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GARDNER PROBLEM REVISITED: FURTHER PROPERTIES OF INDAH RADICAL

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ABSTRACT

Radical theory arises naturally from the study of non-commutative rings and plays a central role in the structural analysis of ring classes. Among the radical classes that have received considerable attention are the prime radical β and the IndaH radical $U(*_k)$, whose relationship is closely related to the Gardner conjecture. While several structural properties of β are well established, the corresponding properties of $U(*_k)$ have remained less clear. In this paper, we investigate the IndaH radical $U(*_k)$ using deductive arguments and structural analysis within the framework of radical theory. In particular, we examine whether $U(*_k)$ satisfies corner-hereditariness, corner-strictness, very corner-hereditariness, and the hereditary phantom corner (HPC) property. We show that $U(*_k)$ possesses all four properties, thereby placing it in closer structural alignment with the prime radical β. The results are obtained under standard assumptions on associative rings and radical classes.

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1. INTRODUCTION

A class of rings γ is called a radical class if γ satisfies [1]: 1. For every ideal I of $R \in \gamma \Rightarrow R/I \in \gamma$, 2. For every ring R, the set $\gamma(R) = \sum\{I|\ I$ is an ideal of $R\} \in \gamma$, and 3. For every ring R, the set $\gamma(R/\gamma(R)) = 0$. Moreover, a radical class γ of rings is called left (right) hereditary if for every ring $R \in \gamma$ then $I \in \gamma$ for every left (right) ideal I of R [1]. It is clear that the set $\{R$ is a ring$|\ (R,\circ)$ forms a group$\}$, where $a \circ b = a + b - ab, \forall a, b \in R$, is a radical class of rings, and this radical class is called the Jacobson radical class [1]. Another concrete example and construction of a radical class of rings can be accessed in [1]. The existence of radical classes in ring theory has become a distinct branch in the development of abstract algebra. This has motivated the study of the theory of radicals of rings. The radical theory of rings is one of the branches of ring theory that still holds many open problems that remain unsolved to this day.

The significance of radical theory in the development of ring theory can be observed through several key aspects. First, it provides structure theorems for semisimple rings with respect to specific radicals. Second, it offers a systematic framework for examining and comparing the properties of rings, including classes of rings defined by closure operations. Third, it enables the construction of rings that highlight and differentiate particular ring properties, such as minimally embeddable rings. Finally, radical theory also uncovers hidden characteristics of rings that can be effectively applied in various contexts within ring theory [1].

On the other hand, the Jacobson radical can be implemented to construct a brace, algebraic structures arising from set-theoretic solutions of the Yang-Baxter equation [2], [3]. As radical theory has developed, several important classes of radicals have been introduced and extensively studied, each contributing to the advancement of ring theory. These include the Köethe nilradical [4]-[7], the prime radical β [8]-[26], the Levitzki radical [27], the Jacobson radical [28]-[37], and the Brown McCoy radical [38], [39]. Moreover, the generalization of radical concepts to other branches of algebra, such as module theory and Lie algebra, has also received considerable attention. In particular, within Lie algebra theory, radicals play a crucial role in ensuring the stability of systems of differential algebraic equations [40]-[44].

Although radical theory has been extensively explored across various areas of abstract algebra, it continues to encompass deep open problems that motivate ongoing research. One of the most prominent open questions is the Gardner problem, proposed in 1988. This problem asks whether the prime radical β coincides with the radical class $U(*_k)$ where $U(*_k) = \{R|R$ has no nonzero ideal in $*_k\}$, and $*_k$ is the essential closure of $*$, and $* = \{R$ is a prime ring such that R/I is not a prime for every nonzero proper ideal I of $R\}$. The concept of $*$ −rings was introduced to capture boundary cases of prime rings that cannot be decomposed into proper nonzero prime factors. Such rings exhibit a high degree of structural rigidity and serve as extreme objects within the class of prime rings. By studying $*$ −rings, we are able to answer open questions about the minimal and maximal special radicals that contain them and thereby clarify the role of atoms in the lattice of special radicals. The radical class $U(*_k)$ is crucial in this topic since it provides both the minimal and maximal special radical framework for $*$ −rings, and in certain universal classes, it coincides with the prime radical β. This makes $*$ −rings fundamental objects for understanding how prime radicals describe the structure of associative rings [45].

Despite nearly four decades of study, the Gardner problem remains unresolved. The importance and implications of a potential solution have been discussed in detail in [46]-[51]. Let $*$ be the class of all $*$ −ring. Let $*_k$ be the essential closure of $*$ and $\alpha = * \cup \beta$, where β is the prime radical. A radical class of rings γ is called left (right) strong if for all left (right) ideals I of R such that $I \in \gamma \Rightarrow I \subseteq \gamma(R)$. Moreover, it is clear that the lower radical $l_\alpha = l_{\bar{\alpha}}$ is left and right strong by Theorem 6 and Corollary 7 in [52]. Furthermore, l_α is an N −radical by Remark 10 in [52]. Recall that $U(*_k) = \{R|$ if $R/I \in *_k$ implies $I = R$ or $I = 0\}$ is the upper radical of the essential closure $*_k$ of $*$ and it is called the IndaH radical [46]. The further properties of the IndaH radical are described as follows. By Corollary 2.4 in [46], $U(*_k)$ is identified as a left hereditary special radical. Moreover, Theorem 2.5 together with Corollary 2.6 in [46] establish that $U(*_k)$ is a left strong radical. In addition, Corollary 2.7 [46] shows that $U(*_k)$ is a special N −radical.

On the other hand, a radical class γ is said to be very corner-hereditary if for every ring A and every idempotent e in A, we have $\gamma(eAe) = aAe \cap \gamma(A)$ [53]. A class γ of rings is corner-hereditary if for every $A \in \gamma$ and every idempotent $e \in A, eAe \in \gamma$. If a radical class γ is left or right hereditary, then it is corner-hereditary by Proposition 3.1 in [53].

Based on the results and issues mentioned above, a question arises regarding the comparison of the properties of the prime radical β and the IndaH radical $U(*_k)$. Previous studies and research findings have established that both the prime radical β and the IndaH radical $U(*_k)$ share a number of fundamental properties. They are left hereditary, right hereditary, left strong, right strong, supernilpotent, special, hypernilpotent, normal, and N −radical. Although the prime radical β has also been shown to satisfy several corner-related properties, as mentioned above in the abstract, the corresponding behavior of the IndaH radical $U(*_k)$ is not yet fully understood. This observation motivates the present study, which investigates whether $U(*_k)$ satisfies these properties and clarifies its position within radical theory.

2. RESEARCH METHODS

This research employs a rigorous literature-based methodology to investigate the structural properties of the IndaH radical $U(*_k)$, drawing upon prior results from radical theory. The study is fundamentally theoretical, utilizing deductive mathematical proofs to derive novel properties from established theorems. The methodological approaches include:

2.1. Direct Proofs and Contrapositive Arguments

For instance, the corner-hereditariness of is directly inferred from Proposition 3.1 in [53], which links left/right hereditary radicals to corner-hereditariness, combined with Corollary 2.4 in [46] confirming $U(*_k)$'s left heredity. However, we also give step-by-step instructions to prove this property and the further related property. To demonstrate that the semisimple class $SU(*_k) = \{R|U(*_k)(R) = 0\}$ is closed under corners, the logic hinges on contraposition if a corner the logic hinges on contraposition. The details of this approach will be explained in each proof.

2.2. Counterexamples

The Brown-McCoy radical (Example 11, [53]) serves as a counterexample to highlight that not all radicals are corner-hereditary, emphasizing $U(*_k)$'s distinctive properties. The counter-example approach is also used to state the scope of the discussion and to indicate the limitations under which corner-hereditariness is considered in this work

2.3. Computational Processes

Several examples in this paper have been successfully identified using a computational algorithm, considering that manual identification would require a significant amount of time. To construct certain radicals of finite rings as presented, we developed an algorithm that was then implemented in a Python-based program using Google Colab. Our code can be accessed at https://github.com/ardiyansyah13/Jacobson_matrices.git.

3. RESULTS AND DISCUSSION

Several radical classes, as well as classes of rings that are not radical classes, are discussed in [1]. We briefly recall some of these examples. First, the class of rings $\mathcal{N} = \{A|a^n = 0, \forall a \in A, n \in \mathbb{Z}^+\}$ forms a radical class, and it is called the Koethe nilradical. Furthermore, the class $\mathcal{J} = \{A|(a \circ b = a + b - ab, \forall a, b \in A, (A,\circ)$ is a group)$\}$ forms a radical class, and it is called the Jacobson radical. However, the class of all nilpotent rings does not form a radical class by the explanation of Example 2.10 in [1]. Now, let A be a ring and let γ be a radical class. It follows from the definition of a radical class that $\gamma(A) = \sum\{I|I$ is an ideal of A such that $I \in \gamma\}$.

In the study of radical classes of rings, a natural question arises concerning the role and significance of a given radical class in determining the structure of a ring. There are two fundamental constructions of radical classes in ring theory: the lower radical construction and the upper radical construction. A detailed exposition of these constructions can be found in Chapter II of [1]. One important example of an upper radical class is the prime radical class β, which is precisely the upper radical class of all prime rings, that is $\beta = U(\pi) = U(\rho)$, where π is the class of all prime rings, and ρ is the class of all semiprime rings [1]. Although,

ρ properly contains π, both classes generate the same upper radical, namely β. Furthermore, it follows from the relevant example in [1] that the class π of all prime rings properly contains the class $*_k$.

Example 1.

1. Every prime ring is a semiprime ring. Now let $\mathbb{R}$ be the fields of real numbers. The ring $\mathbb{R}^n = \{(a_1, \dots, a_n) | a_i \in \mathbb{R}, i \in \{1, \dots, n\}\}$ forms a semiprime ring which is prime. Thus ρ properly contains π.
2. Since every ring in $*_k$ is prime, the class $*_k$ is contained in π. Moreover, the ring of all integer numbers $\mathbb{Z}$ is a prime ring which is not in $*_k$ since the ring quotient $\mathbb{Z}/p\mathbb{Z}$, where p is a prime number, forms a prime ring. Hence, π properly contains $*_k$.
3. Now, we consider the set $Q = \left\{\frac{2a}{2b+1} \middle| a, b \in \mathbb{Z}, \gcd(2a, 2b+1) = 1\right\}$ is a member of $*_k$ since Q is a prime ring which has no non-zero proper prime ideal, Q is a $*$ −ring. Hence, $Q \in *_k$. Moreover, let $R = \{(x, y) | x \in Q, y \in W\}$, where W is a simple idempotent ring of characteristic 0 without unity. The set $\mathcal{H}(R)$ constructed by the intersection of all the nonzero ideals of R is a member of $*$, but the ring R is not a member of $*$ [54].
4. Naturally, every ideal of a member of $*$ is again a member of $*$ [46].

Now, recall the definition of the IndaH radical, which is motivated by the existence of the class $*$ which has been mentioned and studied previously in [46].

Definition 1 [46]. The IndaH radical is defined as the upper radical class $U(*_k) = \{R | R$ has no nonzero ideal in $*_k\}$, where $*_k$ is the essential closure of $*$, and $* = \{R$ is a prime ring such that R/I is not a prime for every nonzero proper ideal I of $R\}$.

It is clear that the IndaH radical is not an empty set since the IndaH radical contains $2\mathbb{Z}_4$. Moreover, the IndaH radical also contains all rings R such that $ab = 0$ for every $a, b \in R$ [1]. Now let γ_1 and γ_2 be different radical classes of rings. In the case of commutative rings, for some radicals, it is quite possible to form the same structure, i.e, $\gamma_1(A) = \gamma_2(A)$, but for rings in general, this is not necessarily the case, i.e, $\gamma_1(A) \subset \gamma_2(A)$ or $\gamma_2(A) \subset \gamma_1(A)$. Moreover, the radical structure of a ring can be examined in detail by comparing it with the structure of the set of all nilpotent elements of the ring. To clarify this discussion, consider the definition of an NI ring [27]. A ring A is called NI if $N(A) = \mathcal{J}(A)$ where $N(A)$ is the set of all nilpotent elements of A and $\mathcal{J}(A)$ is the Jacobson radical of A. Some concrete examples of NI rings can be seen in detail in [27].

Moreover, several examples of concrete rings, especially finite rings, can have their radicals traced by using computational approaches. However, before that, the following definitions and theorems will be used as the concept for developing computational algorithms.

Definition 2 [1]. A radical class γ is called matric-extensible if $\forall n \in N, A \in \gamma \Leftrightarrow$ the set of all $n \times n$ square matrices over A is a member of γ.

Furthermore, the following theorem is a concept that motivates the computation algorithm.

Theorem 1 [1]. *A radical class γ is matrix-extensible if and only if $\gamma(M_n(A)) = M_n(\gamma(A))$.*

Proof. Please see the proof of Theorem 4.9.3 in [1]. ■

Thus, we have the following example.

Example 2. Consider the ring of integers $\mathbb{Z}_n$ modulo n and let $M_3(\mathbb{Z}_n) = \left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} | a_{ij} \in \mathbb{Z}_n; i, j \in \{1,2,3\}\right\}$. It follows from Example 4.9.8 in [1] that the Jacobson radical is matric-extensible. Hence, $\mathcal{J}(M_n(A)) = M_n(\mathcal{J}(A))$ for every ring A by Theorem 1 of this paper. Thus, we can deploy the following algorithm:

Algorithm to Get the Members of $\mathcal{J}(M_3(\mathbb{Z}_n))$ Computationally
Input : Integer $n > 1$
Output: The number of the members of $M_3(\mathbb{Z}_n)$; the number of $\mathcal{J}(M_3(\mathbb{Z}_n))$, and the elements of $\mathcal{J}(M_3(\mathbb{Z}_n))$.

Step 1 : Prime factorization of n.
For each prime number p dividing n, compute the exponent k such that p^k divides n.
Step 2 : Determine the Jacobson radical of $\mathbb{Z}_n$
If p^k divides n and $k > 1$, then the set:
$pZ_n = \{0, p, 2p, \dots\}$ *mod* n forms the Jacobson radical.
If n is square-free (i.e., no prime divides n more than once), then the Jacobson radical is trivial:
$\mathcal{J}(Z_n) = \{0\}$
Step 3: Construct all 3×3 matrices over $\mathcal{J}(\mathbb{Z}_n)$
Generate all combinations of 9 elements (matrix entries) from $\mathcal{J}(Z_n)$.
Reshape each 9-tuple into a 3×3 matrix.
The total number of such matrices is $|\mathcal{J}(Z_n)|^9$.
Step 4 : Display results
Display the number of elements in Z_n.
Display the number of possible matrices in $M_3(Z_n)$.

Implementing the above algorithm, we finally form Table 1 as follows:

Table 1. Computational Exploration on $\mathcal{J}(M_3(\mathbb{Z}_n))$

n	Number of $M_3(\mathbb{Z}_n)$	Number of $\mathcal{J}(M_3(\mathbb{Z}_n))$	Element(s) of $\mathcal{J}(M_3(\mathbb{Z}_n))$
2	512	1	$\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}$
3	19,683	1	$\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}$
4	262,144	512	$\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&0&2\end{pmatrix},\dots,\begin{pmatrix}2&2&2\\2&2&2\\2&2&2\end{pmatrix}\right\}$
5	1,953,125	1	$\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}$
6	10,077,696	1	$\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}$
7	40,353,607	1	$\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}$

On the other hand, the prime radical $\beta(A)$ is the intersection of the prime ideals of the ring A. Until now, it is still under study whether $\beta(A) = U(*_k)(A)$ for every ring A, where $U(*_k)(A)$ is the IndaH radical of A. In this section, we examine the advanced properties of $U(*_k)$ and the results of this research further reinforce the Gardner conjecture, and comparing the properties of the IndaH radical $U(*_k)$ and the prime radical β. The possession of the same properties will further strengthen the conjecture that coincides with the fact that the Gardner conjecture has a positive solution.

Now, we may recall several important definitions associated with classes of radical rings, including the properties of corner-hereditary, very corner-hereditary, corner-strict, and hereditary for phantom corners (HPC), which can be accessed in detail at [53], [55]. For the readers' convenience, the following important definitions, which are previously unstated, are recalled here [53], [55]:

Definition 3. [53] , [55]

1. A radical class γ is called a corner-strict if γ satisfies $e^2 = e \in R$ & $eRe \in \gamma \Rightarrow eRe \subseteq \gamma(R)$.
2. A radical class γ is said to have HPC (hereditary for phantom corner) if $e^2 = e \in R$ & $eRe \in \gamma \Rightarrow R_{22} \in \gamma$, where $R_{22} = \{r - er - re + ere | r \in R\}$.

To enhance the understanding of these definitions, the following are several examples of radical classes along with their corresponding properties.

Example 3.

1. The Jacobson radical is corner-hereditary, and it is also very corner-hereditary by [53].
2. The prime radical β is corner-hereditary, very-corner hereditary, corner-strict, and HPC by [53],[55].
3. The Brown-McCoy radical is not corner-hereditary by [53].
4. The locally nilpotent radical class is corner-hereditary by [53].
5. The Koethe nilradical is corner-hereditary. However, the Koethe nilradical is very corner-hereditary if the Koethe conjecture has a positive answer, that is, the sum of two nil left ideals of a ring is nil by [53].

The next example is clarified by the following lemma.

Lemma 1. *The Levitzki radical class is very corner-hereditary. Furthermore, the Levitzki radical is also a corner-hereditary radical class.*

Proof. It follows from [1] that the Levitzki radical is precisely the locally nilpotent radical. Thus, it is strongly hereditary and left and right strong by Example 3.2.13 and Proposition 3.17.1 in [1], respectively. Moreover, Levitzki's radical is very corner-hereditary [53]. Furthermore, since every very corner-hereditary radical class is a corner-hereditary radical class [53], the Levitzki radical is a corner-hereditary radical class.

Now, we start to investigate the answer to the question on the corner-hereditariness of $U(*_k)$ by giving the following theorem.

Theorem 1. *The radical class $U(*_k)$ is corner-hereditary.*

Proof. The proof of the corner-hereditariness of $U(*_k)$ can also be derived from Proposition 3.1 in [53]. However, we also present the detailed proof to the reader. It follows from Corollary 2.4 in [46] that $U(*_k)$ is left hereditary. Moreover, let R be any member of $U(*_k)$ and let e is an idempotent element in R. It is clear that $\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \cong R \in U(*_k)$, where $R_{11} = eRe, R_{12} = \{er - ere | r \in R\}, R_{21} = \{re - ere | r \in R\}$, and $R_{22} = \{r - er - re + ere | r \in R\}$ by using the Peirce Decomposition explained in [55]. On the other hand, the matrices $\begin{bmatrix} R_{11} & 0 \\ R_{21} & 0 \end{bmatrix}$ is a left ideal of $\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$. Since $U(*_k)$ is left hereditary, $\begin{bmatrix} R_{11} & 0 \\ R_{21} & 0 \end{bmatrix}$ be the member of $U(*_k)$. Now consider the ideal $\begin{bmatrix} 0 & 0 \\ R_{21} & 0 \end{bmatrix}$ of $\begin{bmatrix} R_{11} & 0 \\ R_{21} & 0 \end{bmatrix}$, the quotient ring $\begin{bmatrix} R_{11} & 0 \\ R_{21} & 0 \end{bmatrix} / \begin{bmatrix} 0 & 0 \\ R_{21} & 0 \end{bmatrix} \cong R_{11} = eRe$. Since $U(*_k)$ is closed under homomorphism, $eRe \cong \begin{bmatrix} R_{11} & 0 \\ R_{21} & 0 \end{bmatrix} / \begin{bmatrix} 0 & 0 \\ R_{21} & 0 \end{bmatrix} \in U(*_k)$. Thus, $U(*_k)$ is corner-hereditary.

Furthermore, the following property is a fact of the radical class $U(*_k)$.

Theorem 2. *$U(*_k)$ is a special corner-hereditary since $U(*_k)$ is a special radical.*

Proof. It is obvious since $U(*_k)$ is the upper radical of a special class $*_k$ of rings. ■

It is clear that if a radical class γ is very corner hereditary, then γ is corner hereditary by Proposition 3.7 in [53]. However, the converse is not by the existence of Example 3.8 in [53]. This condition raises a natural question, whether $U(*_k)$ is very corner-hereditary?

To answer the mentioned question, we will separate this process into several steps as follows, since we need $U(*_k)$ to be corner-strict.

Theorem 3. *$U(*_k)$ is corner-strict.*

Proof. Consider the essential closure $*_k$ of the class $*$ of all $*-$rings. The class $*_k$ satisfies the condition: If S is a proper subring of $A \in *_k$ implies $S/\beta(S) \in *_k$ for all rings A and S as described in Theorem 2.5 and Corollary 2.6 in [46]. Moreover, $U(*_k)$ is left strong by Corollary 2.6 in [46]. Furthermore, it follows from Corollary 2.7 in [46] that $U(*_k)$ is $N-$radical. Since $U(*_k)$ is $N-$radical, $U(*_k)$ is right strong by Corollary 3.18.13 in [1]. Thus $U(*_k)$ is left and right strong. It follows from Corollary 3.15 in [53] that $U(*_k)$ is corner-strict. ■

Theorem 3 shows that the corner-strictness of $U(*_k)$ follows from its strong radical properties on both sides. In particular, the result illustrates how standard closure properties of $*_k$, together with the $N-$radical condition are sufficient to ensure corner-strictness.

Theorem 4. *The essential closure $(*_k)$ of the class of all $*$ −ring is very corner-hereditary, and the semisimple class $SU(*_k)$ is corner-hereditary.*

Proof. It follows from Corollary 2.6 in [46] that $U(*_k)$ is N −radical. Thus $U(*_k)$ is right hereditary (Corollary 3.18.13 in [1]). Hence, $U(*_k)$ is left hereditary and right hereditary, which is also right strong. Thus, by using Theorem 3.16 in [53] $(*_k)$ is very corner-hereditary and the semisimple class $SU(*_k)$. ■

Theorem 4 establishes that the essential closure $*_k$ exhibits a stronger hereditary property with respect to corners than ordinary corner-hereditariness. This result highlights the property of $*_k$ under corner constructions and clarifies the role of the semisimple class $SU(*_k)$ in preserving corner-hereditary properties. Moreover, we have the following corollary:

Corollary 1. *The $\pi(SU(*_k))$ is closed under non-zero corner, where $\pi(SU(*_k))$ is the class of prime rings in $SU(*_k)$.*

Proof. Please see Theorem 4.8 in [53]. ■

Corollary 1 shows that the class of prime rings contained in $SU(*_k)$ is stable under non-zero corner formations. This property is consistent with the property of the prime radical β and further supports the structural robustness of the semisimple class associated with $U(*_k)$.

As a final result of this study, we conclude this section by giving the following theorem.

Theorem 5. *The radical class $U(*_k)$ has HPC.*

Proof. It is clear that $U(*_k)$ is a normal radical, and every normal radical has HPC by Theorem 3.17 [53]. ■

Thus, we have Table 2 here.

Table 2. Properties Comparison Between β and $U(*_k)$

Criteria	Prime Radical β	$U(*_k)$
Left Hereditary	Yes	**Yes**
Right Hereditary	Yes	**Yes**
Left strong	Yes	**Yes**
Right strong	Yes	**Yes**
supernilpotent	Yes	**Yes**
special	Yes	**Yes**
hypernilpotent	Yes	**Yes**
normal	Yes	**Yes**
N-radical	Yes	**Yes**
Corner-hereditary	Yes	**Yes**
Corner-strict	Yes	**Yes**
Very corner hereditary	Yes	**Yes**
HPC	Yes	Yes

Theorem 5 completes the analysis of corner-related properties by confirming that $U(*_k)$ satisfies the hereditary phantom corner (HPC) property. Since HPC is a strong structural condition, this result places $U(*_k)$ among well-behaved radical classes and aligns its property with that of classical radicals studied in the literature.

4. CONCLUSION

Based on the comparison shown in Table 2 and evidenced by the properties discussed in the results and discussion section, we can state that the IndaH radical possesses corner-hereditary, corner-strict, very corner-hereditary, and HPC properties, which certainly reinforces the belief that the Gardner conjecture has positive results. This also further motivates the exploration of whether there are additional properties of the prime radical β that can then be compared with the properties of the IndaH radical $U(*_k)$ in graded version for example.

Author Contributions

Puguh Wahyu Prasetyo: Conceptualization, Data Curation, Formal Analysis, Funding Acquisition, Investigation, Methodology, Project Administration, Resources, Supervision, Writing – Original Draft. Muhammad Ardiyansyah: Data Curation, Formal Analysis, Investigation, Methodology, Resources, Software, Validation, Writing – Review and Editing. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare no competing interests.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools (e.g., ChatGPT) were used solely for language refinement, including grammar, spelling, and clarity. The scientific content, analysis, interpretation, and conclusions were developed entirely by the authors. All final text was reviewed and approved by the authors.

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