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LEVEL SOFT SEMIRING FROM FUZZY SUBSEMIRING

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ABSTRACT

This paper introduces the concept of a level soft set and a level soft semiring over a semiring derived from a fuzzy subsemiring. By employing the level subset approach of fuzzy sets, we construct soft structures whose images form subsemirings of the underlying semiring. Several fundamental properties of level soft semirings are established, including their behavior under intersection, union, and AND operations. These results extend classical semiring theory into the fuzzy–soft framework and provide a rigorous algebraic foundation for further theoretical developments. The proposed framework may serve as a basis for future applications in information systems, decision-making, and computational algebra.

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1. INTRODUCTION

The concept of fuzzy groups, first introduced by Rosenfeld [1], has inspired extensive research aimed at extending classical algebraic structures to accommodate uncertainty. Subsequent developments include studies on fuzzy groups and level subgroups [2], as well as investigations into the Cartesian product of intuitionistic fuzzy subgroups [3]. These foundational works marked the beginning of fuzzy algebra, where graded membership values are employed to represent uncertainty within algebraic systems.

Alongside these developments, semiring theory has emerged as an important branch of algebra with applications in various mathematical and computational contexts. A semiring is defined as a non-empty set equipped with two binary operations, namely addition and multiplication, where the additive structure forms a commutative semigroup with an identity element, and the multiplicative structure forms a semigroup. These operations are connected through distributive laws [4], [5], [6]. Unlike rings, semirings do not generally require the existence of additive inverses, which gives rise to distinctive algebraic properties and challenges.

The integration of fuzzy set theory into semiring structures has led to the development of fuzzy semirings, which allow graded membership values to be incorporated into algebraic operations. Several studies have contributed to this area, including works on fuzzy ideals and fuzzy interior ideals [7], intuitionistic fuzzy ideals in Γ-semirings [8], Cartesian products of fuzzy subsemirings [9], cross products of fuzzy semiring ideals [10], fuzzy semiring ideals based on level subsets [11], anti-fuzzy subsemirings [12], and structural characterizations of fuzzy semirings [13]. These studies demonstrate the richness of fuzzy semiring theory and its potential for further generalization.

In parallel with fuzzy algebra, Molodtsov [14] introduced soft set theory as a mathematical framework for handling uncertainty through parameterized families of subsets. This approach was later adapted to semiring structures by Feng et al. [15], who introduced the concept of soft semirings. Further developments by Acar et al. [16], Atagün and Sezgin [17], and Murugadas and Thirumagal [18] expanded soft semiring theory by introducing various classes of soft ideals and investigating algebraic operations such as intersection and union.

Despite these advances, the construction of level soft semirings derived explicitly from fuzzy subsemirings remains relatively underexplored [18], [19], [20], [21], [22], [23], [24], [25]. This gap is particularly significant in the context of semirings, where the absence of additive inverses complicates the selection of fuzzy parameters and distinguishes semiring-based constructions from their group-theoretic counterparts. Consequently, existing approaches do not fully capture the interaction between fuzzy membership levels and soft parameterization within semiring structures.

Motivated by this research gap, the present study aims to develop a formal framework for constructing level soft semirings derived from fuzzy subsemirings. The novelty of this work lies in the integration of the level subset technique of fuzzy sets with soft semiring theory, resulting in a unified fuzzy–soft algebraic structure. Unlike existing studies that treat fuzzy semirings and soft semirings independently, this paper establishes level soft semirings whose images form subsemirings of the underlying semiring. Furthermore, the algebraic behavior of level soft semirings under intersection, union, and AND operations is investigated. These results extend classical semiring theory into the fuzzy and soft set domains and provide a rigorous algebraic foundation for further theoretical developments.

2. RESEARCH METHODS

This study adopts a theoretical and analytical research method to investigate the algebraic properties of level soft semirings derived from fuzzy subsemirings. The research is conducted through a sequence of steps, beginning with a review of fundamental concepts related to semirings, fuzzy subsets, and soft sets. Next, level subsets of fuzzy subsemirings are constructed to define level soft sets. Based on these constructions, level soft semirings are introduced, and their algebraic properties are examined. Finally, the behavior of level soft semirings under operations such as intersection, union, and AND operations is analyzed using rigorous mathematical proofs.

According to Głazek [26], a semiring is an algebraic structure $(\mathcal{S},+,\cdot)$, where $\mathcal{S}$ is a non-empty set equipped with two binary operations: addition and multiplication. The addition operation is associative and commutative, and possesses an identity element, typically denoted by $0_{\mathcal{S}}$. The multiplication operation is associative and, in specific contexts, may also be commutative.

The structure $(\mathcal{S},\cdot)$ forms a semigroup, and the two operations are connected through the distributive laws, which require that for all $a,c,d \in \mathcal{S}$, the following identities are held:

$$a\cdot(c+d)=a\cdot c+a\cdot d \text{ and } (a+c)\cdot d=a\cdot d+c\cdot d.$$

In addition, the zero element $0_{\mathcal{S}}$ acts as an absorbing element under multiplication:

$$a\cdot 0_{\mathcal{S}}=0_{\mathcal{S}}\cdot a=0_{\mathcal{S}} \text{ for all } a\in\mathcal{S}.$$

A non-empty subset $\mathcal{R}\subseteq\mathcal{S}$ is called a **subsemiring** of $\mathcal{S}$ if it is closed under both addition and multiplication, that is, for all $a,c\in\mathcal{R}$, we have $a+c\in\mathcal{R}$ and $a\cdot c\in\mathcal{R}$.

A classic example of a semiring is the set of natural numbers, including zero, $\mathbb{N}_0=\{0,1,2,\cdots\}$, under the usual operations of addition and multiplication. In this structure, addition is associative and commutative with the identity element 0, and multiplication is associated with the identity element 1, and the distributive laws are satisfied. Moreover, 0 acts as an absorbing element under multiplication, meaning any element multiplied by zero yields zero. This example illustrates a semiring lacking additive inverses, distinguishing it from a ring.

Furthermore, subsets such as $2\mathbb{N}_0=\{0,2,4,\cdots\}$ and $6\mathbb{N}_0=\{0,6,12,\cdots\}$, which consist of all non-negative multiples of 2 and 6, respectively, are subsemirings of $\mathbb{N}_0$. These subsets are closed under both addition and multiplication, and they inherit the semiring properties from $\mathbb{N}_0$, making them valid subsemirings. Moreover, since every member of $6\mathbb{N}_0$ is also contained in $2\mathbb{N}_0$, and the operations remain closed within $6\mathbb{N}_0$, it follows that $6\mathbb{N}_0$ is also a subsemiring of $2\mathbb{N}_0$.

This structure generalises the concept of a ring, as it does not require the existence of additive inverses or a multiplicative identity.

In addition, we present the definition of a fuzzy subset as introduced by Abdurrahman et al. [3]. The concepts of level subsets and fuzzy subsemirings are adopted from Abdurrahman [11], [13] and Ahsan et al. [27]. Meanwhile, the notions of soft sets and their associated operations—including intersection, AND, and union—are referenced by Abdurrahman [28], Abdurrahman et al. [29], and John [30]. Furthermore, soft semiring is based on the framework proposed by Feng et al. [15].

Definition 1. A fuzzy subset of a non-empty set $\mathcal{S}$ is defined as a function from $\mathcal{S}$ to the closed interval [0,1].

Definition 2. Let η be a fuzzy subset of a semiring $\mathcal{S}$, and let $\mathscr{b}\in[0,1]$. The level subset of η, denoted by $\eta_{\mathscr{b}}$, is defined as:

$$\eta_{\mathscr{b}}:=\{a\in\mathcal{S}\mid \eta(a)\geq\mathscr{b}\}.$$

Definition 3. Let $(\mathcal{S},+,\cdot)$ be a semiring. A fuzzy subset $\eta:\mathcal{S}\longrightarrow[0,1]$ is called a fuzzy subsemiring *of $\mathcal{S}$ if, for all $a,c\in\mathcal{S}$,*

$$\eta(a+c)\geq\eta(a)\wedge\eta(c), \text{ and } \eta(ac)\geq\eta(a)\wedge\eta(c).$$

Definition 4. Let η and σ be fuzzy subsemirings of a semiring $\mathcal{S}$. The intersection *of* η and σ, denoted by $\eta\widetilde{\sqcap}\sigma$, is defined as:

$$\eta\widetilde{\sqcap}\sigma(a):=\eta(a)\wedge\sigma(a),$$

for all $a\in\mathcal{S}$.

Theorem 1. *A fuzzy subset η of a semiring $\mathcal{S}$ is a fuzzy subsemiring of $\mathcal{S}$ if and only if each level subset $\eta_{\mathscr{b}}$ $(\neq\emptyset)$ is a subsemiring of $\mathcal{S}$, for every $\mathscr{b}\in[0,1]$.*

Definition 5. Let $\mathcal{S}$ be a semiring and $\mathcal{C}$ a set of parameters. A **soft set** over $\mathcal{S}$ concerning the parameter set $\mathcal{C}$ is a mapping

$$\mathfrak{f}_{\mathcal{C}}:\mathcal{C}\longrightarrow\mathcal{P}(\mathcal{S}),$$

where $\mathcal{C}$ is a set of parameters and $\mathcal{P}(\mathcal{S})$ denotes the power set of $\mathcal{S}$. For each $c\in\mathcal{C}$, the image $\mathfrak{f}_{\mathcal{C}}(c)\subseteq\mathcal{S}$ represents the subset *of* $\mathcal{S}$ associated with the parameter c.

This notation will be adopted consistently throughout the remainder of this paper. Unless otherwise stated, any expression of the form $\mathfrak{f}_{\mathcal{C}}$ refers to such a mapping and is considered a soft set over $\mathcal{S}$.

Definition 6. Let $\mathfrak{f}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{C}}$ be two soft sets over the semiring $\mathcal{S}$, where $\mathcal{B}$ and $\mathcal{C}$ are parameter sets. We say that $\mathfrak{f}_{\mathcal{B}}$ is a **soft subset** of $\mathfrak{d}_{\mathcal{C}}$, denoted by

$$\mathfrak{f}_{\mathcal{B}} \sqsubseteq \mathfrak{d}_{\mathcal{C}}.$$

If the following two conditions are satisfied:

1. $\mathcal{B} \subseteq \mathcal{C}$.
2. For every parameter $\mathscr{b} \in \mathcal{B}$, it holds that $\mathfrak{f}_{\mathcal{B}}(\mathscr{b}) \subseteq \mathfrak{d}_{\mathcal{C}}(\mathscr{b})$.

This definition ensures that not only are the parameters of $\mathfrak{f}_{\mathcal{B}}$ contained within those of $\mathfrak{d}_{\mathcal{C}}$, but also that the corresponding subsets of the universe $\mathcal{S}$ are nested accordingly.

Furthermore, the soft sets $\mathfrak{f}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{C}}$ are said to be equal, denoted $\mathfrak{f}_{\mathcal{B}} = \mathfrak{d}_{\mathcal{C}}$, if they are mutually soft subsets of each other, that is,

$$\mathfrak{f}_{\mathcal{B}} \sqsubseteq \mathfrak{d}_{\mathcal{C}} \text{ and } \mathfrak{d}_{\mathcal{C}} \sqsubseteq \mathfrak{f}_{\mathcal{B}}.$$

This implies that $\mathcal{B} = \mathcal{C}$ and $\mathfrak{f}_{\mathcal{B}}(\mathscr{b}) = \mathfrak{d}_{\mathcal{C}}(\mathscr{b})$ for every $\mathscr{b} \in \mathcal{B}$.

Definition 7. Let $\mathcal{S}$ be a semiring and let $\mathfrak{f}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{C}}$ be two soft sets over $\mathcal{S}$, where $\mathcal{B}$ and $\mathcal{C}$ are sets of parameters. Suppose that $\mathcal{D} = \mathcal{B} \cap \mathcal{C} \neq \emptyset$. The intersection of $\mathfrak{f}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{C}}$, denoted by $\mathfrak{f}_{\mathcal{B}} \sqcap \mathfrak{d}_{\mathcal{C}}$, is defined as the soft set $\mathfrak{h}_{\mathcal{D}}: \mathcal{D} \longrightarrow \mathcal{P}(\mathcal{S})$ such that for each $d \in \mathcal{D}$,

$$\mathfrak{h}_{\mathcal{D}}(d) := \mathfrak{f}_{\mathcal{B}}(d) \cap \mathfrak{d}_{\mathcal{C}}(d).$$

This operation yields a new soft set whose parameter set consists of the standard parameters of $\mathfrak{f}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{C}}$, and for each such parameter, the associated value is the intersection of the corresponding subsets of the semiring $\mathcal{S}$. The resulting soft set $\mathfrak{h}_{\mathcal{D}}$ preserves the algebraic structure of the original soft sets within the shared domain and may inherit subsemiring properties under suitable closure conditions.

Definition 8. Let $\mathcal{S}$ be a semiring and let $\mathfrak{f}_{\mathcal{B}}: \mathcal{B} \longrightarrow \mathcal{P}(\mathcal{S})$ and $\mathfrak{d}_{\mathcal{C}}: \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{S})$ be two soft sets over $\mathcal{S}$, where $\mathcal{B}$ and $\mathcal{C}$ are parameter sets. The *AND operation* $\mathfrak{f}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{C}}$, denoted by

$$\mathfrak{f}_{\mathcal{B}} \wedge \mathfrak{d}_{\mathcal{C}},$$

is defined as a soft set $\mathfrak{h}_{\mathcal{B}\times\mathcal{C}}: \mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{S})$, such that for each ordered pair $(\mathscr{b}, c) \in \mathcal{B} \times \mathcal{C}$,

$$\mathfrak{h}_{\mathcal{B}\times\mathcal{C}}(\mathscr{b}, c) = \mathfrak{f}_{\mathcal{B}}(\mathscr{b}) \cap \mathfrak{d}_{\mathcal{C}}(c).$$

This operation induces a new soft set whose parameter domain is the Cartesian product of the original parameter sets and whose value at each parameter pair is the intersection of the corresponding images under $\mathfrak{f}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{C}}$. The resulting structure reflects the joint behaviours of the two soft sets under the intersection operation. It may be interpreted as a binary meet operation in the lattice of soft sets over $\mathcal{S}$.

Definition 9. Let $\mathcal{S}$ be a semiring and let $\mathfrak{f}_{\mathcal{B}}: \mathcal{B} \longrightarrow \mathcal{P}(\mathcal{S})$ and $\mathfrak{d}_{\mathcal{C}}: \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{S})$ be two soft sets over $\mathcal{S}$, where $\mathcal{B}$ and $\mathcal{C}$ are parameter sets. Define $\mathcal{D} = \mathcal{B} \cup \mathcal{C}$. The **union** of $\mathfrak{f}_{\mathcal{B}}$ and $\mathfrak{d}_{\mathcal{C}}$, denoted by

$$\mathfrak{f}_{\mathcal{B}} \sqcup \mathfrak{d}_{\mathcal{C}},$$

is the soft set $\mathfrak{h}_{\mathcal{D}}: \mathcal{D} \longrightarrow \mathcal{P}(\mathcal{S})$ defined by:

$$\mathfrak{h}_{\mathcal{D}}(d) := \begin{cases} \mathfrak{f}_{\mathcal{B}}(d), & d \in \mathcal{B} - \mathcal{C}, \\ \mathfrak{d}_{\mathcal{C}}(d), & d \in \mathcal{C} - \mathcal{B}, \\ \mathfrak{f}_{\mathcal{B}}(d) \cap \mathfrak{d}_{\mathcal{C}}(d), & d \in \mathcal{B} \cap \mathcal{C}. \end{cases}$$

This construction defines a binary operation on the class of soft sets over $\mathcal{S}$, producing a new soft set whose parameter domain is the union of the original domains. The value assigned to each parameter is determined by preserving the original mapping when the parameter is unique to one soft set and intersecting the corresponding images when the parameter is shared.

Definition 10. Let $\mathcal{S}$ be a semiring and $\mathcal{C}$ a set of parameters. Let $\mathfrak{f}_{\mathcal{C}}: \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{S})$ be a soft set over $\mathcal{S}$. The soft set $\mathfrak{f}_{\mathcal{C}}$ is called a **soft semiring** of $\mathcal{S}$ if, for every $c \in \mathcal{C}$, the image $\mathfrak{f}_{\mathcal{C}}(c) \subseteq \mathcal{S}$ is a subsemiring of $\mathcal{S}$; that is, for all $x, z \in \mathfrak{f}_{\mathcal{C}}(c)$, we have:

$$x + z \in \mathfrak{f}_{\mathcal{C}}(c) \text{ and } xz \in \mathfrak{f}_{\mathcal{C}}(c).$$

Prior to analysing the intersection of multiple-level soft semirings, it is necessary to establish a foundational property of semirings: the intersection of two or more subsemirings always results in a subsemiring. This result is pivotal to the following theoretical framework, as each image of a level soft semiring inherently constitutes a subsemiring of the underlying structure.

Accordingly, we present two theorems addressing the behaviours of subsemirings under intersection. The first theorem demonstrates that the intersection of two subsemirings is a subsemiring. The second theorem extends this result to the case of a finite intersection of subsemirings. These results provide a theoretical basis for proving that the intersection of two or more level soft semirings also forms a level soft semiring.

Theorem 2. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be subsemirings of a semiring $\mathcal{S}$. Then the intersection $\mathcal{R}_1 \cap \mathcal{R}_2$ is a subsemiring of $\mathcal{S}$.*

Proof. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are subsemirings of $\mathcal{S}$, it follows that both are non-empty subsets of $\mathcal{S}$ and are closed under the operations of addition and multiplication defined on $\mathcal{S}$. Consider the intersection $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$. We verify the subsemiring conditions as follows:

1. **Non-emptiness**:

   Since both $\mathcal{R}_1$ and $\mathcal{R}_2$ contain the zero element of $\mathcal{S}$, denoted $0_{\mathcal{S}}$, it follows $0_{\mathcal{S}} \in \mathcal{R}_1 \cap \mathcal{R}_2$. Hence, $\mathcal{R} \neq \emptyset$.

2. **Closure under addition**:

   Let $a, c \in \mathcal{R}$. Then $a, c \in \mathcal{R}_1$ and $a, c \in \mathcal{R}_2$. Since both $\mathcal{R}_1$ and $\mathcal{R}_2$ are closed under addition, we have $a + c \in \mathcal{R}_1$ and $a + c \in \mathcal{R}_2$. Thus, $a + c \in \mathcal{R}$.

3. **Closure under multiplication**:

   Similarly, since $a, c \in \mathcal{R}_1 \cap \mathcal{R}_2$, and both $\mathcal{R}_1$ and $\mathcal{R}_2$ are closed under multiplication, it follows that $ac \in \mathcal{R}_1$ and $ac \in \mathcal{R}_2$. Hence, $ac \in \mathcal{R}$.

Therefore, $\mathcal{R}_1 \cap \mathcal{R}_2$ satisfies all the conditions of a subsemiring of $\mathcal{S}$. ■

Theorem 3. *Let $\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_n$ be subsemirings of a semiring $\mathcal{S}$, where $n \geq 2$. Then the intersection $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \cdots \cap \mathcal{R}_n$ is a subsemiring of $\mathcal{S}$.*

Proof. We proceed by mathematical induction on n, the number of subsemirings.

1. **Base case** ($n = 2$):

   The result holds by Theorem 2.

2. **Inductive step:**

   Assume that for some $k \geq 2$, the intersection $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \cdots \cap \mathcal{R}_k$ is a subsemiring of $\mathcal{S}$. Let $\mathcal{R}_{k+1}$ be another subsemiring of $\mathcal{S}$. Consider the intersection:

   $$\mathcal{R}_1 \cap \mathcal{R}_2 \cap \cdots \cap \mathcal{R}_k \cap \mathcal{R}_{k+1} = (\mathcal{R}_1 \cap \mathcal{R}_2 \cap \cdots \cap \mathcal{R}_k) \cap \mathcal{R}_{k+1}$$

   By the inductive hypothesis, $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \cdots \cap \mathcal{R}_k$ is a subsemiring of $\mathcal{S}$, and by Theorem 2, the intersection of two subsemirings is a subsemiring. Therefore, the entire intersection is a subsemiring of $\mathcal{S}$.

By the principle of mathematical induction, the intersection of any finite number of subsemirings of $\mathcal{S}$ is a subsemiring of $\mathcal{S}$. ■

3. RESULTS AND DISCUSSION

Using the foundational properties of subsemiring intersections established in Section 2, we now analyse the algebraic behaviour of level soft semirings under intersection, union, and AND operations. In this study, we focus on constructing a soft set derived from a fuzzy subset defined over a semiring $\mathcal{S}$. Unlike groups, a semiring does not require the existence of additive inverses for its elements. This fundamental difference significantly influences the method used to select fuzzy parameters.

In group structures, the presence of identity and inverse elements enables the use of membership values associated with the identity element as robust and representative parameters. Abdurrahman et al. [29] have extensively discussed this method in the context of level soft sets and level soft groups, where parameters are derived from values directly related to the group's identity and inverse elements.

However, this approach is not directly applicable to fuzzy semirings, as additive inverses are not guaranteed for all elements. Therefore, in this context, we adopt a more flexible approach by defining the parameter set as the image of η, denoted by $\eta(\mathcal{S})$. This approach enables us to capture the relationship between elements in S and their membership degrees in the fuzzy subset, without relying on the presence of an inverse element.

Theorem 4. *Let η be a fuzzy subset of a semiring $\mathcal{S}$, and define the parameter set as:*

$$\boldsymbol{\mathcal{B}} = \eta(\mathcal{S}).$$

Let $\mathfrak{f}_{\boldsymbol{\mathcal{B}}} \subseteq \boldsymbol{\mathcal{B}} \times \boldsymbol{\mathcal{P}}(\mathcal{S})$ be a relation. For each $\mathscr{b} \in \mathcal{B}$, define:

$$\mathfrak{f}_{\boldsymbol{\mathcal{B}}}(\mathscr{b}) := \eta_{\mathscr{b}} = \{a \mid \eta(a) \geq \mathscr{b}\}.$$

Then, $\mathfrak{f}_{\boldsymbol{\mathcal{B}}}$ is a soft set over $\mathcal{S}$.

Proof. A detailed proof is provided in Proposition 4 of Abdurrahman et al.[29]. ■

The soft $\mathfrak{f}$ over $\mathcal{S}$, as defined in Theorem 4, is hereafter referred to as the level soft set over $\mathcal{S}$. For clarity, unless otherwise stated, the notation $\mathfrak{f}$ will always denote a level soft set derived from a fuzzy subset throughout this section. In the sequel, this level soft set will serve as the basic construction for defining and analysing level soft semirings derived from fuzzy subsemirings.

Theorem 5. *Let $\mathcal{S}$ be a semiring and let $\eta: \mathcal{S} \longrightarrow [0,1]$ be a fuzzy subsemiring of $\mathcal{S}$. Let $\mathcal{B} = \eta(\mathcal{S})$. Define the mapping $\mathfrak{f}_{\boldsymbol{\mathcal{B}}}: \mathcal{B} \longrightarrow \boldsymbol{\mathcal{P}}(\mathcal{S})$ by*

$$\mathfrak{f}_{\boldsymbol{\mathcal{B}}}(\mathscr{b}) := \eta_{\mathscr{b}} = \{a \in \mathcal{S} \mid \eta(a) \geq \mathscr{b}\}, \text{ for each } \mathscr{b} \in \mathcal{B}.$$

Then each $\mathfrak{f}_{\boldsymbol{\mathcal{B}}}$ is a subsemiring of $\mathcal{S}$. Consequently, $\mathfrak{f}_{\boldsymbol{\mathcal{B}}}$ is a soft semiring over $\mathcal{S}$.

Proof. Let $\eta: \mathcal{S} \longrightarrow [0,1]$ be a fuzzy subsemiring of a semiring $\mathcal{S}$, as defined in Definition 3, which requires that for all $a, c \in \mathcal{S}$, the inequalities

$$\eta(a + c) \geq \eta(a) \wedge \eta(c), \text{ and } \eta(ac) \geq \eta(a) \wedge \eta(c)$$

hold. Let $\mathcal{B} = \eta(\mathcal{S})$ be the image of η, and define the mapping $\mathfrak{f}_{\boldsymbol{\mathcal{B}}}: \mathcal{B} \longrightarrow \boldsymbol{\mathcal{P}}(\mathcal{S})$ by

$$\mathfrak{f}_{\boldsymbol{\mathcal{B}}}(\mathscr{b}) := \eta_{\mathscr{b}} = \{a \in \mathcal{S} \mid \eta(a) \geq \mathscr{b}\}, \text{ for each } \mathscr{b} \in \mathcal{B}.$$

Based on Theorem 4, this mapping $\mathfrak{f}_{\boldsymbol{\mathcal{B}}}$ is a level soft set over $\mathcal{S}$. Now, take any $a, c \in \eta_{\mathscr{b}}$. According to Definition 2, which defines the level subset of a fuzzy set, this means

$$\eta(a) \geq \mathscr{b} \text{ and } \eta(c) \geq \mathscr{b}.$$

Then, by the fuzzy subsemiring condition (Definition 3), we have

$$\eta(a + c) \geq \eta(a) \wedge \eta(c) \geq \mathscr{b} \text{ and } \eta(ac) \geq \eta(a) \wedge \eta(c) \geq \mathscr{b}.$$

Thus,

$$a + c \in \eta_{\mathscr{b}} \text{ and } ac \in \eta_{\mathscr{b}}.$$

Therefore, $\eta_{\mathscr{b}}$ is closed under addition and multiplication, and hence is a subsemiring of $\mathcal{S}$. Since each image $\mathfrak{f}_{\boldsymbol{\mathcal{B}}}(\mathscr{b}) = \eta_{\mathscr{b}}$ is a subsemiring of $\mathcal{S}$, and $\mathfrak{f}_{\boldsymbol{\mathcal{B}}}$ is a level soft set over $\mathcal{S}$, it follows from Definition 10 that $\mathfrak{f}_{\boldsymbol{\mathcal{B}}}$ is a soft semiring over $\mathcal{S}$. ■

Although this result follows naturally from the definition of level soft sets, it is nontrivial in the semiring setting due to the absence of additive inverses, which distinguishes it from analogous constructions in group-based frameworks.

Motivated by the construction and properties established in Theorem 5, we now formally introduce the notion of a level soft semiring.

Definition 11. *Let $\mathcal{S}$ be a semiring and let $\eta: \mathcal{S} \longrightarrow [0,1]$ a fuzzy subsemiring of $\mathcal{S}$. Let*

$$\mathfrak{f}_{\mathcal{B}}(\mathscr{b}) := \eta_{\mathscr{b}} = \{a \mid \eta(a) \geq \mathscr{b}\}$$

for each $\mathscr{b} \in \mathcal{B} = \eta(\mathcal{S})$. If each $\mathfrak{f}_{\mathcal{B}}(\mathscr{b})$ is a subsemiring of $\mathcal{S}$, then $\mathfrak{f}_{\mathcal{B}}$ is defined as a ***level soft semiring*** *over $\mathcal{S}$.*

Here, the parameter $\mathscr{b}$ represents a membership level in the image of the fuzzy subsemiring and should not be confused with an arbitrary soft parameter. It is worth noting that this construction differs from classical soft semirings in that the parameter set is intrinsically determined by the image of the underlying fuzzy subsemiring, rather than being chosen arbitrarily.

Definition 11 establishes the framework for a level soft semiring by ensuring that each level subset derived from a fuzzy subsemiring forms a subsemiring, thereby giving an algebraic interpretation to fuzzy membership levels.

Theorem 6. *Each level soft set derived from a fuzzy subsemiring constitutes a soft subsemiring.*

Proof. Let $\eta: \mathcal{S} \longrightarrow [0,1]$ be a fuzzy subsemiring of a semiring $\mathcal{S}$. For each $\mathscr{b} \in [0,1]$, define the level subset

$$\mathfrak{f}_{[0,1]}(\mathscr{b}) := \{a \in \mathcal{S} \mid \eta(a) \geq \mathscr{b}\}.$$

Since η satisfies the fuzzy subsemiring of $\mathcal{S}$, for any $a, c \in \mathfrak{f}_{[0,1]}(\mathscr{b})$, we have $\eta(a) \geq \mathscr{b}$ and $\eta(c) \geq \mathscr{b}$. By Definition 3, this implies:

$$\eta(a + c) \geq \eta(a) \wedge \eta(c) \geq \mathscr{b} \text{ and } \eta(a + c) \geq \eta(a) \wedge \eta(c) \geq \mathscr{b}.$$

Hence,

$$a + c \in \mathfrak{f}_{[0,1]}(\mathscr{b}) \text{ and } ac \in \mathfrak{f}_{[0,1]}(\mathscr{b}),$$

which shows that $\mathfrak{f}_{[0,1]}(\mathscr{b})$ is closed under both addition and multiplication. Therefore, each level subset $\mathfrak{f}_{[0,1]}(\mathscr{b})$ forms a subsemiring of $\mathcal{S}$.

Now, define a mapping

$$\mathfrak{f}_{[0,1]}: [0,1] \longrightarrow \mathcal{P}(\mathcal{S}) \text{ by } \mathfrak{f}_{[0,1]}(\mathscr{b}) = \{a \in \mathcal{S} \mid \eta(a) \geq \mathscr{b}\} \text{ for all } \mathscr{b} \in [0,1].$$

Since each image $\mathfrak{f}_{[0,1]}(\mathscr{b})$ is a subsemiring of $\mathcal{S}$, the function $\mathfrak{f}_{[0,1]}$ defines a soft subsemiring over $\mathcal{S}$. ■

Theorem 6 rigorously confirms that each level soft set derived from a fuzzy subsemiring inherently satisfies the structural requirements of a soft subsemiring. This result affirms the algebraic consistency of the level-based construction and guarantees the preservation of semiring properties across all levels of fuzzy membership. Building upon this foundation, we now present Corollary 1, which highlights the existence of at least one level subset that satisfies the subsemiring conditions. This corollary serves as a direct consequence of the fuzzy subsemiring definition and reinforces the applicability of the level soft semiring framework.

Corollary 1. *Let $\eta: \mathcal{S} \longrightarrow [0,1]$ be a fuzzy subsemiring of a semiring $\mathcal{S}$. Then there exists at least one $\mathscr{b} \in \eta(\mathcal{S})$ such that the level subset*

$$\mathfrak{f}_{\eta(\mathcal{S})}(\mathscr{b}) := \{a \in \mathcal{S} \mid \eta(a) \geq \mathscr{b}\}$$

is a subsemiring of $\mathcal{S}$.

Proof. This result follows directly from Definition 3 of a fuzzy subsemiring and Theorem 6, which guarantees that every level subset $\mathfrak{f}_{\eta(\mathcal{S})}(\mathscr{b})$ derived from η, forms a subsemiring $\mathcal{S}$. Since $\eta(\mathcal{S})$ is nonempty, at least one such $\mathscr{b}$ must exist. ■

In semiring theory, fuzzy subsemirings provide a graded structure that captures partial membership of elements participating in algebraic operations. When constructing soft sets based on the level subsets of a fuzzy subsemiring, it becomes essential to examine whether the resulting structure preserves the semiring properties. Specifically, we consider whether each image under the soft set mapping forms a subsemiring. The following corollary confirms that such a construction yields a level soft semiring, thereby extending the fuzzy framework into the domain of soft algebraic structures.

Corollary 2. *Let η be a fuzzy subsemiring of a semiring $\mathcal{S}$. Then the level soft $\mathfrak{f}_{\eta(\mathcal{S})}$ over $\mathcal{S}$ is a level soft semiring over $\mathcal{S}$.*

Proof. Let $\eta: \mathcal{S} \longrightarrow [0,1]$ be a fuzzy subsemiring of a semiring $\mathcal{S}$, and let $\mathfrak{f}_{\eta(\mathcal{S})}$ be a level soft over $\mathcal{S}$. Then by Theorem 5, we have that $\mathfrak{f}_{\eta(\mathcal{S})}$ is a level soft semiring over $\mathcal{S}$. ■

Theorem 7. *Let η and σ be fuzzy subsemirings of a semiring $\mathcal{S}$. Then the level soft set $\mathfrak{f}_{\eta\widetilde{\cap}\sigma(\mathcal{S})}$ over $\mathcal{S}$ is a level soft semiring over $\mathcal{S}$.*

Proof. Assume that η and σ are fuzzy subsemirings of the semiring $\mathcal{S}$, by Definition 3, for all $a, x \in \mathcal{S}$, the following inequalities hold:

$$\eta(a+x) \geq \eta(a) \wedge \eta(x), \eta(ax) \geq \eta(a) \wedge \eta(x),$$
$$\sigma(a+x) \geq \sigma(a) \wedge \sigma(x), \sigma(ax) \geq \sigma(a) \wedge \sigma(x).$$

Define the fuzzy intersection $\phi = \eta\widetilde{\cap}\sigma$ as in Definition 4:

$$\phi(a) := \eta\widetilde{\cap}\sigma(a), \forall a \in \mathcal{S}.$$

We demonstrate that the fuzzy intersection ϕ satisfies the conditions of a fuzzy subsemiring over $\mathcal{S}$. For any $a, x \in \mathcal{S}$, we have:

$$\begin{aligned}
\phi(a+x) &= \eta\widetilde{\cap}\sigma(a+x) \\
&= \eta(a+x) \wedge \sigma(a+x) \\
&\geq \big(\eta(a) \wedge \eta(x)\big) \wedge \big(\sigma(a) \wedge \sigma(x)\big) \\
&= \big(\eta(a) \wedge \sigma(a)\big) \wedge \big(\eta(x) \wedge \sigma(x)\big) \\
&= \phi(a) \wedge \phi(x),
\end{aligned}$$

and

$$\begin{aligned}
\phi(ax) &= \eta\widetilde{\cap}\sigma(ax) \\
&= \eta(ax) \wedge \sigma(ax) \\
&\geq \big(\eta(a) \wedge \eta(x)\big) \wedge \big(\sigma(a) \wedge \sigma(x)\big) \\
&= \big(\eta(a) \wedge \sigma(a)\big) \wedge \big(\eta(x) \wedge \sigma(x)\big) \\
&= \phi(a) \wedge \phi(x).
\end{aligned}$$

Hence, ϕ satisfies the conditions of a fuzzy subsemiring according to Definition 3.

Next, define the level soft set $\mathfrak{f}_{\phi(\mathcal{S})}$ as in Theorem 4, by:

$$\mathfrak{f}_{\phi(\mathcal{S})}(\mathscr{b}) := \{a \in \mathcal{S} \mid \phi(a) \geq \mathscr{b}\}, \forall \mathscr{b} \in \phi(\mathcal{S}).$$

Since ϕ is a fuzzy subsemiring, then by Theorem 5, each level subset $\mathfrak{f}_{\phi(\mathcal{S})}(\mathscr{b})$ is closed under both addition and multiplication. Consequently, each level subset is derived from the fuzzy intersection $\mathfrak{f}_{\phi(\mathcal{S})}$ forms a subsemiring of $\mathcal{S}$.

Hence, by Definition 11, the level soft set $\mathfrak{f}_{\phi(\mathcal{S})}$ is a level soft semiring over $\mathcal{S}$. ■

Theorem 8. *Let $\eta_1, \eta_2, \cdots, \eta_n$ be fuzzy subsemirings of a semiring $\mathcal{S}$. Then the level soft set*

$$\mathfrak{f}_{(\eta_1\widetilde{\cap}\eta_2\widetilde{\cap}\cdots\widetilde{\cap}\eta_n)(\mathcal{S})}$$

over $\mathcal{S}$ is a level soft semiring over $\mathcal{S}$.

Proof. We prove this theorem by mathematical induction on n, the number of fuzzy subsemirings.

1. **Base case** $(n = 2)$:

   By Theorem 5, the intersection $\eta_1\widetilde{\cap}\eta_2$ is a fuzzy subsemiring of $\mathcal{S}$, and the corresponding level soft set is a level soft semiring over $\mathcal{S}$.

2. **Inductive step**:
   Assume that for some $k \geq 2$, the intersection

$$\eta_1 \widetilde{\cap} \eta_2 \widetilde{\cap} \cdots \widetilde{\cap} \eta_k$$

is a fuzzy subsemiring of $\mathcal{S}$, and the level soft set

$$\mathfrak{f}_{(\eta_1 \widetilde{\cap} \eta_2 \widetilde{\cap} \cdots \widetilde{\cap} \eta_k)(\mathcal{S})}$$

is a level soft semiring over $\mathcal{S}$.

Let η_{k+1} be another fuzzy subsemiring of $\mathcal{S}$. By Definition 4, the intersection

$$\phi(a) := \eta_1(a) \widetilde{\cap} \eta_2(a) \widetilde{\cap} \cdots \widetilde{\cap} \eta_k(a) \widetilde{\cap} \eta_{k+1}(a), \forall a \in \mathcal{S}.$$

Since both $\eta_1 \widetilde{\cap} \eta_2 \widetilde{\cap} \cdots \widetilde{\cap} \eta_k$ and η_{k+1} are fuzzy subsemirings of $\mathcal{S}$, it follows from the same reasoning as in Theorem 7 that ϕ is also a fuzzy subsemiring of $\mathcal{S}$.

By the principle of mathematical induction, the intersection of any finite number of fuzzy subsemirings yields a fuzzy subsemiring, and the corresponding level soft set is a level soft semiring over $\mathcal{S}$. ∎

Having established the construction of level soft semirings from fuzzy subsemirings, we now investigate the stability of these structures under standard soft set operations.

Theorem 9. *Let $\mathfrak{f}_{\eta(\mathcal{S})}$ and $\mathfrak{f}_{\sigma(\mathcal{S})}$ be level soft semirings over a semiring $\mathcal{S}$, where the intersection of their parameter sets $\eta(\mathcal{S}) \cap \sigma(\mathcal{S}) \neq \emptyset$. Therefore, the intersection of the level soft sets $\mathfrak{f}_{\eta(\mathcal{S})} \sqcap \mathfrak{f}_{\sigma(\mathcal{S})}$ constitutes a level soft semiring over $\mathcal{S}$. Moreover, the following identity holds:*

$$\mathfrak{f}_{\eta(\mathcal{S})} \sqcap \mathfrak{f}_{\sigma(\mathcal{S})} = \mathfrak{f}_{\eta \widetilde{\cap} \sigma(\mathcal{S})}.$$

Proof. Assume that $\mathfrak{f}_{\eta(\mathcal{S})}$ and $\mathfrak{f}_{\sigma(\mathcal{S})}$ are level soft semirings over a semiring $\mathcal{S}$, where the intersection of their parameter sets $\eta(\mathcal{S}) \cap \sigma(\mathcal{S}) \neq \emptyset$. By Definition 11 for each parameter $\mathscr{b} \in \eta(\mathcal{S}) \cap \sigma(\mathcal{S})$, the corresponding images, $\mathfrak{f}_{\eta(\mathcal{S})}(\mathscr{b})$ and $\mathfrak{f}_{\sigma(\mathcal{S})}(\mathscr{b})$ are subsemirings of $\mathcal{S}$.

Given that both images are subsemirings, their intersection

$$\mathfrak{f}_{\eta(\mathcal{S})}(\mathscr{b}) \cap \mathfrak{f}_{\sigma(\mathcal{S})}(\mathscr{b})$$

necessarily forms a subsemiring of $\mathcal{S}$, as established in Theorem 2. According to Definition 7, the intersection of the soft sets $\mathfrak{f}_{\eta(\mathcal{S})}$ and $\mathfrak{f}_{\sigma(\mathcal{S})}$ is defined pointwise as:

$$\left(\mathfrak{f}_{\eta(\mathcal{S})} \sqcap \mathfrak{f}_{\sigma(\mathcal{S})}\right)(\mathscr{b}) = \mathfrak{f}_{\eta(\mathcal{S})}(\mathscr{b}) \cap \mathfrak{f}_{\sigma(\mathcal{S})}(\mathscr{b}), \text{ for all } \mathscr{b} \in \eta(\mathcal{S}) \cap \sigma(\mathcal{S}).$$

Thus, each image of the soft set $\mathfrak{f}_{\eta(\mathcal{S})} \sqcap \mathfrak{f}_{\sigma(\mathcal{S})}$ is a subsemiring of $\mathcal{S}$, implying that the intersection is a level soft semiring over $\mathcal{S}$.

Furthermore, for any $\mathscr{b} \in \eta(\mathcal{S}) \cap \sigma(\mathcal{S})$, we have:

$$\begin{aligned}
\mathfrak{f}_{\eta(\mathcal{S})} \sqcap \mathfrak{f}_{\sigma(\mathcal{S})}(\mathscr{b}) &= \{a \in \mathcal{S} \mid \eta(a) \geq \mathscr{b}\} \cap \{a \in \mathcal{S} \mid \sigma(a) \geq \mathscr{b}\} \\
&= \{a \in \mathcal{S} \mid \eta(a) \geq \mathscr{b} \text{ and } \sigma(a) \geq \mathscr{b}\} \\
&= \{a \in \mathcal{S} \mid \eta(a) \wedge \sigma(a) \geq \mathscr{b}\} \\
&= \left\{a \in \mathcal{S} \,\middle|\, \eta \widetilde{\cap} \sigma(a) \geq \mathscr{b}\right\} \\
&= \mathfrak{f}_{\eta \widetilde{\cap} \sigma(\mathcal{S})}(\mathscr{b}).
\end{aligned}$$

Hence, the identity $\mathfrak{f}_{\eta(\mathcal{S})} \sqcap \mathfrak{f}_{\sigma(\mathcal{S})} = \mathfrak{f}_{\eta \widetilde{\cap} \sigma(\mathcal{S})}$ holds. ∎

Theorem 9 shows that the class of level soft semirings is closed under the intersection operation, provided that their parameter sets overlap. This result ensures that the algebraic structure of level soft semirings remains stable under natural soft set operations.

Theorem 10. *Let $\mathfrak{f}_{\eta_1(\mathcal{S})}, \mathfrak{f}_{\eta_2(\mathcal{S})}, \cdots, \mathfrak{f}_{\eta_n(\mathcal{S})}$ be level soft semirings over a semiring $\mathcal{S}$, and suppose that the intersection of their parameter sets is nonempty:*

$$\eta_1(\mathcal{S}) \cap \eta_2(\mathcal{S}) \cap \cdots \cap \eta_n(\mathcal{S}) \neq \emptyset.$$

Therefore, the intersection

$$\mathfrak{f}_{\eta_1(\mathcal{S})} \sqcap \mathfrak{f}_{\eta_2(\mathcal{S})} \sqcap \cdots \sqcap \mathfrak{f}_{\eta_n(\mathcal{S})}$$

constitutes a level soft semiring over $\mathcal{S}$*.*

Proof. Let $\mathscr{b} \in \eta_1(\mathcal{S}) \cap \eta_2(\mathcal{S}) \cap \cdots \cap \eta_n(\mathcal{S})$. Since each $\mathfrak{f}_{\eta_i(\mathcal{S})}$ is a level soft semiring, it follows that for each $i = 1, 2, \cdots, n$, the image $\mathfrak{f}_{\eta_i(\mathcal{S})}(\mathscr{b})$ is a subsemiring of $\mathcal{S}$.

By Theorem 8, the intersection

$$\bigcap_{i=1}^{n} \mathfrak{f}_{\eta_i(\mathcal{S})}(\mathscr{b})$$

is also a subsemiring of $\mathcal{S}$. Therefore, for each $\mathscr{b}$ in the standard parameter set, the image of the intersection soft set is a subsemiring.

Hence, the soft set

$$\mathfrak{f}_{\eta_1(\mathcal{S})} \sqcap \mathfrak{f}_{\eta_2(\mathcal{S})} \sqcap \cdots \sqcap \mathfrak{f}_{\eta_n(\mathcal{S})}$$

satisfies the definition of a level soft semiring over $\mathcal{S}$. ■

Theorem 10 establishes that the class of level soft semirings is closed under the intersection operation when the parameter sets have a nonempty overlap. Having addressed this case, it is natural to consider the complementary situation in which the parameter sets of two-level soft semirings are disjoint. In such a setting, the intersection operation is no longer applicable, and the union operation becomes the appropriate mechanism for combining the soft structures. This leads to Theorem 11, which investigates whether the union of two level soft semirings with disjoint parameter sets preserves the level soft semiring structure.

Theorem 11. *Let* $\mathfrak{f}_{\eta(\mathcal{S})}$ *and* $\mathfrak{f}_{\sigma(\mathcal{S})}$ *be level soft semirings over a semiring* $\mathcal{S}$*, where the parameter sets* $\eta(\mathcal{S})$ *and* $\sigma(\mathcal{S})$ *are disjoint. Then the union*

$$\mathfrak{f}_{\eta(\mathcal{S})} \sqcup \mathfrak{f}_{\sigma(\mathcal{S})}$$

is a level soft semiring over $\mathcal{S}$*.*

Proof. Let $\mathfrak{f}_{\eta(\mathcal{S})}$ and $\mathfrak{f}_{\sigma(\mathcal{S})}$ be level soft semirings over $\mathcal{S}$, and define $\mathfrak{f}_{\mathcal{D}} = \mathfrak{f}_{\eta(\mathcal{S})} \sqcup \mathfrak{f}_{\sigma(\mathcal{S})}$, where $\mathcal{D} = \eta(\mathcal{S}) \cup \sigma(\mathcal{S})$. By Definition 9 of soft set union, for each parameter $\mathscr{s} \in \mathcal{D}$, we have:

1. If $\mathscr{s} \in \eta(\mathcal{S}) - \sigma(\mathcal{S})$, then $\mathfrak{f}_{\mathcal{D}}(\mathscr{s}) = \mathfrak{f}_{\eta(\mathcal{S})}(\mathscr{s})$, which is a subsemiring of $\mathcal{S}$,
2. If $\mathscr{s} \in \sigma(\mathcal{S}) - \eta(\mathcal{S})$, then $\mathfrak{f}_{\mathcal{D}}(\mathscr{s}) = \mathfrak{f}_{\sigma(\mathcal{S})}(\mathscr{s})$, which is a subsemiring of $\mathcal{S}$.

Since $\eta(\mathcal{S}) \cap \sigma(\mathcal{S}) = \emptyset$, there is no overlap, and each image $\mathfrak{f}_{\mathcal{D}}(\mathscr{s})$ is inherited directly from either $\mathfrak{f}_{\eta(\mathcal{S})}$ or $\mathfrak{f}_{\sigma(\mathcal{S})}$, both of which are level soft semirings. Therefore, for all $\mathscr{s} \in \mathcal{D}$, $\mathfrak{f}_{\mathcal{D}}(\mathscr{s})$ is a subsemiring of $\mathcal{S}$.

Hence, $\mathfrak{f}_{\mathcal{D}} = \mathfrak{f}_{\eta(\mathcal{S})} \sqcup \mathfrak{f}_{\sigma(\mathcal{S})}$ is a level soft semiring over $\mathcal{S}$. ■

Theorem 11 addresses the behavior of level soft semirings under the union operation when their parameter sets are disjoint. While union and intersection represent fundamental ways of combining soft structures, they do not capture all possible interactions between level soft semirings. To obtain a more general framework that unifies these operations, we now consider the AND operation, which allows the combination of level soft semirings across parameter tuples. This motivates Theorem 12, where the preservation of the level soft semiring structure under the AND operation is investigated.

Theorem 12. *Let* $\mathcal{S}$ *be a semiring, and let* $\mathfrak{f}_{\eta(\mathcal{S})}$ *and* $\mathfrak{f}_{\sigma(\mathcal{S})}$ *be level soft semirings over* $\mathcal{S}$*, as defined in* Definition 10*. Consider the* ***AND operation*** *applied to these two soft structures, resulting in a new soft set defined on the Cartesian product of their parameter domains. Then the resulting soft set*

$$\mathfrak{f}_{\eta(\mathcal{S})} \wedge \mathfrak{f}_{\sigma(\mathcal{S})}$$

is a level soft semiring over $\mathcal{S}$*.*

Proof. Let $\mathfrak{f}_{\eta(\mathcal{S})}$ and $\mathfrak{f}_{\sigma(\mathcal{S})}$ be level soft semirings over a semiring $\mathcal{S}$, meaning that for each parameter $\mathscr{b} \in \text{Dom}\left(\mathfrak{f}_{\eta(\mathcal{S})}\right)$, the image $\mathfrak{f}_{\eta(\mathcal{S})}(\mathscr{b}) \subseteq \mathcal{S}$ is a subsemiring of $\mathcal{S}$, and similarly for each $\mathscr{d} \in \text{Dom}\left(\mathfrak{f}_{\sigma(\mathcal{S})}\right)$, the image $\mathfrak{f}_{\sigma(\mathcal{S})}(\mathscr{d}) \subseteq \mathcal{S}$ is also a subsemiring of $\mathcal{S}$.

According to Definition 8, the AND operation between $\mathfrak{f}_{\eta(\mathcal{S})}$ and $\mathfrak{f}_{\sigma(\mathcal{S})}$ yields a new soft set

$$\mathfrak{f}_{\eta(\mathcal{S})\wedge\sigma(\mathcal{S})}\colon \text{Dom}\left(\mathfrak{f}_{\eta(\mathcal{S})}\right) \times \text{Dom}\left(\mathfrak{f}_{\sigma(\mathcal{S})}\right) \longrightarrow \mathcal{P}(\mathcal{S}),$$

defined by

$$\mathfrak{f}_{\sigma(\mathcal{S})\wedge\sigma(\mathcal{S})}(\mathscr{b},\mathscr{d}) = \mathfrak{f}_{\eta(\mathcal{S})}(\mathscr{b}) \cap \mathfrak{f}_{\sigma(\mathcal{S})}(\mathscr{d}).$$

Since both $\mathfrak{f}_{\eta(\mathcal{S})}(\mathscr{b})$ and $\mathfrak{f}_{\sigma(\mathcal{S})}(\mathscr{d})$ are subsemirings of $\mathcal{S}$, and the intersection of two subsemirings is again a subsemiring (see Theorem 2). It follows that

$$\mathfrak{f}_{\sigma(\mathcal{S})\wedge\sigma(\mathcal{S})}(\mathscr{b},\mathscr{d})$$

is a subsemiring of $\mathcal{S}$ for all $(\mathscr{b},\mathscr{d}) \in \text{Dom}\left(\mathfrak{f}_{\eta(\mathcal{S})}\right) \times \text{Dom}\left(\mathfrak{f}_{\sigma(\mathcal{S})}\right)$.

Hence, according to Definition 11, the soft set $\mathfrak{f}_{\sigma(\mathcal{S})\wedge\sigma(\mathcal{S})}$ constitutes a level soft semiring over $\mathcal{S}$.■

Theorem 12 establishes that the level soft semiring structure is preserved under the AND operation when applied to two level soft semirings. This result serves as a fundamental building block for more general constructions. In many practical and theoretical situations, it is necessary to combine more than two-level soft semirings simultaneously. Therefore, it is natural to extend the binary AND operation to an iterative form involving multiple-level soft semirings. This extension is formalized in Theorem 13, which generalizes the result of Theorem 12 to the case of n-level soft semirings.

Theorem 13. *Let $\mathfrak{f}_{\eta_1(\mathcal{S})}, \mathfrak{f}_{\eta_2(\mathcal{S})}, \cdots, \mathfrak{f}_{\eta_n(\mathcal{S})}$ be level soft semirings over a semiring $\mathcal{S}$, where $n \geq 2$. Let the soft set*

$$\mathfrak{f}_{\eta_1(\mathcal{S})} \wedge \mathfrak{f}_{\eta_2(\mathcal{S})} \wedge \cdots \wedge \mathfrak{f}_{\eta_n(\mathcal{S})}$$

be defined through the iterative application of the AND operation iteratively over all n soft semirings. Then the resulting soft set is a level soft semiring over $\mathcal{S}$.

Proof.

1. **Base Case** $(n = 2)$:

   From Theorem 12, we know that the AND operation between two level soft semirings

   $$\mathfrak{f}_{\eta_1(\mathcal{S})} \wedge \mathfrak{f}_{\eta_2(\mathcal{S})}$$

   results in a level soft semiring over $\mathcal{S}$, since the intersection of their images at each parameter pair yields a subsemiring.

2. **Inductive Step:**

   Assume that for some $k > 2$, the soft set

   $$\mathfrak{f}_{\eta_1(\mathcal{S})} \wedge \mathfrak{f}_{\eta_2(\mathcal{S})} \wedge \cdots \wedge \mathfrak{f}_{\eta_k(\mathcal{S})}$$

   is a level soft semiring over $\mathcal{S}$. Let $\mathfrak{f}_{\eta_{k+1}(\mathcal{S})}$ be another level soft semiring.
   Define the new soft set by:

   $$\mathfrak{f}_{\eta_1(\mathcal{S})} \wedge \cdots \wedge \mathfrak{f}_{\eta_k(\mathcal{S})} \wedge \mathfrak{f}_{\eta_{k+1}(\mathcal{S})}\colon \text{Dom}\left(\mathfrak{f}_{\eta_1(\mathcal{S})}\right) \times \cdots \times \text{Dom}\left(\mathfrak{f}_{\eta_{k+1}(\mathcal{S})}\right) \longrightarrow \mathcal{P}(\mathcal{S}),$$

   with image:

   $$\left(\mathfrak{f}_{\eta_1(\mathcal{S})} \wedge \cdots \wedge \mathfrak{f}_{\eta_k(\mathcal{S})} \wedge \mathfrak{f}_{\eta_{k+1}(\mathcal{S})}\right)(\mathscr{b}_1, \mathscr{b}_2, \cdots, \mathscr{b}_k, \mathscr{b}_{k+1}) = \left(\bigcap_{i=1}^{k} \mathfrak{f}_{\eta_i(\mathcal{S})}(\mathscr{b}_i)\right) \cap \mathfrak{f}_{\eta_{k+1}(\mathcal{S})}(\mathscr{b}_{k+1}).$$

   By the inductive hypothesis, $\bigcap_{i=1}^{k} \mathfrak{f}_{\eta_i(\mathcal{S})}(\mathscr{b}_i)$ is a subsemiring of $\mathcal{S}$, and since $\mathfrak{f}_{\eta_{k+1}(\mathcal{S})}(\mathscr{b}_{k+1})$ is also a subsemiring; their intersection is a subsemiring (by Theorem 2). Thus, the resulting soft set is a level soft semiring over $\mathcal{S}$.

Thus, by mathematical induction, the iterative AND operation over any finite family of level soft semirings results in a level soft semiring over $\mathcal{S}$. ■

These results show that level soft semirings retain their algebraic structure under fundamental soft set operations. These closure properties highlight that level soft semirings retain a robust algebraic structure under standard soft set operations, extending known closure results from fuzzy semirings and soft semirings to the level-based fuzzy–soft semiring framework.

4. CONCLUSION

This paper develops a level-based fuzzy–soft algebraic framework by constructing level soft semirings derived from fuzzy subsemirings over a semiring. Using the level subset approach, level soft sets are shown to induce subsemiring structures for each parameter, and their algebraic behavior under intersection, union, and AND operations is systematically investigated. The results demonstrate that the class of level soft semirings is stable under these fundamental soft set operations, providing a rigorous theoretical extension of fuzzy semiring and soft semiring theories. While the present study focuses on structural properties, the proposed framework offers a foundation for future research on homomorphisms, quotient structures, and potential applications in broader mathematical and computational contexts.

Author Contributions

Saman Abdurrahman: Conceptualisation, Formal Analysis, and Writing - Original Draft. Thresye: Literature Review, Theoretical Analysis, Revision, and Editing. All authors have read and approved the final manuscript.

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Declarations

The authors declare no competing interests.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools (e.g., ChatGPT) were used solely for language refinement, including grammar, spelling, and clarity. The scientific content, analysis, interpretations, and conclusions were entirely developed by the authors.

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