BAREKENG: Journal of Mathematics and Its Applications
September 2026 Volume 20 Issue 3 Page 2075-2084
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol20iss3pp2075-2084

SOME PROPERTIES OF N-INTEGRAL ON SET-VALUED FUNCTIONS

Corina Karim[1*], Cornelia Yosefine Halim[2]

[1,2] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Brawijaya*
Jln. Veteran No 10-11, Ketawanggede, Lowokwaru, Malang, 65145, Indonesia

*Corresponding author's e-mail: * co_mathub@ub.ac.id*

Article Info

Article History:
Received: 21st July 2025
Revised: 8th January 2026
Accepted: 16th March 2026
Available online: 8th April 2026

Keywords:
N-integral
Set-Valued functions.

ABSTRACT

The Upper and Lower N-integrals were introduced in 2019 based on the concept of δ-fine partitions, which also underlies the Henstock–Kurzweil integral. While the integration of set-valued functions was initially developed through measure-theoretic approaches, later studies extended the Henstock–Kurzweil integral to the set-valued setting and compared it with measure-based integrals. In this paper, we study the N-integral for set-valued functions within this framework. We prove that the N-integral satisfies fundamental properties such as boundedness and linearity, and we establish conditions under which it coincides with the Henstock–Kurzweil integral. Our results extend and complement several earlier results on the integration of set-valued functions and Henstock–Kurzweil-type integrals.

How to cite this article:

C. Karim and C. Y. Halim., "SOME PROPERTIES OF N-INTEGRAL ON SET-VALUED FUNCTIONS", *BAREKENG: J. Math. & App.*, vol. 20, iss. 3, pp. 2075-2084, Sep, 2026.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*
Research Article · Open Access

1. INTRODUCTION

Research on the integration of set-valued functions has evolved through several foundational methods. One of the first such contributions came from the economist Aumann [1], who constructed an integral for set-valued functions based on measure theory concepts. This was followed by an alternative definition in [2], also based on measure theory. Both methods rely on measure-theory and have had a lasting impact on subsequent research within the field. Wibowo and Muslikh [3] then studied the Henstock-Kurzweil integral for set-valued functions, only more recently. The Henstock-Kurzweil integral is constructed based on $\delta - fine$ parts, providing a freer framework than the standard Riemann or Lebesgue integral. Building on this idea, Racca and Cabral [4] introduced Upper and Lower N-integrals in 2019. This work discovers the properties of the N-Integral for set-valued functions within a framework of a similar nature, demonstrating that it fulfills critical properties including boundedness, linearity, and characterizes its relationship with the Henstock-Kurzweil integral.

The N-integral represents an innovative category of integral applicable to set-valued functions, situated within the continuum between Riemann integrable functions and Henstock-Kurzweil integrable functions [5]. It possesses the capability to integrate all improper Riemann integrable functions, irrespective of their Lebesgue integrability status [6]. In the case of a Henstock-Kurzweil integrable function, the criterion for N-integrability is characterized by the existence of a null set S such that for any sufficiently fine partial division D_v, the aggregate of the discrepancies between function values and the lengths of the intervals remains less than epsilon [7]. The N-integral also constitutes a novel integral for real-valued functions defined over a closed and bounded interval, positioned between Riemann integrable functions and Henstock-Kurzweil integrable functions. It can integrate improper Riemann integrable functions that do not conform to Lebesgue integrability. A function is deemed N-integrable if and only if it satisfies the conditions of Henstock-Kurzweil integrability and exhibits continuity almost everywhere [6].

This paper presents a significant scholarly contribution by meticulously examining the attributes of the N-integral concerning set-valued functions within the framework of real Banach spaces, which is an area that has not been comprehensively explored in existing academic discourse. While preceding studies, such as [8], have introduced an innovative set-valued integral for scalar-valued functions in relation to set-valued measures, along with an investigation of its convergence theorems and its interrelations with contractive mappings in n-normed spaces, these inquiries did not extend to the conceptualization or the characteristics of the N-integral for multifunction. The distinguishing feature of the current research is that it goes beyond the mere application of the N-integral paradigm in a general context, it is specifically dedicated to demonstrating essential structural properties, such as boundedness, linearity, and the equivalence to Henstock-Kurzweil integrability, for set-valued functions. Although [4] acknowledged the N-integral as a novel instrument capable of integrating all improper Riemann integrable functions, including those that do not fall under the category of Lebesgue integrable functions, those conclusions were predominantly confined to scalar instances. Moreover, earlier investigations [7], [9] have examined the conditions under which N-integrability corresponds with Henstock-Kurzweil integrability, with a particular focus on scalar continuity almost everywhere. Nevertheless, this manuscript broadens that equivalence into the realm of set-valued contexts, an area where such connections had not been previously substantiated. This paper continues the properties previously established in [10] by developing a foundational theoretical framework for optimizing approximate integration of multifunctions, utilizing both fixed and free points within the domain, a direction hinted at but not formally developed in earlier studies.

2. RESEARCH METHODS

Let Y as a real Banach space and Y* as the dual space of Y; see [11] for details. We write ck(Y) to denote the family of all nonempty compact convex subsets of Y. We use

$$B_{Y^*} = \{y^* \in Y^*; \| |y^*| \| \leq 1\},$$

to represent the closed unit ball of Y*, where |y*| denotes the absolute value (modulus) of the linear functional y*, defined by

$$|y^*|(y) = |y^*(y)|, \forall y \in Y.$$

Consequently, the norm ‖ |y*| ‖ is given by

$$|||y^*||| = \sup_{\|y\|\leq 1} |y^*(y)|,$$

which coincides with the usual operator norm of y* in Y*.

Family of $ck(Y)$ is equipped with Hausdorff metric, which is defined as

$$hd(P_1, P_2) = \max\{d(P_1, P_2), d(P_2, P_1)\}, P_1, P_2 \in ck(Y),$$

with $d(P_1, P_2) = \sup_{p_1 \in P_1} \inf_{p_2 \in P_2} \|p_1 - p_2\|$ and $d(P_2, P_1) = \sup_{p_2 \in P_2} \inf_{p_1 \in P_1} \| p_2 - p_1 \|$. We have $(ck(Y), hd)$ to be a complete metric space [12].

Definition 1. For every $T \in ck(Y)$, denote $s(\cdot, T)$ to represent the support functions of T, it defined on Y^* as

$$s(y^*, T) = \sup\{\langle y^*, y\rangle; y \in T\}, \forall\, y^* \in Y^*.$$

Here, ⟨y*, y⟩ denotes the canonical duality pairing between Y* and Y, that is, ⟨y*, y⟩ = y*(y).

Moreover, for any $P, Q \in ck(Y)$, the Hausdorff distance can be characterized in terms of support functions as,

$$hd(P, Q) = \sup_{y^* \in B_Y^*} |s(y^*, P) - s(y^*, Q)|.$$

Example 1. Let $P = conv\{(0,0), (1,0)\}$ be a line segment from (0,0) to (1,0) on the x-axis, and $Q = conv\{(0,0), (0,1)\}$ be a line segment from (0,0) to (0,1) on the y-axis. These are both compact convex subsets of $\mathbb{R}^2$, so $P, Q \in ck(\mathbb{R}^2)$.

For any $y^* \in \mathbb{R}^2$, the support functions of a set $T \subset \mathbb{R}^2$ is:

$$s(y^*, T) = \sup\{\langle y^*, y\rangle; y \in T\}.$$

Let $y^* = (a, b) \in \mathbb{R}^2$. Then:

For P, since all points in P lie on the x-axis between (0,0) and (1,0), we get:

$$s\big((a, b), P\big) = \sup\{a \cdot x + b \cdot 0\,; x \in [0,1]\} = \sup\{ax\,; x \in [0,1]\} = \max\{0, a\}.$$

For Q, since all points in Q lie on the y-axis between (0,0) and (0,1), we get:

$$s\big((a, b), Q\big) = \sup\{a \cdot 0 + b \cdot y\,; y \in [0,1]\} = \sup\{by\,; y \in [0,1]\} = \max\{0, b\}.$$

The Hausdorff metric is defined as:

$$hd(P, Q) = \sup_{y^* \in B_{Y^*}} |s(y^*, P) - s(y^*, Q)|$$

where B_{Y^*} is the unit ball in the dual space $Y^* = \mathbb{R}^2$, i.e., all $y^* = (a, b)$ such that $\sqrt{a^2 + b^2} \leq 1$.

As a few sample directions:

If $y^* = (1,0)$: $s(y^*, P) = 1, s(y^*, Q) = 0$, difference=1.
If $y^* = (0,1)$: $s(y^*, P) = 0, s(y^*, Q) = 1$, difference=1.
If $y^* = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$: $s(y^*, P) = \frac{1}{\sqrt{2}}, s(y^*, Q) = \frac{1}{\sqrt{2}}$, difference=0.

So

$$hd(P, Q) = \sup_{y^* \in B_{Y^*}} |s(y^*, P) - s(y^*, Q)| = 1.$$

In this example, P and Q are orthogonal line segments. Their support functions differ maximally in directions aligned with their respective axes. The Hausdorff distance between them is 1.

Remark 1. On [13], for each $P, Q \in ck(Y)$ and $y^* \in B_Y^*$, note that

i. $s(y^*, \theta P) = \theta s(y^*, P)$ *for every* $\theta \geq 0$,
ii. $s(y^*, P + Q) = s(y^*, P) + s(y^*, Q)$.

Lemma 1. Let Y be a Banach space, $p, q \in Y$ and $P, Q \in ck(Y)$. We define $hd(P, Q) = max\,\{d(P, Q), d(Q, P)\}$ such that:

i. $hd(\{p\}, \{q\}) = \| p - q \|$,
ii. $hd(\lambda P, \lambda Q) = |\lambda| hd(P, Q)$, *for every* $\lambda \in \mathbb{R}$.

Definition 2. Let Y be a Banach Space and $P \subset Y$, we define a convex hull of P by

$$chl(P) = \cap_{P \subset X} X,$$

where X is convex set.

Lemma 2. Let Y be a Banach space, $P, Q \subset Y$, we use $chl(P)$ to denote convex hull of P, then:

i. $chl(P+Q) = chl(P) + chl(Q)$,
ii. $hd(chl(P), chl(Q)) \leq hd(P,Q)$.

Theorem 1. (Shapley-Folkman-Starr). *Let* $n \in \mathbb{N}$, $P_i \in ck(Y)$ *for* $i = 1,2,\ldots,n$, *we have*

$$hd\left(\sum_{i=1}^{n} P_i, \sum_{i=1}^{n} chl(P_i)\right) \leq \frac{\sqrt{n}}{2} max_{i=1,2,\ldots,n} diam(P_i),$$

with n *is dimension of* Y *and* $diam(P_i) = sup_{p,q \in P_i} \|p - q\|$.

Next, we introduced some terms that were used for discussing about definition N-integrals based on [6]. These terms are fundamental to developing a precise and organized picture of what can be meant by an integral of set-valued functions. Precise use of terminology not only helps describe the structure of the N-integral mathematically but also provides readers with a systematic way to follow the development of the theory.

Let $D_v = \{(t_i, [z_j, z_{j+1}]); t_i \in [z_j, z_{j+1}]\}$ partial division on $[m,n]$, describe $I(D_v)$ as

$$I(D_v) = \{[z_j, z_{j+1}];\ (t_i, [z_j, z_{j+1}]) \in D_v\}.$$

Let S be a null set on $[m,n]$ and D_v is partial division on $[m,n]$, define D_{v_S} as

$$D_{v_S} = \{(t_i, [z_j, z_{j+1}]); t_i \in S\}$$

and D_v^* as

$$D_v^* = \{(t_i^*, [z_j, z_{j+1}]); t_i^* \notin S\}.$$

Definition 3. [6]. Let a real valued function f on $[m,n]$ is said to be Riemann integrable to C if for every $\epsilon > 0$ there exists a gauge δ on $[m,n]$, such that for any $\delta - fine$ division of $[m,n]$, $D_{v_S} = \{(t_i, [z_j, z_{j+1}]); t_i \in S\}$, and $D_v^* = \{(t_i^*, [z_j, z_{j+1}]); t_i^* \notin S\}$ on $[m,n]$ obtain

$$\left|(D_v^*)\sum f(t_i^*)(z_{j+1} - z_j) - C\right| < \epsilon,$$

f is said to be N-integrable.

We then say C to be the N-integral of f over $[m,n]$ and write

$$\int_m^n f = C.$$

Henstock-Kurzweil integration on set-valued function first introduced by Muslikh [3]. There also a comparison between Henstock-Kurzweil, Aumann and Debreu integration.

Definition 4 [3]. Set-valued function $F: [m,n] \to ck(Y)$ said to be Henstock-Kurzweil integrable on $[m,n]$ if there exists $H \in ck(Y)$ that satisfying for each $\epsilon > 0$ exists a gauge ρ on $[m,n]$ such that

$$hd\left((D_v)\sum_{i=1}^{n} F(t_i)(z_{j+1} - z_j), H\right) < \epsilon$$

for each partial division $\delta - fine$ $\{(t_i, [z_0, z_1]), \ldots, (t_n, [z_{n-1}, z_n])\}$ on $[m,n]$ or we can write as

$$H = \int_m^n F\, dt.$$

3. RESULTS AND DISCUSSION

This section introduces the concept of the N-integral of set-valued functions with a particular emphasis on investigating the characteristics of the N-integral, notably its linearity and boundedness. In this context, we further analyze the interrelation between the N-integral of set-valued functions and the Henstock-Kurzweil integral for set-valued functions. These comparisons highlight how the N-integral extends classical integration concepts to multifunction, while preserving key analytical properties. The results presented aim to deepen understanding of generalized integration in the context of multifunction and their measurable selections. Fundamental concepts associated with set-valued functions can be referenced in [13], whereas the foundational theorems pertinent to the N-integral of set-valued functions are delineated in [10].

The N-integral, situated within a more extensive classification of set-valued integrals that encompasses Aumann integrals, preserves linearity which are a fundamental characteristic essential for the integration of multifunction in scenarios such as fuzzy measures and stochastic processes and establishes a comprehensive framework for set-valued integration; however, its efficacy is contingent upon certain conditions, including measurability and the characteristics of the underlying measure, which may exhibit variability across different applications [14], [15], [16], [17], [18], [19], [20], [21], [22]. Throughout this paper, we adopt the notation ρ in place of $\delta - fine$ for simplicity.

Definition 5. Let Y be a Banach space, set-valued $G: [m,n] \to ck(Y)$ is N-integrable on $[m,n]$ if there exists a null set S on $[m,n]$ and a set-valued $A \in ck(Y)$ satisfying for every $\epsilon > 0$ there exist ρ division $D_v = \{(t_i^*, [z_j, z_{j+1}])\}; t_i^* \in S\}$ on $[m,n]$ and any $D_v^* = \{(t_i^*, [z_j, z_{j+1}])\}; t_i^* \notin S\}$ such that

$$hd\left((D_v^*) \sum_{i=1}^{n} G(t_i)(z_{j+1} - z_j), A \right) < \epsilon,$$

where A is integral value, we can write as

$$A = (N) \int_m^n G\, dt,$$

with $(N) \int_m^n$ represent N-integral.

Example 2. Let $Y = \mathbb{R}$ (a Banach space with norm $|\cdot|$), on $[0,1]$. If $G(t) = [0,t] \subset \mathbb{R}$ we define a set-valued function $G: [0,1] \to ck(\mathbb{R})$ by

$$G(t) = [0,t] = \{ x \in \mathbb{R}; 0 \le x \le t\},$$

for every $t \in [0,1]$. Let S be a null set.

Take a tagged division $D_v = \{(t_i^*, [t_i, t_{i+1}])\}_{i=1}^{n}$ of $[0,1]$, where $t_i^* \in [t_i, t_{i+1}]$. The mesh is fine enough depending on a gauge ρ.

For each subinterval $[t_i, t_{i+1}]$, compute:

$$G(t_i^*) \cdot (t_{i+1} - t_i) = [0, t_i^*] \cdot (t_{i+1} - t_i) = \{x(t_{i+1} - t_i) | x \in [0, t_i^*]\} = [0, t_i^*(t_{i+1} - t_i)].$$

Then the sum over all intervals:

$$\sum_{i=1}^{n} G(t_i^*) \cdot (t_{i+1} - t_i) = \text{Minkowski sum of interval } [0, t_i^*(t_{i+1} - t_i)].$$

This gives a set-valued sum:

$$\left[0, \sum_{i=1}^{n} t_i^*(t_{i+1} - t_i)\right].$$

Let

$$A = \left[0, \int_0^1 t\, dt\right] = \left[0, \frac{1}{2}\right].$$

We now check:

$$hd\left(\sum_{i=1}^{n} G(t_i^*)\cdot(t_{i+1}-t_i), A\right) < \epsilon,$$

as the partition gets finer, the upper bound of the sum approaches $\frac{1}{2}$, and using Definition 1, the Hausdorff distance between the approximating interval and $A = \left[0,\frac{1}{2}\right]$, become arbitrarily small.

Thus, $G(t) = [0,t]$ is N-integrable on $[0,1]$ and

$$A = (N)\int_0^1 G(t)dt = \left[0,\frac{1}{2}\right].$$

Theorem 2. *Let $G: [m,n] \to ck(Y)$ is set-valued function on $[m,n]$.*

i. *If $G: [m,n] \to ck(Y)$ N-integrable on $[m,n]$, then $chl(G)$ N-integrable on $[m,n]$ with*
$$\int_m^n chl(G)dt = chl\left(\int_m^n (G)dt\right).$$

ii. *If $G: [m,n] \to ck(Y)$ bounded and satisfy (i), then G N-integrable on $[m,n]$ with*
$$\int_m^n chl(G)dt = \int_m^n G\,dt\,.$$

Proof.

i. For any $\epsilon > 0$, there is gauge ρ on $[m,n]$ which is satisfy N integrable on Definition 5 and Lemma 2 (i), we have

$$(D_v^*)\sum_{i=1}^{n} chl(G(t_i))(z_{j+1}-z_j) = chl\left((D_v^*)\sum_{i=1}^{n} G(t_i)(z_{j+1}-z_j)\right).$$

With Lemma 2 (ii), we have

$$hd\left((D_v^*)\sum_{i=1}^{n} chl(G(t_i))(z_{j+1}-z_j), chl\left(\int_m^n G\,dt\right)\right) = hd\left(chl\left((D_v^*)\sum_{i=1}^{n} G(t_i)(z_{j+1}-z_j)\right), chl\left(\int_m^n G\,dt\right)\right)$$
$$\leq hd\left((D_v^*)\sum_{i=1}^{n} G(t_i)(z_{j+1}-z_j), \int_m^n G\,dt\right)$$
$$< \epsilon.$$

ii. For any $\epsilon > 0$, there is gauge $\rho = \rho(\frac{\epsilon}{2}t_i)$ on $[m,n]$ that satisfy $chl(G)$ N-integrable on Definition 5. Choose $\rho > \frac{\epsilon}{2\,r\sqrt{n}}$ where $G(t) \subset B_r(0)$ for any $t \in [m,n]$, with Theorem 1, we have

$$hd\left((D_v^*)\sum_{i=1}^{k} G(t_i)(z_{j+1}-z_j), \int_m^n chl(G)dt\right)$$
$$\leq hd\left((D_v^*)\sum_{i=1}^{k} G(t_i)(z_{j+1}-z_j), (D_v^*)\sum_{i=1}^{k} chl\left(G(t_i)(z_{j+1}-z_j)\right)\right)$$
$$+ hd\left((D_v^*)\sum_{i=1}^{k} chl\left(G(t_i)(z_{j+1}-z_j)\right) + \int_m^n chl(G)dt\right)$$
$$\leq \frac{\sqrt{n}}{2} max_{i=1,2,\ldots,k} diam\left(G(t_i)(z_{j+1}-z_j)\right) + \frac{\epsilon}{2}.$$

Since $G(t) \subset B_r(0)$, then $diam(G(t_i)) \leq diam(B_r(0)) = 2r$, so

$$hd\left((D_v^*)\sum_{i=1}^{k} G(t_i)(z_{j+1}-z_j), \int_m^n chl(G)dt\right) \leq \sqrt{n}.r\rho + \frac{\epsilon}{2} < \epsilon$$

Corollary 1. Let S is null set on $[m,n]$. If $G:[m,n] \to ck(Y)$ N-integrable on $[m,n]$, then N-integral value of G and $chl(G)$ coincide, both are element of $ck(Y)$.

Theorem 3. *If set-valued $G_1:[m,n] \to ck(Y)$and $G_2:[m,n] \to ck(Y)$ N-integrable on $[m,n]$, then:*

i. $G_1 + G_2$ *N-integrable on* $[m,n]$ *and*

$$\int_m^n (G_1 + G_2)dt = \int_m^n G_1 dt + \int_m^n G_2\, dt.$$

ii. γG_1 *N-integrable on* $[m,n]$, *for* $\gamma \in \mathbb{R}$ *and*

$$\int_m^n \gamma G_1 dt = \gamma \int_m^n G_1 dt.$$

Proof.

i. Let $\epsilon > 0$. For G_1 and G_2 set-valued, there exist an integral value $A_1, A_2 \in ck(Y)$ and gauge $\rho'_\epsilon, \rho''_\epsilon$ on $[m,n]$ such that

$$hd\left((D_v^*)\sum_{i=1}^{n} G_1(t_i)(z_{j+1} - z_j), A_1\right) < \frac{\epsilon}{2} \tag{1}$$

for every division ρ_{G_1} $D_v = \{(t_i, [z_j, z_{j+1}]) | i = 1,2, \dots, n\}$ on $[m,n]$ and

$$hd\left((D_v^*)\sum_{i=1}^{n} G_2(t_i)(z_{j+1} - z_j), A_1\right) < \frac{\epsilon}{2} \tag{2}$$

for every division ρ_{G_2} $D_v = \{(t_i, [z_j, z_{j+1}]) | i = 1,2, \dots, n\}$ on $[m,n]$.

If division ρ $D_v = \{(t_i, [z_j, z_{j+1}]) | i = 1,2, \dots, n\}$ from $[m,n]$, then ρ'_ϵ $D_v = \{(t_i, [z_j, z_{j+1}])\}$ and ρ''_ϵ $D_v = \{(t_i, [z_j, z_{j+1}])\}$ from $[m,n]$, so

$$\begin{aligned}
&hd\left((D_v^*)\sum_{i=1}^{n}\big(G_1(t_i) + G_2(t_i)\big)(z_{j+1} - z_j), (A_1 + A_2)\right)\\
&= hd\left((D_v^*)\sum_{i=1}^{n} G_1(t_i)(z_{j+1} - z_j) + (D_v^*)\sum_{i=1}^{n} G_2(t_i)(z_{j+1} - z_j), (A_1 + A_2)\right)\\
&= \sup_{x^* \in B_{X^*}} \left| s\left(x^*, (D_v^*)\sum_{i=1}^{n} G_1(t_i)(z_{j+1} - z_j) + (D_v^*)\sum_{i=1}^{n} G_2(t_i)(z_{j+1} - z_j)\right) \right.\\
&\left. - s(x^*, A_1 + A_2) \right|\\
&\le \sup_{x^* \in B_{X^*}} \left| s\left(x^*, (D_v^*)\sum_{i=1}^{n} G_1(t_i)(z_{j+1} - z_j) - s(x^*, A_1)\right) \right|\\
&+ \sup_{x^* \in B_{X^*}} \left| s\left(x^*, (D_v^*)\sum_{i=1}^{n} G_2(t_i)(z_{j+1} - z_j) - s(x^*, A_2)\right) \right|\\
&< hd\left((D_v^*)\sum_{i=1}^{n} G_1(t_i)(z_{j+1} - z_j), A_1\right) + hd\left((D_v^*)\sum_{i=1}^{n} G_2(t_i)(z_{j+1} - z_j), A_2\right)\\
&< \epsilon.
\end{aligned}$$

It shows that $G_1 + G_2$ N-integrable on $[m,n]$ and

$$\int_m^n (G_1 + G_2)dt = \int_m^n G_1 dt + \int_m^n G_2 dt.$$

ii. For any $\epsilon > 0$, there exist gauge ρ on $[m, n]$ such that

$$hd\left((D_v^*)\sum_{i=1}^{n} G(t_i)(z_{j+1} - z_j), A\right) < \frac{\epsilon}{|\gamma|},$$

for every division ρ $D_v^* = \{(t_i^*, [z_j, z_{j+1}]); t_i^* \notin S\}$. With Lemma 1 we have

$$hd\left((D_v^*)\sum_{i=1}^{n} \gamma G_1(t_i)(z_{j+1} - z_j), \gamma A\right) = |\gamma| hd\left((D_v^*)\sum_{i=1}^{n} G_1(t_i)(z_{j+1} - z_j), A\right)$$

$$\leq |\gamma| \frac{\epsilon}{|\gamma|} = \epsilon.$$

It shows that γG_1 N-integrable on $[m, n]$, with

$$\int_m^n \gamma G_1 dt = \gamma \int_m^n G_1 dt.$$

Definition 6. [3]. Set-valued function $G: [m, n] \to ck(Y)$ said to be Henstock-Kurzweil integrable on $[m, n]$ if there exist $H \in ck(Y)$ that satisfying for each $\epsilon > 0$ exists a gauge ρ on $[m, n]$ such that

$$hd\left((D_v)\sum_{i=1}^{n} G(t_i)(z_{j+1} - z_j), H\right) < \epsilon$$

for each partial division ρ $\{(t_i, [z_0, z_1]), \dots, (t_n, [z_{n-1,} z_n])\}$ on $[m, n]$ or we can write as

$$H = \int_m^n G\, dt.$$

Theorem 4. *If $F: [m, n] \to ck(Y)$ is N-integrable on $[m, n]$ then it is Henstock-Kurzweil integrable on $[m, n]$.*

Proof. Let $F: [m, n] \to ck(Y)$ is N-integrable on $[m, n]$ with A is integral value and S is null set on $[m, n]$. For any $\epsilon > 0$, there exists gauge ρ_1 such that for any division ρ_1 $D_v = \{(t_i, [z_j, z_{j+1}])\}$ from $[m, n]$ we have

$$hd\left((D_v^*)\sum_{i=1}^{n} G(t_i)(z_{j+1} - z_j), A\right) < \frac{\epsilon}{2}.$$

Since S zero measure, there exist gauge ρ_2 such that for any division ρ_2 D_S from $[m, n]$, we have

$$hd\left((D_{v_S})\sum_{i=1}^{n} G(t_i)(z_{j+1} - z_j), 0\right) < \frac{\epsilon}{2}.$$

Then use $\rho(t) = \min\{\rho_1(t), \rho_2(t)\}$. So, for any division ρ D_v from $[m, n]$ we have

$$hd\left((D_v)\sum_{i=1}^{n} G(t_i)(z_{j+1} - z_j), H\right)$$

$$\leq hd\left((D_v^*)\sum_{i=1}^{n} G(t_i)(z_{j+1} - z_j), A\right) + hd\left((D_{v_S})\sum_{i=1}^{n} G(t_i)(z_{j+1} - z_j), 0\right) < \epsilon.$$

It showed that F is set-valued function that Henstock-Kurzweil integrable.

4. CONCLUSION

The N-integral of set-valued functions possesses key mathematical properties, including linearity, and boundedness. These features make the integral behave as expected under processes like scaling as well as additivity. In addition, it is proved that a set-valued function $F: [m, n] \rightarrow ck(Y)$ is N-integrable on the interval $[m, n]$ if it is integrable according to the Henstock-Kurzweil integral. This correspondence indicates the general utility and the stability of the N-integral, locating it on the hierarchy of theories of integration as well as confirming its compatibility with established theories of integration.

Author Contributions

Corina Karim: Conceptualization, Theoretical Framework Development, Methodology, Formal Analysis, Validation, Project Administration, and Validation. Cornelia Yosefine Halim: Investigation, Development of Mathematical Proofs, Formal Analysis, Writing Draft. All authors discussed the mathematical development of the N-integral as a generalization of the Riemann and Henstock–Kurzweil integrals and approved the final manuscript.

Funding Statement

We would like to express our sincere gratitude to the Faculty of Mathematics and Natural Sciences, Universitas Brawijaya. This research was supported by an internal research grant under contract No. 3438/UN10.F09/PN/2023.

Acknowledgment

The authors would like to express their sincere gratitude to the Faculty of Mathematics and Natural Sciences and the Analysis Laboratory, Department of Mathematics, Universitas Brawijaya, for the facilities and support provided during this research. The authors also gratefully acknowledge the valuable assistance and contributions from colleagues and reviewers whose constructive comments helped improve the quality of this manuscript.

Declarations

The authors declare that there are no conflicts of interest associated with this research.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools (e.g., ChatGPT) were used solely for language refinement (grammar, spelling, and clarity). The scientific content, analysis, interpretation, and conclusions were developed entirely by the authors. The authors reviewed and approved all final text.

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