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THE EXPLICIT FORMULAS OF PARAMETRIZATION OF COADJOINT ORBITS OF THE HEISENBERG LIE GROUP

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ABSTRACT

This research focuses on the Heisenberg Lie group. The aim is to determine the coadjoint orbits and their parametrizations. The method used in this research involves constructing the parametrization of coadjoint orbit for Heisenberg Lie group corresponding to the Heisenberg Lie algebra of dimension 2n+1. Furthermore, the obtained results are specialized to the cases of n=1, 2, and 3 which correspond to the Heisenberg Lie algebras of dimensions 3, 5, and 7. The main results are the explicit formulas of coadjoint orbits for the Heisenberg Lie group H_1, H_2, and H_3 which are expressed by the equations (〖Ad〗^* H_1) l_(α,β,γ)={l_(α^',β^',γ^'):α^',β^',γ^'∈R}, (〖Ad〗^* H_2) l_(α,β,γ)={l_(α^',β^',γ^'):α^',β^'∈R^2,γ^'∈R}, and (〖Ad〗^* H_3) l_(α,β,γ)={l_(α^',β^',γ^'):α^',β^'∈R^3,γ^'∈R}. In addition, their associated parametrizations are given by the explicit formulas ψ(γZ^*,u)=∑_(i=1)^n▒(u_i X_i^*+u_(n+i) Y_i^*) +γZ^* for n=1, 2, and 3. As a further study, various types of Lie groups can be explored to determine coadjoint orbits and their parametrization. Two Lie groups that are interesting to investigate further regarding their coadjoint orbits and parametrization are the diamond and Jacobi groups.

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1. INTRODUCTION

Lie group theory is a significant area of mathematics that intersects algebra and geometry, providing a framework for understanding the structure and geometric properties of various mathematical objects [1], [2], [3]. This theory has profound implications in both pure and applied mathematics, as well as in theoretical physics. Lie groups are essential in understanding the fundamental structures of algebra and topology, including classical linear groups, topological groups, and analytic manifolds [4], [5], [6], [7]. In mathematics, Lie group theory plays a crucial role in understanding the structure and geometric properties of various other mathematical objects [8], [9]. This role includes, among other things, Lie symmetry and solutions to reaction-diffusion systems that arise in biomathematics [10] and Lie theory in several economic and financial problems [11]. Of course, to understand these applications, it is necessary to further comprehend the basic concepts of Lie groups. One of these concepts is matrix Lie groups and their properties.

Matrix Lie groups are defined as closed subgroups of the set of all invertible matrices, making them an essential object of study within both algebra and geometry. These groups can be represented as sets of matrices that respect certain group properties, such as closure under matrix multiplication and inversion, while also forming smooth manifolds, thereby connecting algebraic group theory with differential geometry [12], [13]. Some of the most well-known examples of matrix Lie groups include the Heisenberg Lie group H_n, which plays a significant role in quantum mechanics and harmonic analysis; the orthogonal matrix Lie group $O(n)$, which consists of matrices preserving the Euclidean inner product, and is widely used in physics and computer science; and the special orthogonal matrix Lie group $\mathrm{SO}(n)$, which is a subgroup of $\mathrm{O}(n)$ that further restricts matrices to those with determinant 1, representing rotations in n-dimensional space [8]. The concept of matrix Lie groups is not only crucial for understanding the geometry and structure of matrix representations, but it also provides a framework for studying their actions on associated algebraic structures, particularly Lie algebras and dual Lie algebras. Each matrix Lie group is associated with a Lie algebra, which captures the local, infinitesimal structure of the group, and the dual Lie algebra, which provides a corresponding space of linear functionals. The interaction between a matrix Lie group and its Lie algebra offers profound insights into both the algebraic and geometric properties of the group.

The study of the actions of matrix Lie groups on their Lie algebras and dual Lie algebras leads to the development of Lie group representation theory, a field that bridges the gap between abstract algebra and geometry. Representation theory aims to express abstract Lie groups as concrete groups of matrices, making their structure more accessible and understandable. The action of matrix Lie groups on Lie algebras is intricately linked to the theory of adjoint orbits, which describe how the group acts on its own Lie algebra via conjugation. Similarly, the action of matrix Lie groups on the dual space of the Lie algebra gives rise to the theory of coadjoint orbits, which plays a central role in both symplectic geometry and the orbit method in representation theory. This connection between group actions and orbits is foundational to understanding the broader implications of Lie group theory in both mathematics and theoretical physics.

Research on the theory of coadjoint orbits of Lie groups has been extensively conducted by numerous scholars over the years, contributing to a deep understanding of their structure and significance in various branches of mathematics and physics [14], [15], [16]. These studies have provided a foundation for exploring the relationship between Lie groups and their coadjoint orbits, with significant implications for representation theory, symplectic geometry, and quantum mechanics. The exploration of coadjoint orbits has also led to important advancements in understanding how Lie groups act on their duals, offering insights into the nature of these actions and their representations [17], [18], [19], [20], [22]. On the other hand, Corwin and Greenleaf [8] represents a pivotal development in the field, as they introduced a systematic approach to parameterizing coadjoint orbits of matrix Lie groups. Their method has become a cornerstone in the study of specific Lie groups, particularly matrix Lie groups, where they provided a detailed framework for the parameterization of orbits, including those of the Heisenberg Lie group, as well as other matrix Lie groups. The significance of their contributions lies in the ability to concretely describe and compute the structure of these coadjoint orbits, which is crucial for understanding their geometric and algebraic properties. Motivated by these foundational studies, this article seeks to further explore the concept of coadjoint orbits and delve into the specific process of parameterizing them, with a particular focus on the Heisenberg Lie group [17]. The objective of this article is to derive explicit formulas for the coadjoint orbits associated with the Heisenberg Lie group and to present a detailed parameterization of these orbits. By doing so, this work aims to enhance the understanding of the underlying symplectic geometry of the Heisenberg group and contribute to the broader theory of coadjoint orbits in the context of nilpotent Lie groups.

Since this research prioritizes concrete computations to obtain the explicit formulas of coadjoint orbits of the Heisenberg Lie groups, we only focus on the calculation for the case of $n = 1, 2, 3$ which corresponds Lie algebras of dimension $3, 5$, and 7 so that the formula obtained is easier to understand for those who are interested in studying the representation theory of Lie groups by the orbit method. We thank Corwin and Greenleaf [21] and Kirillov [17] motivation and the general formula of the coadjoint orbit and its parametrization. In addition, we also give interpretations the obtained coadjoint orbit formulas for the case of $n = 1, 2, 3$ including their parameterizations.

2. RESEARCH METHODS

The research method used in this study is descriptive research by analyzing articles related to Heisenberg Lie groups, especially about their coadjoint orbits and their parametrizations. In particular, this reseach focuses on the works of Corwin and Greenleaf [21] and Kirillov [17]. We compute explicitly their results for coadjoint orbit formulas for cases $n = 1,2,3$. At this stage, the authors studied the literature on these works, particularly those related to the parameterization of the coadjoint orbits of a Lie group. This research follows a systematic approach divided into several key stages.

1. The first step in this research methodology is to identify the research problem, which focuses on how to parameterize the coadjoint orbits of the Heisenberg Lie group. This stage involves a thorough review of existing literature and highlights the significance of studying coadjoint orbits in the context of Lie group representation theory and symplectic geometry.
2. The second step is identified, the next step is to conduct a comprehensive review of the literature, specifically focusing on previous studies related to coadjoint orbits and their parameterization, particularly within the Heisenberg Lie group and other matrix Lie groups. This review serves to establish a solid theoretical foundation for the study. Based on the literature review, the next phase involves the development of the theoretical framework. At this stage, the foundational concepts of Lie groups, Lie algebras, and coadjoint orbits are systematically organized, including Kirillov's method, which forms the theoretical basis for the representation of Lie groups through coadjoint orbits.
3. The third step is parameterized the obtained coadjoint orbits. In this step, we give explicit formulas of coadjoint orbit parametrizations.

The detail of our approach can be illustrated in the Fig. 1.

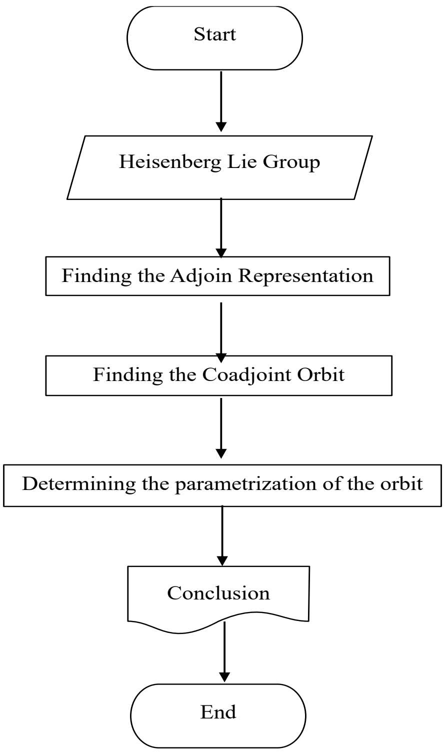

Figure 1. Research Flow Chart

2.1 Preliminaries

In this sub-section, we shall briefly review some basic notions of the Heisenberg Lie group, adjoint representations, coadjoint orbits, and their parametrizations.

Definition 1. [21] The Heisenberg Lie algebra $\mathfrak{h}_n$ of dimension $2n+1$ is the vector space whose basis is $\mathfrak{B} = \{(X)_{i=1}^n, (Y)_{j=1}^n, Z\}$ and whose Lie brackets are given by $[X_k, Y_k] = Z$ and other zeroes for $1 \le k \le n$. Furthermore, an element $\sum_{k=1}^n (x_k X_k + y_k Y_k) + pZ$ of $\mathfrak{h}_n$ corresponds to the matrix of dimension $(n+2)\times(n+2)$ as realized in the following form:

$$\sum_{k=1}^{n}(x_kX_k + y_kY_k) + pZ = \begin{pmatrix} 0 & x_1 & \dots & x_n & p \\ 0 & \ddots & \dots & 0 & y_1 \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & y_n \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{1}$$

Example 1. An element of 3 dimensional Heisenberg Lie algebra whose basis is $S = \{X_1, Y_1, Z\}$ can be represented as 3×3 matrix as follows:

$$x_1X_1 + y_1Y_1 + pZ = \begin{pmatrix} 0 & x_1 & p \\ 0 & 0 & y_1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

Indeed, the matrices X_1, X_2, and Z are given in the following forms:

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We denote $\sum_{k=1}^n (x_k X_k + y_k Y_k) + pZ$ by $(\bar{x}, \bar{y}, p)$ with $\bar{x} = (x_1, x_2, \dots, x_n), \bar{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and $p \in \mathbb{R}$. Let $\bar{x}.\bar{y} = \sum_{k=1}^n x_k y_k$ be the usual dot product on $\mathbb{R}^n$. Then we have the following:

Definition 2. [21] The Heisenberg Lie group H_n is the matrix Lie group whose element can be represented as the exponential coordinates of the element in Eq. (1). Namely we have:

$$\exp(\bar{x}, \bar{y}, p) = \begin{pmatrix} 1 & x_1 & \dots & x_n & p + \frac{1}{2}\bar{x}.\bar{y} \\ 0 & 1 & \dots & 0 & y_1 \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & y_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{3}$$

Example 2. An element of 5 dimensional Heisenberg Lie algebra with the matrix representative

$$A = \begin{pmatrix} 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

can be realized as the exponential matrix

$$\exp(A) = \begin{pmatrix} 1 & 1 & 2 & 16 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 3. [17] Let G be a Lie group whose Lie algebra is given by $\mathfrak{g}$. For $g \in G$, let C_g be the conjugation map given by

$$C_g : G \ni x \mapsto C_g(x) = gxg^{-1} \tag{4}$$

and the derivative of C_g is denoted by $(C_g)_*$. Then, the adjoint representation of G is the mapping

$$\mathrm{Ad}(g) := (C_g)_* : \mathfrak{g} \ni X \mapsto \mathrm{Ad}(g)X \in \mathfrak{g}. \tag{5}$$

Furthermore, in the case of G is a matrix Lie group then Eq. (5) is given by simpler formula as conjugation. Namely, we have $\mathrm{Ad}(g)X = gXg^{-1}$ for every $g \in G$ and $X \in \mathfrak{g}$.

Definition 4. [17]. Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{g}^*$ be a dual vector space of $\mathfrak{g}$. The coadjoint representation of G on the space $\mathfrak{g}^*$ is defined by

$$\mathrm{Ad}^*(g): \mathfrak{g}^* \ni l \mapsto \mathrm{Ad}^*(g)l \in \mathfrak{g}^*. \tag{6}$$

Let $l \in \mathfrak{g}^*$, the coadjoint orbit of G is the set given by:

$$\mathrm{Ad}^*(G)l = \{\mathrm{Ad}^*(g)l | g \in G\} \subseteq \mathfrak{g}^*. \tag{7}$$

Let G be a matrix Lie group whose Lie algebra is $\mathfrak{g}$ and its dual vector space is $\mathfrak{g}^*$. Let X be an element of $\mathfrak{g}$ and let l be an element of $\mathfrak{g}^*$ then:

$$l(W) = \langle l, W\rangle = \mathrm{tr}(lW). \tag{8}$$

Furthermore, Eq. (6) can be given by $\mathrm{Ad}^*(g) := \left(\mathrm{Ad}(g^{-1})\right)^*$ and by following Eq. (8) then we have:

$$\langle \mathrm{Ad}^*(g)l, W\rangle = \left\langle \left(\mathrm{Ad}(g^{-1})\right)^* l, W\right\rangle = \langle l, \mathrm{Ad}(g^{-1})W\rangle \tag{9}$$

with $g \in G$, $W \in \mathfrak{g}$, and $l \in \mathfrak{g}^*$ [17].

3. RESULTS AND DISCUSSION

In this section, we present the key findings of the study, followed by an in-depth discussion of their implications within the context of the Heisenberg Lie group and its coadjoint orbits. The results include explicit parametrizations of the coadjoint orbits, along with their mathematical properties. We also explore how these findings compare with existing literature and discuss the broader significance of these results in the framework of symplectic geometry and representation theory. Our main results are stated in two following propositions.

3.1 Coadjoint Orbits of the 3, 5, and 7-Dimensional Heisenberg Lie Group

The general formula of coadjoint orbits of the Heisenberg group and its parametrization can be found in [21], but we confirm for the special cases of Heisenberg Lie group of dimensions $n = 3, 5, 7$. These dimensions were chosen because they can still be calculated concretely, whereas higher dimensions (≥ 9) are very difficult to calculate in real terms. This is important since it can attract young researchers to understand a representation theory of Lie groups. We believe that our results can give more concrete computations to understand the coadjoint orbits and parametrization of the Heisenberg Lie group. The first result is the explicit formulas of the coadjoint orbits of the Heisenberg Lie groups corresponding to the Heisenberg Lie algebras of dimension 3, 5, and 7.

Proposition 1. *Let H_1, H_2, and H_3 be Heisenberg Lie groups of dimensions 3, 5, and 7 respectively corresponding to their Lie algebras $\mathfrak{h}_1, \mathfrak{h}_2$, and $\mathfrak{h}_3$. Let $\mathfrak{h}_1^*, \mathfrak{h}_2^*$, and $\mathfrak{h}_3^*$ be their dual vector spaces of $\mathfrak{h}_1, \mathfrak{h}_2$, and $\mathfrak{h}_3$ respectively. Then the coadjoint orbits of H_1 at a point $l_{\alpha,\beta,\gamma} \in \mathfrak{h}_1^*$ have two formulas as follows:*

1. $(\mathrm{Ad}^* H_1) l_{\alpha,\beta,\gamma} = \{l_{\alpha',\beta',\gamma'} | \alpha', \beta', \gamma' \in \mathbb{R}\}$, $\gamma \neq 0, \alpha' = \alpha + \gamma y, \beta' = \beta - \gamma x,$ and $\gamma' = \gamma$.
2. $(\mathrm{Ad}^* H_1) l_{\alpha,\beta,\gamma} = \{l_{\alpha,\beta,0} | \alpha, \beta \in \mathbb{R}\}$,
   where $l_{\alpha,\beta,\gamma} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & \beta & 0 \end{pmatrix} = \alpha X^* + \beta Y^* + \gamma Z^* \in \mathfrak{h}_1^*.$
   In similar way, the coadjoint orbits of H_2 at a point $k_{\alpha,\beta,\gamma} \in \mathfrak{h}_2^*$ can be written in the following forms:
3. *The coadjoint orbits of H_2 consists of*: $(\mathrm{Ad}^* H_2) k_{\bar{\alpha},\bar{\beta},\gamma} = \{k_{\bar{\alpha}',\bar{\beta}',\gamma'} | \bar{\alpha}', \bar{\beta}' \in \mathbb{R}^2, \gamma' \in \mathbb{R}\}$, *for* $\gamma \neq 0$, $\bar{\alpha}' = \bar{\alpha} + \gamma\bar{y}$, $\bar{y} = (y_1, y_2) \in \mathbb{R}^2, \beta' = \bar{\beta} - \gamma\bar{x}, \bar{x} = (x_1, x_2)\mathbb{R}^2$, $\gamma' = \gamma$, *and the last*

coadjoint orbits of H_2 *is of the* $(\mathrm{Ad}^*H_2)k_{\bar{\alpha},\bar{\beta},\gamma} = \{k_{\bar{\alpha},\bar{\beta},0} | \bar{\alpha}, \bar{\beta} \in \mathbb{R}^2\}$, *where* $k_{\bar{\alpha},\bar{\beta},\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ \gamma & \beta_1 & \beta_2 & 0 \end{pmatrix} \in \mathfrak{h}_2^*, \bar{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2, \bar{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2, \gamma \in \mathbb{R}$.

Finally, we also have the coadjoint orbits of H_3 *at a point* $m_{\bar{\eta},\bar{\zeta},\gamma} \in \mathfrak{h}_3^*$ *can be written in the following forms*:

4. *The coadjoint orbits of* H_3 *consists of*: $(\mathrm{Ad}^*H_3)m_{\bar{\eta},\bar{\zeta},\gamma} = \{m_{\bar{\eta}',\bar{\zeta}',\delta'} | \bar{\eta}', \bar{\zeta}' \in \mathbb{R}^3, \delta \in \mathbb{R}\}$, *for* $\delta \neq 0$, $\bar{\eta}' = \bar{\eta} + \delta\bar{q}$, $\bar{q} = (q_1, q_2, q_3), \bar{\zeta}' = \bar{\zeta} - \delta\bar{p}, \bar{p} = (p_1, p_2, p_3)$ $\delta' = \delta$, *and the last coadjoint orbits of* H_3 *is of the* $(\mathrm{Ad}^*H_3)m_{\bar{\eta},\bar{\zeta},\delta} = \{m_{\bar{\eta},\bar{\zeta},0} | \bar{\eta}, \bar{\zeta} \in \mathbb{R}^3\}$, *where* $m_{\bar{\eta},\bar{\zeta},\delta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \eta_1 & 0 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 0 \\ \delta & \zeta_1 & \zeta_2 & \zeta_3 & 0 \end{pmatrix} \in \mathfrak{h}_3^*, \bar{\eta} = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3, \bar{\zeta} = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3, \delta \in \mathbb{R}$.

Proof. Firstly, we shall prove for case $n = 1$. We apply Eqs. (6) and (9). Let $g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ be an element of H_1 and $W = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = aX + bY + cZ$ be an element of the Lie algebra $\mathfrak{h}_1$ with $B = \{X, Y, Z\}$ is a basis standard basis for $\mathfrak{h}_1$. Furthermore, for any element $l_{\alpha,\beta,\gamma} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & \beta & 0 \end{pmatrix} = \alpha X^* + \beta Y^* + \gamma Z^*$ of a dual vector space $\mathfrak{h}_1^*$, then we shall compute its coadjoint orbits directly as follows:

$$\langle \mathrm{Ad}^*(g) l_{\alpha,\beta,\gamma}, W \rangle = \langle (\mathrm{Ad}(g^{-1}))^* l_{\alpha,\beta,\gamma}, W \rangle = \langle l_{\alpha,\beta,\gamma}, \mathrm{Ad}(g^{-1})W \rangle$$

$$= l_{\alpha,\beta,\gamma}(\mathrm{Ad}(g^{-1})W) = l_{\alpha,\beta,\gamma}(g^{-1}Wg)$$

$$= l_{\alpha,\beta,\gamma}\left(\begin{pmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$= l_{\alpha,\beta,\gamma}\left(\begin{pmatrix} 0 & a & ay - bx + c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right) = \mathrm{trace}\left\{ \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & \beta & 0 \end{pmatrix} \begin{pmatrix} 0 & a & bx - ay + c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$= \alpha a + \beta b + \gamma c + a\gamma y - b\gamma x = \mathrm{trace}\left\{ \begin{pmatrix} 0 & 0 & 0 \\ \alpha + \gamma y & 0 & 0 \\ \gamma & \beta - \gamma x & 0 \end{pmatrix} \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$= \langle l_{\alpha',\beta',\gamma'} | W \rangle, \forall W \in \mathfrak{h}_1.$$

Recalling that $\langle l_{\alpha',\beta',\gamma'} | W \rangle = l_{\alpha',\beta',\gamma'}(W)$. Therefore, $\mathrm{Ad}^*(g) l_{\alpha,\beta,\gamma} = l_{\alpha',\beta',\gamma'}$ where $\alpha' = \alpha + \gamma y, \beta' = \beta - \gamma x$, and $\gamma' = \gamma$.

If we fix $\gamma \neq 0$, then we have 2-dimensional coadjoint orbits in the plane of the form $(\mathrm{Ad}^*H_1)l_{\alpha,\beta,\gamma} = \{l_{\alpha',\beta',\gamma'} | \alpha', \beta', \gamma' \in \mathbb{R}\}$, with $\alpha' = \alpha + \gamma y, \beta' = \beta - \gamma x$, and $\gamma' = \gamma$ as required. Furthermore, if we fix $\gamma = 0$, then we have points in the plane in the form $(\mathrm{Ad}^*H_1)l_{\alpha,\beta,\gamma} = \{l_{\alpha,\beta,0} | \alpha, \beta \in \mathbb{R}\}$.

Secondly, we shall prove for case $n = 2$. We also apply the equations (6) and (9). Let $h = \begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ be an element of H_2 and $U = \begin{pmatrix} 0 & a_1 & a_2 & c \\ 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = a_1X_1 + a_2X_2 + b_1Y_1 + b_2Y_2 + cZ$ be an element of the Lie algebra $\mathfrak{h}_2$ with $B' = \{X_1, X_2, Y_1, Y_2, Z\}$ is a standard basis for $\mathfrak{h}_2$. Let us fix an element

$k_{\overline{\alpha},\overline{\beta},\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ \gamma & \beta_1 & \beta_2 & 0 \end{pmatrix} = \alpha_1 X_1^* + \alpha_2 X_2^* + \beta_1 Y_1^* + \beta Y_2^* + \gamma Z^* \in \mathfrak{h}_2^*, \overline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2, \overline{\beta} =$ $(\beta_1, \beta_2) \in \mathbb{R}^2, \gamma \in \mathbb{R}$. of a dual vector space $\mathfrak{h}_2^*$, then following for case of coadjoint orbits of H_1 then we have:

$$\begin{aligned}
\langle \mathrm{Ad}^*(h) k_{\overline{\alpha},\overline{\beta},\gamma}, U \rangle &= \langle (\mathrm{Ad}(h^{-1}))^* k_{\overline{\alpha},\overline{\beta},\gamma}, U \rangle = \langle k_{\overline{\alpha},\overline{\beta},\gamma}, \mathrm{Ad}(h^{-1}) U \rangle \\
&= k_{\overline{\alpha},\overline{\beta},\gamma}(\mathrm{Ad}(h^{-1})U) = k_{\overline{\alpha},\overline{\beta},\gamma}(h^{-1} U h) \\
&= k_{\overline{\alpha},\overline{\beta},\gamma}\left(\begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & a_1 & a_2 & c \\ 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \\
&= k_{\overline{\alpha},\overline{\beta},\gamma}\left(\begin{pmatrix} 0 & a_1 & a_2 & c + a_1 y_1 + a_2 y_2 - b_1 x_1 - b_2 x_2 \\ 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \\
&= \mathrm{trace}\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ \gamma & \beta_1 & \beta_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & a_1 & a_2 & c + a_1 y_1 + a_2 y_2 - b_1 x_1 - b_2 x_2 \\ 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \\
&= \alpha_1 a_1 + a_1 \gamma y_1 + \alpha_2 a_2 + a_2 \gamma y_2 + b_1 \beta_1 - b_1 \gamma x_1 + b_2 \beta_2 - b_2 \gamma x_2 + c\gamma \\
&= \mathrm{trace}\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_1 + a_1 \gamma y_1 & 0 & 0 & 0 \\ \alpha_2 + a_2 \gamma y_2 & 0 & 0 & 0 \\ \gamma & \beta_1 - b_1 \gamma x_1 - b_2 \gamma x_2 & \beta_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & a_1 & a_2 & c \\ 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \\
&= \langle k_{\overline{\alpha}',\overline{\beta}',\overline{\gamma}'}, U \rangle = k_{\overline{\alpha}',\overline{\beta}',\overline{\gamma}'}(U), \ \forall\, U \in \mathfrak{h}_2.
\end{aligned}$$

Therefore, $\mathrm{Ad}^*(h) k_{\overline{\alpha},\overline{\beta},\gamma} = k_{\overline{\alpha}',\overline{\beta}',\overline{\gamma}'} \in \mathfrak{h}_2^*$ where $\overline{\alpha}' = \overline{\alpha} + \gamma \overline{y},\ \overline{y} = (y_1, y_2) \in \mathbb{R}^2, \overline{\beta}' = \overline{\beta} - \gamma \overline{x}, \overline{x} = (x_1, x_2) \in \mathbb{R}^2,\ \ \gamma' = \gamma, \overline{\alpha} = (\alpha_1, \alpha_2), \overline{\beta} = (\beta_1, \beta_2)$.

If we fix $\gamma \neq 0$, then we have 4-dimensional coadjoint orbits in the space of the form $(\mathrm{Ad}^* H_2) k_{\overline{\alpha},\overline{\beta},\gamma} = \{ k_{\overline{\alpha}',\overline{\beta}',\overline{\gamma}'} \mid \overline{\alpha}', \overline{\beta}' \in \mathbb{R}^2, \gamma' \in \mathbb{R} \}$ as required. Furthermore, if we fix $\gamma = 0$, then we have points of the form $(\mathrm{Ad}^* H_2) k_{\overline{\alpha},\overline{\beta},\gamma} = \{ k_{\overline{\alpha},\overline{\beta},0} \mid \overline{\alpha}, \overline{\beta} \in \mathbb{R}^2 \}$.

Thirdly, we shall compute all coadjoint orbits of H_3. To do that, let w and w^{-1} be elements of H_3 with the following matrices realizations:

$w = \begin{pmatrix} 1 & \overline{p} & r \\ 0 & 1 & \overline{q} \\ 0 & 0 & 1 \end{pmatrix}$, where $\overline{p} = (p_1, p_2, p_3)$ and $\overline{q} = (q_1, q_2, q_3)$ are elements of $\mathbb{R}^3$. Indeed, w is the 5×5 matrix. Moreover, let $V = \begin{pmatrix} 0 & \overline{a} & c \\ 0 & 0 & \overline{b} \\ 0 & 0 & 0 \end{pmatrix}$ be an element of $\mathfrak{h}_3$ and let $m_{\overline{\eta},\overline{\zeta},\delta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \eta_1 & 0 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 0 \\ \delta & \zeta_1 & \zeta_2 & \zeta_3 & 0 \end{pmatrix}$ be any element of $\mathfrak{h}_3^*$ with $\overline{\eta} = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3, \overline{\zeta} = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3, \delta \in \mathbb{R}$.

Then by direct computations, we also have:

$$\begin{aligned}
\langle \mathrm{Ad}^*(w) m_{\overline{\eta},\overline{\zeta},\delta}, V \rangle &= \langle (\mathrm{Ad}(w^{-1}))^* m_{\overline{\eta},\overline{\zeta},\delta}, V \rangle = \langle m_{\overline{\eta},\overline{\zeta},\delta}, \mathrm{Ad}(w^{-1}) V \rangle \\
&= m_{\overline{\eta},\overline{\zeta},\delta}(\mathrm{Ad}(w^{-1})V) = m_{\overline{\eta},\overline{\zeta},\delta}(w^{-1} V w)
\end{aligned}$$

$$= m_{\bar{\eta},\bar{\zeta},\delta}\left(\begin{pmatrix}1 & \bar{p} & r\\ 0 & 1 & \bar{q}\\ 0 & 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}0 & \bar{a} & c\\ 0 & 0 & \bar{b}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & \bar{p} & r\\ 0 & 1 & \bar{q}\\ 0 & 0 & 1\end{pmatrix}\right)$$

$$= m_{\bar{\eta},\bar{\zeta},\delta}\left(\begin{pmatrix}0 & p_1 & p_2 & p_3 & c + a_1q_1 + a_2q_2 + a_3q_3 - b_1p_1 - b_2p_2 - b_3q_3\\ 0 & 0 & 0 & 0 & q_1\\ 0 & 0 & 0 & 0 & q_2\\ 0 & 0 & 0 & 0 & q_3\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\right)$$

$$= \text{trace}\left\{\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ \eta_1 & 0 & 0 & 0 & 0\\ \eta_2 & 0 & 0 & 0 & 0\\ \eta_3 & 0 & 0 & 0 & 0\\ \delta & \zeta_1 & \zeta_2 & \zeta_3 & 0\end{pmatrix}\begin{pmatrix}0 & p_1 & p_2 & p_3 & c + a_1q_1 + a_2q_2 + a_3q_3 - b_1p_1 - b_2p_2 - b_3q_3\\ 0 & 0 & 0 & 0 & q_1\\ 0 & 0 & 0 & 0 & q_2\\ 0 & 0 & 0 & 0 & q_3\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\right\}$$

$$= \eta_1a_1 + a_1\delta q_1 + \eta_2a_2 + a_2\delta q_2 + \eta_3a_3 + a_3\delta q_3 + b_1\zeta_1 - b_1\delta p_1 + b_2\zeta_2 - b_2\delta p_2 + b_3\zeta_3 - b_3\delta p_3 + c\delta$$

$$= \text{trace}\left\{\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ \eta_1 + \delta q_1 & 0 & 0 & 0 & 0\\ \eta_2 + \delta q_2 & 0 & 0 & 0 & 0\\ \eta_3 + \delta q_3 & 0 & 0 & 0 & 0\\ \delta & \zeta_1 - \delta p_1 & \zeta_2 - \delta p_2 & \zeta_3 - \delta p_3 & 0\end{pmatrix}\begin{pmatrix}0 & \bar{a} & c\\ 0 & 0 & \bar{b}\\ 0 & 0 & 0\end{pmatrix}\right\}$$

$$= \langle m_{\bar{\eta}',\bar{\zeta}',\delta'}, V\rangle = m_{\bar{\eta}',\bar{\zeta}',\delta'}(V), \qquad \forall\, V \in \mathfrak{h}_3.$$

Therefore, $\text{Ad}^*(w)m_{\bar{\eta},\bar{\zeta},\delta} = m_{\bar{\eta}',\bar{\zeta}',\delta'} \in \mathfrak{h}_3^*$ where $\bar{\eta}' = \bar{\eta} + \delta\bar{q}$, $\bar{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$, $\bar{\zeta}' = \bar{\zeta} - \delta\bar{p}$, $\bar{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$, $\delta' = \delta$, $\bar{\eta} = (\eta_1, \eta_2, \eta_3)$, $\bar{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$.

If we fix $\delta \neq 0$, then we have 6-dimensional coadjoint orbits in the space of the form $(\text{Ad}^*H_3)m_{\bar{\eta},\bar{\zeta},\delta} = \{m_{\bar{\eta},\bar{\zeta},\delta} | \bar{\eta}, \bar{\zeta} \in \mathbb{R}^3\}$ as required. Furthermore, if we fix $\delta = 0$, then we have points of the form $(\text{Ad}^*H_3)m_{\bar{\eta},\bar{\zeta},\delta} = \{m_{\bar{\eta},\bar{\zeta},0} | \bar{\eta}, \bar{\zeta} \in \mathbb{R}^3\}$. ■

3.2 Parametrization Coadjoint Orbits of the 3, 5, and 7-Dimensional Heisenberg Lie Groups

In this sub-section, we state the second result of parametrizations of coadjoint orbits of the Heisenberg Lie groups corresponding to the Heisenberg Lie algebras of dimension $3, 5$, and 7. The general formula of coadjoint orbits of Heisenberg Lie group is obtained in [21], however we bring down it to earth by direct computations for low dimensions.

Proposition 2. *Let H_1, H_2, and H_3 be Heisenberg Lie groups of dimensions 3, 5, and 7 whose their Lie agebras are $\mathfrak{h}_1, \mathfrak{h}_2$, and $\mathfrak{h}_3$ corresponding to their dual vector spaces $\mathfrak{h}_1^*, \mathfrak{h}_2^*$, and $\mathfrak{h}_3^*$. Let S and T be disjoint sets of indices satisfying $S \cup T = \{1,2,3\}$. Then parametrizing variables of coadjoint orbits of H_1 in $U = \{f \in \mathfrak{h}_1^* \,; \langle f, Z\rangle = f(Z) \neq 0\ \}$ are given by $u_1 = -\gamma x$ and $u_2 = \gamma y$. Moreover, if $B = \{X, Y, Z\}$ is a basis for $\mathfrak{h}_1$, it is also obtained that*

$$\tau: (U \cap V_T) \times V_S \ni (\gamma Z^*, \iota) \mapsto \gamma Z^* + u_1Y + u_2X \in U,$$

with $V_S = span\,\{X^*, Y^*\} \subseteq \mathfrak{h}_1^*$ and $V_T = span\,\{Z^*\}$.

Furthermore, parametrizing variables of coadjoint orbits of H_2 in $U' = \{\psi \in \mathfrak{h}_2^* \,; \langle \psi, Z'\rangle = \psi(Z') \neq 0\ \}$ are given by $v_1 = -\gamma x_1, v_2 = -\gamma x_2$ and $v_3 = \gamma y_1, v_4 = \gamma y_2$. Moreover, if $B' = \{X_1, X_2, Y_1, Y_2, Z'\}$ is a basis for $\mathfrak{h}_1$, it is also obtained that

$$\chi: (U' \cap V'_{T'}) \times V'_{S'} \ni (\gamma Z'^*, \iota') \mapsto \gamma Z'^* + v_1Y_1 + v_2Y_2 + v_3X_1 + v_4X_2 \in U',$$

with $V'_{S'} = span\,\{X_1^*, X_2^*, Y_1^*, Y_2^*\} \subseteq \mathfrak{h}_2^*$, $V'_{T'} = span\,\{Z'^*\}$, and $S' \cup T' = \{1,2,3,4,5\}$.

The last one, let $B'' = \{P_1, P_2, P_3, Q_1, Q_2, Q_3, R\}$ be a basis for $\mathfrak{h}_3$ and let $S'' \cup T'' = \{1,2,3, \dots, 7\}$ be the union of set indeces S'' and T''. Then , parametrizing variables of coadjoint orbits of H_3 in $U'' = \{\varphi \in \mathfrak{h}_3^* \,; \langle \varphi, R\rangle = \varphi(R) \neq 0\ \}$ are given by $e_1 = -\delta p_1, e_2 = -\delta p_2, e_3 = -\delta p_3$ and $e_4 = \delta q_1, e_5 = \delta q_2$, and $e_6 = \delta q_3$. Moreover, the map $\varepsilon: (U'' \cap V''_{T''}) \times V''_{S''} \to U''$ is given by

$$\varepsilon(\delta R^*, \iota'') = \delta R^* + \sum_{i=1}^{3} e_i Q_i + \sum_{j=4}^{6} e_j P_j$$

with $V''_{S''} = span\,\{P_1^*, P_2^*, P_3^*, Q_1^*, Q_2^*, Q_3^*\} \subseteq \mathfrak{h}_3^*$, $V''_{T''} = span\,\{R^*\}$.

Proof. Firstly, we shall prove for case $n = 1$. We observe the coadjoint orbits of H_1 as proved Proposition 1. Let H_1 act on $\mathfrak{h}_1^*$ by the coadjoint action $\mathrm{Ad}^* H_1$. Namely, for $g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \simeq (x, y, z) \in H_1$ and $l_{\alpha,\beta,\gamma} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & \beta & 0 \end{pmatrix} = \alpha X^* + \beta Y^* + \gamma Z^*$ of a dual vector space $\mathfrak{h}_1^*$, then we obtained in Proposition 1 that

$$\mathrm{Ad}^*(g) l_{\alpha,\beta,\gamma} = \mathrm{Ad}^*(x, y, z) l_{\alpha,\beta,\gamma} = \gamma Z^* + (\alpha + \gamma y) X^* + (\beta - \gamma x) Y^*.$$

We recall for $\gamma \neq 0$, then we have 2-dimensional coadjoint orbits in the plane of the form $(\mathrm{Ad}^* H_1) l_{\alpha,\beta,\gamma} = \{l_{\alpha',\beta',\gamma'} | \alpha', \beta', \gamma' \in \mathbb{R}\}$, with $\alpha' = \alpha + \gamma y, \beta' = \beta - \gamma x,$ and $\gamma' = \gamma$ as required. Furthermore, since the dimension of $\mathfrak{h}_1^*$ equal to 3, then $S = \{2,3\}$. This implies $U = \{f \in \mathfrak{h}_1^*\ ; \langle f, Z\rangle = f(Z) \neq 0$ with $V_S = span\,\{X^*, Y^*\} \subseteq \mathfrak{h}_1^*$ and $V_T = span\,\{Z^*\}$. Now, let $l_{0,0,\gamma}$ be element of $U \cap V_T = \{\gamma Z^*\ ; \gamma \neq 0\}$. Then

$$\mathrm{Ad}^*(g) l_{0,0,\gamma} = \gamma Z^* + (\gamma y) X^* + (-\gamma x) Y^*.$$

The projection of $\mathrm{Ad}^*(g) l_{0,0,\gamma}$ to V_S yields the parametrization variables of coadjoint orbits *of* H_1 in $U = \{f \in \mathfrak{h}_1^*\ ; \langle f, Z\rangle = f(Z) \neq 0\ \}$ as of forms $u_1 = -\gamma x$ and $u_2 = \gamma y$ as required. As consequences, we get the map

$$\tau: (U \cap V_T) \times V_S \ni (\gamma Z^*, \iota) \mapsto \gamma Z^* + u_1 Y + u_2 X \in U.$$

Secondly, we shall prove for case $n = 2$. Let H_2 acting on $\mathfrak{h}_2$ by coadjoint action $\mathrm{Ad}^* H_2$. Let $\{X_1, X_2, Y_1, Y_2, Z\}$ be the standard basis for $\mathfrak{h}_2$ and let $\{X_1^*, X_2^*, Y_1^*, Y_2^*, Z^*\}$ be the standard basis for $\mathfrak{h}_2^*$. Let us denote $\exp\left(\sum_{i=1}^{2}(x_i X_i + y_i Y_i) + zZ\right)$ with $(x, y, z) \in \mathbb{R}^5$. From Proposition 1, we obtained that:

$$\mathrm{Ad}^*(x, y, z)\left(\sum_{i=1}^{2}(\alpha_i X_i^* + \beta_i Y_i^*) + \gamma Z^*\right) = l_{\alpha+y\gamma, \beta-x\gamma, \gamma}.$$

Since each nontrivial orbit $(\gamma \neq 0)$ has dimension 4 and $\mathfrak{h}_2^*$ has dimension 5, the exceptional basis indices must be $S = \{1,2,3,4\}$. Thus $U' = \{l_{\alpha,\beta,\gamma} \in \mathfrak{h}_2^* | l_{\alpha,\beta,\gamma}(Z) \neq 0\}$, $V_S = span\{X_1^*, X_2^*, Y_1^*, Y_2^*\}$, and $V'_{T'} = span\{Z^*\}$. For $l_{0,0,\gamma} \in U' \cap V'_{T'} = \{\gamma Z^* | \gamma \neq 0\}$, the value of the coadjoint action is given by

$$\mathrm{Ad}^*(x, y, z) l_{0,0,\gamma} = \sum_{i=1}^{2}(y_i \gamma X_i^* - x_i \gamma Y_i^*) + \gamma Z^*.$$

By projecting $\sum_{i=1}^{2}(y_i \gamma X_i^* - x_i \gamma Y_i^*) + \gamma Z^*$ to V_S, it can be seen that the parametrizing variable for orbits in U' are $v_1 = y_1 \gamma$, $v_2 = y_2 \gamma$, $v_3 = -x_1 \gamma$, and $v_4 = -x_2 \gamma$. Furthermore, if $\gamma \neq 0$, then map $\chi: (U' \cap V'_{T'}) \times V'_{S'} \to U'$ is given by

$$\chi: (U' \cap V'_{T'}) \times V'_{S'} \ni (\gamma Z'^*, \iota') \mapsto \gamma Z'^* + v_1 Y_1 + v_2 Y_2 + v_3 X_1 + v_4 X_2 \in U'$$

as required.

Thirdly, the last one for case $n = 3$. We observe the coadjoint orbits of H_3 as proved in Proposition 1. Let H_3 act on $\mathfrak{h}_3^*$ by the coadjoint action $\mathrm{Ad}^* H_3$. Namely, for $= \begin{pmatrix} 1 & \bar{p} & r \\ 0 & 1 & \bar{q} \\ 0 & 0 & 1 \end{pmatrix}$, where $\bar{p} = (p_1, p_2, p_3)$ and $\bar{q} = (q_1, q_2, q_3)$ are elements of $\mathbb{R}^3$ and $m_{\bar{\eta},\bar{\zeta},\delta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \eta_1 & 0 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 0 \\ \delta & \zeta_1 & \zeta_2 & \zeta_3 & 0 \end{pmatrix}$ be any element of $\mathfrak{h}_3^*$ with $\bar{\eta} = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3, \bar{\zeta} = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3, \delta \in \mathbb{R}.$, then we obtained in Proposition 1 that:

$$\begin{aligned}\mathrm{Ad}^*(w)m_{\bar{\eta},\bar{\zeta},\delta} &= \mathrm{Ad}^*(\bar{p},\bar{q},r)m_{\bar{\eta},\bar{\zeta},\delta}\\ &= \delta R^* + (\eta_1+\delta q_1)P_1^* + (\eta_2+\delta q_2)P_2^* + (\eta_3+\delta q_3)P_3^* + (\zeta_1-\gamma p_1)Q_1^* + (\zeta_2-\gamma p_2)Q_2^*\\ &+ (\zeta_3-\gamma p_3)Q_3^*.\end{aligned}$$

By Proposition 1, for $\gamma \neq 0$, then we have 6-dimensional coadjoint orbits in space of the form $\mathrm{Ad}^*(w)m_{\bar{\eta},\bar{\zeta},\delta} = m_{\bar{\eta}',\bar{\zeta}',\delta'} \in \mathfrak{h}_3^*$ where $\bar{\eta}' = \bar{\eta} + \delta\bar{q}$, $\bar{q} = (q_1,q_2,q_3) \in \mathbb{R}^3, \bar{\zeta}' = \bar{\zeta} - \delta\bar{p}, \bar{p} = (p_1,p_2,p_3) \in \mathbb{R}^3$, $\delta' = \delta, \bar{\eta} = (\eta_1,\eta_2,\eta_3), \bar{\zeta} = (\zeta_1,\zeta_2,\zeta_3)$. Furthermore, since the dimension of $\mathfrak{h}_3^*$ equal to 7, then $S = \{2,3,4,5,6,7\}$. This implies $U'' = \{\varphi \in \mathfrak{h}_3^* \ ; \langle\varphi,R\rangle = \varphi(R) \neq 0\ \}$ with $V''_{S''} = span\,\{P_1^*,P_2^*,P_3^*,Q_1^*,Q_2^*,Q_3^*\} \subseteq \mathfrak{h}_3^*$, $V''_{T''} = span\,\{R^*\}$. By letting let $m_{0,0,\delta}$ be element $U'' \cap V''_{T''} = \{\delta R^*\ ; \delta \neq 0\}$. Then

$$\mathrm{Ad}^*(w)m_{0,0,\delta} = \delta R^* + (\delta q_1)P_1^* + (\delta q_2)P_2^* + (\delta q_3)P_3^* + (-\gamma p_1)Q_1^* + (-\gamma p_2)Q_2^* + (-\gamma p_3)Q_3^*.$$

In matrix realization, we have:

$$\mathrm{Ad}^*(w)m_{0,0,\delta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ \delta q_1 & 0 & 0 & 0 & 0\\ \delta q_2 & 0 & 0 & 0 & 0\\ \delta q_3 & 0 & 0 & 0 & 0\\ \delta & -\delta p_1 & -\delta p_2 & -\delta p_3 & 0\end{pmatrix} \in \mathfrak{h}_3^*.$$

The projection of $\mathrm{Ad}^*(w)m_{0,0,\delta}$ to $V''_{S''}$ yields the parametrization variables of coadjoint orbits *of* H_3 in $U'' = \{\varphi \in \mathfrak{h}_3^* \ ; \langle\varphi,R\rangle = \varphi(R) \neq 0\ \}$ as of $e_1 = -\delta p_1, e_2 = -\delta p_2, e_3 = -\delta p_3$ and $e_4 = \delta q_1, e_5 = \delta q_2$, and $e_6 = \delta q_3$.

Moreover, the map $\varepsilon: (U'' \cap V''_{T''}) \times V''_{S''} \to U''$ is given by

$\varepsilon(\delta R^*, \iota'') = \delta R^* + \sum_{i=1}^{3} e_i Q_i + \sum_{j=4}^{6} e_j P_j$. ∎

4. CONCLUSION

This section summarizes the key findings of the research, highlighting the significance of the derived parametrizations of coadjoint orbits for the Heisenberg Lie group. The conclusions drawn from this study reflect both the theoretical contributions to Lie group representation theory and the practical implications for future research in symplectic geometry and related fields. Based on the results and discussion, it can be concluded that:

1. The explicit form of the coadjoint orbit of the 3, 5, and 7-dimensional Heisenberg Lie group is the set of matrices that are subsets of the dual Heisenberg Lie algebra, expressed as $(\mathrm{Ad}^*H_n)l_{\alpha,\beta,\gamma} = \{l_{\alpha',\beta',\gamma'} | \alpha',\beta' \in \mathbb{R}^n, \gamma' \in \mathbb{R}\}$ for $\gamma \neq 0$ and $(\mathrm{Ad}^*H_n)l_{\alpha,\beta,0} = \{l_{\alpha,\beta,0} | \alpha,\beta \in \mathbb{R}^n\}$ for $\gamma = 0$ with $n = 1$, 2, and 3.
2. The explicit form of parametrization of the coadjoint orbit of the 3, 5, and 7-dimensional Heisenberg Lie group is the mapping ψ: $(U \cap V_T) \times V_S \ni (\gamma Z{\wedge}*, u) \mapsto \psi(\gamma Z{\wedge}*, u) \in U$, expressed as $\psi(\gamma Z^*, u) = \sum_{i=1}^{n}(u_i X_i^* + u_{n+i} Y_i^*) + \gamma Z^*$ with $\gamma \neq 0$ and $n = 1$, 2, and 3.
3. For future research, the orbit methods can be applied to construct a unitary irreducible representation (UIR) of Lie groups including how to find a UIR of Heisenberg Lie group of dimensions $2n + 1$.

Author Contributions

Muhammad Zaky Zachary : Conceptualization, Methodology, Writing-Original Draft, Software, Validation. Edi Kurniadi: Data Curation, Resources, Draft Preparation. Sisilia Sylviani: Formal Analysis, Validation. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare that has no conflicts of interest to report study.

Declaration of Generative AI and AI-assisted Technologies

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