oscillatory behavior exhibits algebraic decay and eventual convergence due to damping or dissipation effects. The decay rate is directly influenced by the asymptotic properties of the Mittag–Leffler function, which depend on the fractional order α. These findings provide a deeper understanding of fractional-order damped oscillatory systems and offer a more generalized framework for analyzing dissipative processes in engineering, physics, and control systems.

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1. INTRODUCTION

Fractional calculus, an extension of classical calculus, has gained significant attention due to its ability to describe complex physical and engineering phenomena, particularly those exhibiting memory effects and anomalous diffusion [1]. Unlike traditional integer-order differentiation and integration, fractional calculus allows derivatives and integrals of arbitrary real or complex orders, making it highly applicable in various fields such as viscoelasticity, control systems, and diffusion processes [2], [3], [4]. Reference [5] presents a broad survey covering both theoretical and applied aspects of fractional calculus. The concept of fractional differentiation can be traced back to a correspondence in 1695 between L'Hôpital and Leibniz, in which Leibniz speculated on the meaning and possibility of derivatives of non-integer order [6], [7]. Over time, several definitions of fractional derivatives have been introduced, including those of Riemann-Liouville, Caputo, and Grünwald-Letnikov, each serving different analytical and numerical purposes [8], [9].

A key development in fractional calculus is the Mittag-Leffler function, first introduced by Gösta Mittag-Leffler in 1903, which plays a fundamental role in solving fractional differential equations (FDEs) [10], [11]. This function generalizes exponential function and provides a powerful tool for modeling various dynamical processes, particularly those governed by fractional-order differential equations. It has been widely applied in fields such as anomalous diffusion, relaxation phenomena, and wave propagation [12]. The asymptotic behavior of the Mittag-Leffler function allows it to capture the memory and hereditary properties inherent in fractional order systems, which makes it essential to accurately describe dissipative and oscillatory processes [13].

One of the classic mathematical physics principles is the study of damped harmonic oscillations, where the system undergoes a restoring force proportional to its displacement and an opposing damping force [14]. Traditional models of integer order describe damping effects using exponential decay, but in many physical systems, such as viscoelastic materials and electrical circuits, damping does not follow a simple exponential law [15]. Instead, a fractional-order approach provides a more accurate representation by incorporating memory effects, leading to algebraic decay rather than purely exponential behavior [1]. In this context, fractional differential equations offer a more general and realistic framework for analyzing damped oscillators, where the fractional derivative order influences the decay rate and oscillatory behavior [2], [6].

Although fractional oscillatory systems have been extensively studied, much of the existing literature primarily emphasizes numerical methods rather than explicit analytical solutions [9]. Furthermore, the role of the Mittag-Leffler function in describing the behavior of damped oscillatory systems remains underexplored, especially in connection with the Laplace transform method [13]. This study addresses these gaps by deriving a closed-form analytical solution to the fractional differential equation of a damped oscillator using the Laplace transform and the Mittag-Leffler function. By employing the Caputo derivative of order $0 < \alpha < 1$, the proposed formulation offers deeper insights into the effects of fractional-order damping on oscillatory dynamics.

This study seeks to address that gap by deriving explicit analytical expressions for the linear fractionally damped oscillator using the Caputo derivative and Laplace transform approach. The analytical formulation, solution, and discussion are presented in the subsequent sections.

2. RESEARCH METHODS

As mentioned in the previous chapter, our focus is limited to deriving analytical solutions for the fractional equations under consideration. In other words, we do not aim to explore real-world applications of these solutions. Our main interest lies in analyzing several notable properties of the solutions, particularly their exponential or sinusoidal characteristics and the influence of the damping coefficient α.

In this section, we begin by presenting several foundational concepts, including the Gamma function, fractional integrals, and fractional derivatives. The Mittag–Leffler function is then introduced as a generalization of the complex exponential function. We conclude the section with a brief review of the Laplace transform, emphasizing its application to expressions involving the Caputo fractional derivative.

2.1 Gamma Function

The Gamma function $\Gamma(\alpha)$ is a generalized form of the factorial notation $n!$ for all natural numbers, i.e. $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ [9]. The formal definition for this function is given below.

Definition 1 Suppose $\alpha \in \mathbb{C}$, with $Re(\alpha) > 0$, then

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \tag{1}$$

is called the Gamma function or the second type of Euler integral.

We can derive another important result based on Definition 1. Using the substitution method on Eq. (1) yields the following important result

$$\Gamma(\alpha+1) = \int_0^\infty e^{-t} t^{\alpha+1-1} dt = \lim_{b\to\infty} \int_0^b e^{-t} t^{\alpha} dt = \alpha\Gamma(\alpha).$$

Clearly, we obtain for $\alpha \in \mathbb{N}$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \Leftrightarrow \Gamma(\alpha+1) = \alpha!$$

2.2 Fractional Integral

The fractional integral is generalization of Cauchy's theorem for closed recurrent integrals [3]. The following definition is arguably the most important theory before proceeding to write down the definition for fractional form for the integral.

Definition 2 [8] Suppose f is continuous on the closed interval $[0,t], t > 0$ with $f \in \mathbb{C}$, then

$$\int_0^t \left(\int_0^{\xi_n} \left(\dots \left(\int_0^{\xi_2} f(\xi_1) d\xi_1 \right) \dots \right) d\xi_{n-1} \right) d\xi_n = \frac{1}{(n-1)!} \int_0^t (t-\xi)^{n-1} f(\xi) d\xi.$$

The above definition is applied for an arbitrary order of $\alpha \in \mathbb{R}^+$. Now, we are ready to define the integral form with fractional order α.

Definition 3 [9] Suppose $f \in \mathbb{C}[0,t]$ with $\alpha, t \in \mathbb{R}^+$, then

$$I^\alpha [f(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi,$$

is called the fractional Riemann-Liouville integral of order α where $\Gamma(*)$ is Gamma function. We will make use of Definition 3 in order to define the fractional derivative.

2.3 Fractional Derivative

Several types of fractional derivatives have been developed, but the most commonly used are Riemann-Liouville type fractional derivatives and Caputo type fractional derivatives [1]. Other fractional derivative definitions, like the Caputo–Fabrizio approach, have been introduced to eliminate the singular kernel issue present in classical formulations [16]. Caputo type fractional derivatives are defined as follows.

Definition 4 [11] [17] Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ and $n-1 < \alpha < n$, then

$$D^\alpha f(t) = I^{n-\alpha} D^n [f(t)] = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi,. \tag{2}$$

Recent developments have extended the Caputo derivative to more generalized forms, including definitions that involve differentiation with respect to another function [18].

2.4 Differential Equation

In order to establish the definition for a fractional differential equation, we need to review the definition for the differential equation. Consider the following definition that covers both ordinary and partial differential equations.

Definition 5 [14] A differential equation is an equation involving one or more independent variables, one or more dependent variables, and one or more functions that have one or more derivatives $\left(\text{e.g. } y', y'', \ldots, y^{(n)} \text{ with } n \in \mathbb{N}\right)$, where $y^{(n)}$ is the nth derivative of y.

In similar ways, we may conclude that the differential equations are very sensitive regarding what happens to both dependent and independent variables. Consider the following definition for fractional type of differential equation.

Definition 6 [1] A fractional differential equation (FDE) is differential equation of fractional or rational order involving one or more independent variables x and one or more dependent variables.

The general form of the FDE of order α is as follows

$$c_1 D^{\alpha_1} x^n(t) + c_2 D^{\alpha_2} x^{n-1}(t) + \cdots + c_m D^{\alpha_m} x(t) = f(t), \quad (3)$$

with $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 0$ are the orders of the derivative, $c_1, c_2, \ldots, c_m$ are the real constants, $x(t)$ is the solution function and $f(t)$ is the condition function in $t \geq 0$. In general, the linear FDE of order α can written as the following equation

$$D^{\alpha_1} x(t) + C_1 D^{\alpha_2} x(t) + \cdots + C_m D^{\alpha_m} x(t) = f(t) \quad (4)$$

where $x(t)$ is a continuous-differentiable function with $(t \geq 0)$ and C_m are constant coefficients with $m \in \mathbb{N}$.

2.5 Mittag-Leffler Function

The Gamma function is a generalized form of the factorial function, while the Mittag-Leffler function is a generalized form of the exponential function. The Mittag-Leffler function was introduced by Gosta Mittag-Leffler in 1902 to 1905 and written in five articles on the analytic representation of single-valued branches of monogenic functions [10]. The exponential function e^z plays a fundamental role in integer-order differential equations and can be expressed as a power series

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i+1)}.$$

Definition 7 [10] The one parameter Mittag-Leffler function is given by

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}, \quad (5)$$

with $\alpha, \beta > 0, i \in \mathbb{N}$ and $z \in \mathbb{C}$.

Next, we may proceed to obtain the first derivative of $E_{\alpha,\beta}(z)$. Upon obtaining the formula for second, third, and so on – derivatives, we may rewrite there results in the following theorem.

Theorem 1 [10] *Suppose $\alpha, \beta > 0$ and $k \in \mathbb{N}$, then the k –th order derivative of Mittag-Leffler function is given by*

$$E_{\alpha,\beta}^{(k)}(z) = \sum_{i=0}^{\infty} \frac{(i+k)!}{i!} \frac{z^i}{\Gamma\big((i+k)\alpha + \beta\big)}. \quad (6)$$

We then use Theorem 1 to obtain the fractional derivative for a particular function, i.e. $y(t) = E_{\alpha,1}(\mu t^\alpha)$.

Theorem 2 [6] *Suppose $\alpha > 0$ and μ is a constant with $y(t) = E_{\alpha,1}(\mu t^\alpha), t \geq 0$. Then*

$$D^\alpha y(t) = \mu y(t).$$

From Theorem 2 we can obtain the generalized derivative form for order $m\alpha$ in the following expressions

$$D^\alpha y(t) = \sum_{i=1}^{\infty} \frac{\mu^i t^{(i-1)\alpha}}{\Gamma\big((i-1)\alpha + 1\big)}$$

$$\vdots$$

$$D^{(m-1)\alpha}y(t) = \sum_{i=m-1}^{\infty} \frac{\mu^i t^{(i-(m-1))\alpha}}{\Gamma\left(\left(i-(m-1)\right)\alpha+1\right)}$$

$$D^{m\alpha}y(t) = \sum_{i=m}^{\infty} \frac{\mu^i t^{(i-m)\alpha}}{\Gamma\left((i-m)\alpha+1\right)} = \sum_{i=0}^{\infty} \frac{\mu^{i+m} t^{i\alpha}}{\Gamma(i\alpha+1)}$$

Hence, we obtain

$$D^{m\alpha}y(t) = \mu^m y(t), \qquad m \in \mathbb{N}. \tag{7}$$

2.6 Laplace Transformation

Laplace transform is an operational method that transforms a differential equation from time dimension t into a new dimension with independent variable s, where $s \in \mathbb{C}$. In the other hand, the inverse Laplace transform is a transformation from s to t dimension [13].

Definition 8 [19] Suppose $f(t)$ is a function of t. The Laplace transform of $f(t)$, expressed in $F(s)$ or $\mathcal{L}[f(t)]$, is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt\,, \qquad s \in \mathbb{C}, \tag{8}$$

if the integral exists.

We can derive several important properties of Laplace Transform based on Definition 8.

Theorem 3 [14] *If* $\mathcal{L}[f(t)] = F(s)$, *then*

$$\mathcal{L}[f'(t)] = sF(s) - f(0). \tag{9}$$

Eq. (9) can be generalized for the nth derivative as follows

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n F(s) - \sum_{i=0}^{n-1} s^{n-i} f^{(i)}(0). \tag{10}$$

We now turn our attention into the Laplace transform with the involvement of fractional derivatives.

Theorem 4 [1] [2] *Suppose* $\mathcal{L}[f(t)] = F(s)$ *and* D^{α} *is a Caputo fractional derivative operator, then*

$$\mathcal{L}[D^{\alpha}f(t)] = s^{\alpha}F(s) - \sum_{i=0}^{n-1} s^{\alpha-i} f^{(i)}(0), \tag{11}$$

with $n-1 < \alpha \leq n$.

Now, we are ready to connect the Mittag-Leffler function and Laplace transform as stated in the following theorem.

Theorem 5 [2] *If* $\alpha > 0, \mu \in \mathbb{R}$ *and* $s^{\alpha} > |\mu|$, *then*

$$E_{\alpha}(-\mu t^{\alpha}) = \mathcal{L}^{-1}\left[\frac{s^{\alpha}}{s(s^{\alpha}+\mu)}\right]. \tag{12}$$

The importance of the Mittag–Leffler function in fractional-order modeling has been demonstrated in numerous applied studies. Its wide-ranging applications appear in the solutions of problems in biology, engineering, physics, and Earth sciences [20]. More specifically, the Mittag–Leffler function serves as a fundamental component in the formulation of solutions to various models, including kinetic equations, random walk processes, and Lévy flights [21].

3. RESULTS AND DISCUSSION

We begin this section by presenting the mathematical model in both its classical (integer-order) and fractional-order forms. We then compare the underdamped and overdamped solutions, accompanied by the corresponding figures that support and illustrate the analysis.

3.1 Damped Oscillator Equation

Consider the following oscillator equation with damping

$$M\frac{d^2y}{dt^2}+\gamma\frac{dy}{dt}+ky=f(t). \tag{13}$$

here, $y(t)$ represents the displacement of the body or the deviation of the spring, $\frac{dy}{dt}$ denotes the velocity of oscillation, and $\frac{d^2y}{dt^2}$ represents the acceleration. The parameters are defined as follows: M is the mass of the body (kg), γ is the damping coefficient, k is the spring constant (N/m), and $f(t)$ is the external force applied at time t.

Eq. (13) becomes a homogeneous and autonomous ordinary differential equation when $f(t)=0$, for which the solution can be obtained using the superposition principle in exponential form. A fractional-order version of Eq. (13), using the Caputo derivative of order α, is given by [15]

$$MD^{2\alpha}y(t)+\gamma D^{\alpha}y(t)+ky(t)=f(t). \tag{14}$$

If $f(t)=0$, then

$$D^{2\alpha}y(t)+2\lambda D^{\alpha}y(t)+\omega_0^2y(t)=0, \tag{15}$$

where $\omega_0^2=\frac{k}{M}$ represents the square of the natural angular frequency (rad/s), and $\lambda=\frac{\gamma}{2M}$ denotes the damping coefficient.

3.2 Classical (Integer Order) Solution

By applying a similar substitution to Eq. (13) with $f(t)=0$, we obtain the classical second-order differential equation:

$$\frac{d^2y}{dt^2}+2\lambda\frac{dy}{dt}+\omega_0^2y(t)=0 \tag{16}$$

Comparing this with the fractional-order model in Eq. (15), we observe that Eq. (16) is recovered when $\alpha=1$. The analytical solution to the classical damped oscillator equation can be obtained using the Laplace transform by substituting Eq. (10) into Eq. (16), resulting in

$$y(t)=e^{-\lambda t}\left[y(0)\cos(\omega t)+\left(\frac{\lambda y(0)+y'(0)}{\omega}\right)\sin(\omega t)\right],\qquad \omega=\sqrt{\omega_0^2-\lambda^2}. \tag{17}$$

When $\lambda>0$ and $\omega_0>\lambda$ (equivalently, $-\omega_0<-\lambda$), the solution $y(t)$ approaches zero as $t\to\infty$. In this case, the system exhibits underdamped oscillations, where the amplitude decays gradually over time. Conversely, if $\omega_0<\lambda$ or $-\omega_0>-\lambda$, the solution given in Eq. (17) takes the form

$$y(t)=e^{-\lambda t}\left[y(0)\cosh(|\omega|t)+\left(\frac{\lambda y(0)+y'(0)}{|\omega|}\right)\sinh(|\omega|t)\right] \tag{18}$$

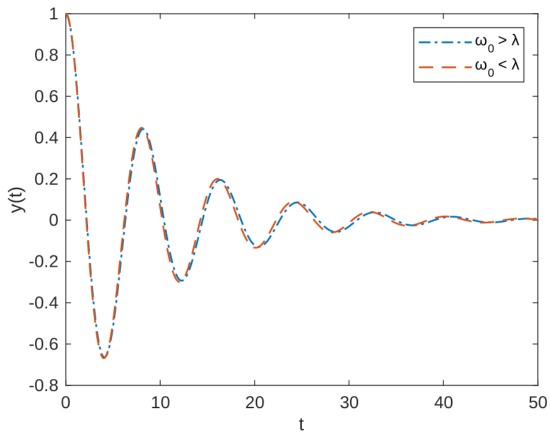

Figure 1. Comparison of Solutions of the Classical Damped Oscillator Eq. (17) for TWO CASES: Underdamped ($\omega_0>\lambda$) and Overdamped ($\omega_0<\lambda$).

The solution Eq. (18) forms a slowly decreasing function that converges to 0. Its decaying characteristic is in accordance with the property of hyperbolic sine and cosine function, i.e.

$$e^{-\lambda t} \cdot \cosh(|\omega| t) = \frac{e^{(|\omega|-\lambda)t} + e^{-(|\omega|-\lambda)t}}{2} \to 0$$

and

$$e^{-\lambda t} \cdot \sinh(|\omega| t) = \frac{e^{(|\omega|-\lambda)t} - e^{-(|\omega|-\lambda)t}}{2} \to 0.$$

We conclude that, in any case regarding the nonnegative value of λ and ω_0, the solutions of Eq. (16) always converge to 0.

Therefore, irrespective of the relative magnitudes of λ and ω_0, as long as both are nonnegative, the solution to Eq. (16) asymptotically tends to zero. This aligns with the physical intuition that in the presence of damping, the system's energy dissipates over time, ultimately stabilizing at equilibrium. The classical model thus serves as a limit case of the broader fractional-order framework, which introduces additional flexibility in characterizing memory and hereditary effects in damped dynamical systems.

Fig. 1 presents the time-domain responses of the classical damped harmonic oscillator as governed by Eq. (17), under two distinct damping regimes. The red dashed curve corresponds to the overdamped case with parameters $\lambda = 0.1$, $\omega_0 = 0.6$, while the blue dashed curve represents the underdamped case with $\lambda = 0.6$, $\omega_0 = 0.1$. Despite the same initial conditions, the two scenarios exhibit qualitatively different dynamical behaviors. In the underdamped case $(\omega_0 > \lambda)$, the solution exhibits oscillatory behavior with an exponentially decaying envelope, indicative of sustained, yet diminishing, energy in the system. The presence of a nonzero natural frequency results in sinusoidal motion modulated by the damping coefficient.

In contrast, the overdamped case $(\lambda > \omega_0)$ is characterized by a slower, non-oscillatory return to equilibrium. The response is governed by a combination of two exponentially decaying terms, resulting in a monotonic decay without overshoot. However, due to the relatively small difference between λ and ω_0, the transient behaviors appear visually similar in the initial time span. Notably, as time progresses, both responses asymptotically approach zero, aligning with theoretical predictions that damping, regardless of regime, ultimately drives the system to rest. This comparison highlights how the relative magnitudes of λ and ω_0 govern the transition between oscillatory and non-oscillatory damping in second-order linear systems.

3.3 Fractional Order Damped Oscillator Solution

The analytical solution of the damped oscillator equation in fractional order is obtained using the Laplace transform, based on the fractional derivative property given in Eq. (11). Consider the following transformed equation

$$\mathcal{L}^{-1}[Y(s)] = y(0)\left[\ell_1 \mathcal{L}^{-1}\left(\frac{s^\alpha}{s(s^\alpha - a_1)}\right) + \ell_2 \mathcal{L}^{-1}\left(\frac{s^\alpha}{s(s^\alpha - a_2)}\right)\right]. \tag{19}$$

Utilizing Theorem 5, the inverse Laplace property for the fractional exponential kernel, the time-domain solution becomes

$$y(t) = y(0)[\ell_1 E_\alpha(a_1 t^\alpha) + \ell_2 E_\alpha(a_2 t^\alpha)]. \tag{20}$$

Eq. (20) can be expanded based on Definition 7, resulting in the following series representation

$$y(t) = y(0)\left[\ell_1 \sum_{i=0}^{\infty} \frac{(a_1 t^\alpha)^i}{\Gamma(\alpha i + 1)} + \ell_2 \sum_{i=0}^{\infty} \frac{(a_2 t^\alpha)^i}{\Gamma(\alpha i + 1)}\right]. \tag{21}$$

Here, the parameters are defined as

$$a_{1,2} = \left(-\lambda \pm \sqrt{\lambda^2 - \omega^2}\right)^{\frac{1}{\alpha}}, \ \ell_1 = \frac{-2\lambda - a_1}{a_2 - a_1} \ \text{ and } \ \ell_2 = \frac{-2\lambda + a_2}{a_2 - a_1}.$$

In Eq. (21), two types of damping behavior are considered: weak damping (underdamped) and strong damping (overdamped). These cases are analyzed using various values of the fractional order in the interval $(0 < \alpha \leq 1)$.

3.3.1 Underdamped Case

We consider the case where $\lambda = 0.1$ and $\omega_0 = 0.6$, representing a weakly damped (underdamped) system since the damping coefficient is smaller than the natural frequency. Figure 2 displays the simulation results of Eq. (21) for several values of the fractional order $0 < \alpha \leq 1$, specifically $\alpha = 0.5, 0.9, 0.99$ and 1. The results show that the fractional order α α significantly influences the oscillatory behavior and decay rate of the system. For $\alpha = 0.5$, the solution decays slowly and smoothly without oscillation, reflecting the strong memory effect characteristic of the Mittag-Leffler function at low fractional orders.

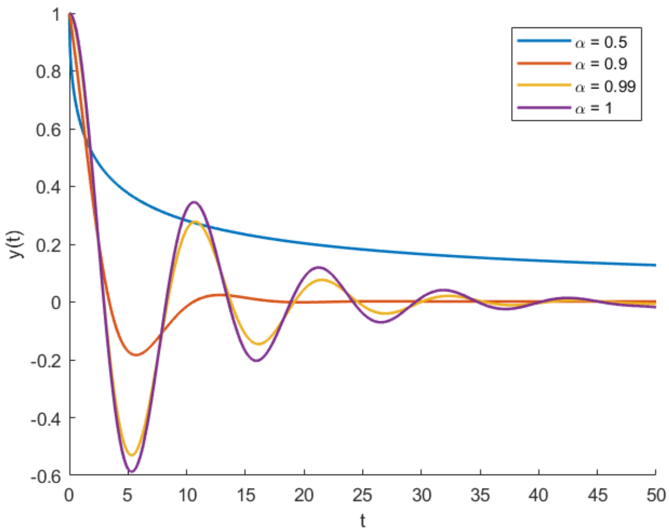

Figure 2. Response of a Weakly Damped Fractional Oscillator with $\lambda = 0.1$, $\omega_0 = 0.6$ for Various Fractional Order α.

As α increases, the damping effect becomes weaker, and oscillations begin to emerge. For instance, at $\alpha = 0.9$, the solution shows moderate damping with visible oscillations, while at $\alpha = 0.99$, the oscillatory behavior becomes more prominent with faster decay. When $\alpha = 1$, the solution corresponds to the classical (non-fractional) damped oscillator, characterized by an exponential decay modulated by a sinusoidal function, and it exhibits the fastest decay among all cases analyzed.

3.3.2 Overdamped Case

Fig. 3 presents the simulation results of Eq. (21) for various fractional orders $0 < \alpha \leq 1$, in the strongly damped (overdamped) case with $\lambda = 0.15$ and $\omega_0 = 0.1$. In this regime, the damping coefficient dominates the system dynamics, causing the solution to decay rapidly without oscillation. The system takes longer to reach equilibrium, and the oscillatory behavior is effectively suppressed. The figure includes four curves corresponding to $\alpha = 0.5, 0.9, 0.99$ and 1. As in the underdamped case, the solution behavior-governed by the Mittag-Leffler function depends strongly on the value of the fractional order α.

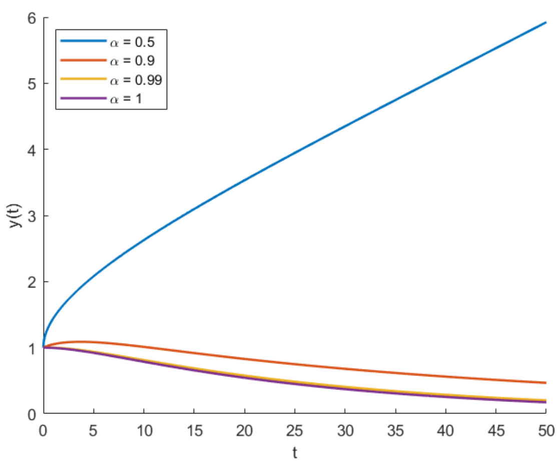

Figure 3. Response of a Strongly Damped Fractional Oscillator with $\lambda = 0.15$, $\omega_0 = 0.1$ for Various Fractional Orders α.

These observations emphasize the role of the fractional order α in shaping the temporal evolution of the system. The Mittag-Leffler function enables a unified analytical description of the system's memory-dependent dynamics, which cannot be captured by conventional integer-order models.

4. CONCLUSION

This study has presented a closed-form analytical solution to the fractional-order damped oscillator equation by employing the Laplace transform alongside the Mittag-Leffler function. The model incorporates the Caputo fractional derivative with order $0 < \alpha < 1$, extending the classical framework to account for systems with memory-dependent damping behavior. The resulting solution reveals that the fractional order α plays a critical role in shaping the system's dynamic response. For lower values of α, the system exhibits slower decay with suppressed oscillations, while as α approaches 1, the behavior aligns closely with that of conventional exponential-sinusoidal damped oscillations. This analytical framework offers a more generalized and flexible approach to modelling damped systems governed by fractional dynamics. The findings provide a theoretical foundation for further studies and may serve as a basis for future extensions involving non-homogeneous systems, external forcing, or nonlinear effects.

Author Contributions

Gusrian Putra: Conceptualization, Software, Validation, Resources, Formal Analysis, Methodology. Meysi Supmawati: Methodology, Writing–Original Draft, Validation, Writing–Review and Editing. Lutfi Mardianto: Validation, Writing–Review and Editing, Investigation. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors have no conflicts of interest.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools were used solely for language refinement (grammar, spelling, and clarity). The scientific content, analysis, interpretation, and conclusions were developed entirely by the authors. The authors reviewed and approved all final text.

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