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OSCILLATION PROPERTIES OF SOLUTIONS TO CONFORMABLE FRACTIONAL DELAY DIFFERENTIAL SYSTEMS

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ABSTRACT

This article examines the oscillation of solutions to a particular class of conformable fractional nonlinear delay differential systems of order α and 0<α≤1. By employing the equivalence transformation and the associated Riccati substitution technique, we are able to produce some new necessary conditions for the oscillation of all of the solutions of the differential system. Several results reported are extended, unified, and improved over established results. Two examples are provided to show the importance of the main results.

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1. INTRODUCTION

The investigation of oscillatory behavior in delay differential systems has become a major issue in both theoretical research and practical applications. These systems, which are distinguished by the fact that their future state is dependent not only on the current but also on previous states, occur naturally across a wide range of fields. Applications include population biology (where delays are often associated with gestation or maturation periods), engineering control systems (where feedback lags are widespread), and economics, physics, and epidemiology (where processes inherently entail temporal delays). As a result, a significant body of research has been conducted to investigate the oscillation criterion for these systems, with a particular emphasis on scalar and two-dimensional situations (see, for example[1], [2], [3], [4], [5], [6], [7], [8], [9]).

Despite significant progress in the research of two-dimensional delay differential systems, which includes detailed examination of their stability, boundedness, and oscillatory behavior, comparatively few results have been reported for systems with three or more dimensions. This disparity can be due to the greater analytical complexity and mathematical hurdles involved in higher-dimensional systems, such as the complex coupling of components and the more sophisticated behavior arising from multiple delay factors. Nonetheless, research into three-dimensional delay differential systems is gaining traction because they provide a more realistic framework for simulating complex phenomena with multiple interdependent variables that change over time. Some pioneering studies have addressed this issue (see [10], [11], [12], [13], [14], [15], [16]),providing basic findings and analytical tools that pave the way for future research. These publications demonstrate that, while the discipline is still evolving, significant progress has been made in understanding the qualitative dynamics, including oscillation conditions, stability, and the presence of periodic or chaotic solutions in such systems. Continued study in this area is critical for furthering mathematical theory and enhancing the accuracy of models used in scientific and engineering applications.

Although significant progress has been achieved in the oscillation theory of both integer-order and classical fractional differential equations, the study of nonlinear delay differential systems involving the conformable fractional derivative remains relatively underdeveloped. Most of the available results are confined to special cases, leaving a clear need for more general oscillation criteria that can unify, extend, and improve upon existing findings (see [17], [18], [19]). This gap is particularly important because conformable fractional derivatives, with their balance of simplicity and analytical power, provide a natural framework for exploring systems where memory and delay effects play a crucial role.

In this research, we examine the oscillatory behavior of solutions to a class of conformable fractional nonlinear delay differential systems of order α with $0 < \alpha \leq 1$. Our approach relies on equivalence transformations in combination with the Riccati substitution technique, which together enable the derivation of new necessary conditions for oscillation. The results obtained are broad enough to generalize and unify earlier work, while also offering sharper insights into the qualitative dynamics of such systems. The specific objectives of this study are threefold. First, it endeavors to construct a rigorous analytical framework for investigating oscillatory behavior in conformable fractional nonlinear delay differential systems. Second, it focuses on establishing novel oscillation criteria by means of equivalence transformations and Riccati-type substitutions. Third, it seeks to substantiate the theoretical developments through carefully chosen illustrative examples that underscore both the applicability and robustness of the results. The anticipated outcomes of this research are expected to deepen the theoretical understanding of fractional-order systems with delays and to provide a foundation for subsequent advancements in applied mathematics and allied disciplines. Moreover, the findings may offer practical insights in contexts where oscillatory dynamics in memory-dependent systems play a significant role.

2. RESEARCH METHODS

For application in circuit theory, L.L.Rauch [20] proposed a model describing a circuit with a pentode lamp, which is a third-order system of the form

$$\begin{aligned} \dot{\sigma} &= a(f(\sigma) - (1+b)\sigma - z), \\ \dot{y} &= -c(f(\sigma) - \sigma - z), \\ \dot{z} &= -d(y+z), \end{aligned}$$

$f(\sigma)$is a non-linearity of saturation type.

A physical application is considered. Vreeke et al. [21] studied the system of ordinary differential equations to model the two-temperature feedback nuclear reactor problem. Spanikova et al. [22] studied the oscillatory behavior of a neutral-type system. In 1908, Langevin first formulated the following equation

$$\frac{d}{dt}\left(\frac{d}{dt}+\lambda\right)x(s)+p(s)x(s)=0 \text{ for } s\geq r_0,$$

which has proven to be a powerful tool for modeling the evolution of physical systems in fluctuating environments. Recently, Tariboon et al. [23] studied second-order linear impulsive differential equations.

On the other hand, because of their relevance in dynamical systems, fractional differential equations have attracted the attention of several researchers; for further details, see the monographs ([24], [25]). Even though the widely used fractional derivatives are the Riemann-Liouville and Caputo derivatives, which are nonlocal. In recent years, limit-based conformable fractional derivative has played a vital role in the field of fractional calculus. The fundamental concepts of differentiating and integrating in conformable fractional calculus have been developed in ([26], [27], [28], [29]) references cited therein. The conformable fractional derivative has proven to be an effective tool for modeling a variety of mechanical systems, including the fractional harmonic oscillator, the fractional damped oscillator, and the dynamics of a one-dimensional forced oscillation [30]. These models deliver a more precise depiction of systems characterized by memory and non-local phenomena. Furthermore, Tariboon et al. [31] investigated the oscillatory behavior associated with impulsive conformable fractional differential equations, providing valuable insights into the impact of impulsive effects on fractional-order systems.

As evidenced by the existing literature, oscillation theory for delay differential systems has been extensively developed for integer-order models and, more recently, for classical fractional derivatives such as the Riemann–Liouville and Caputo types. In the context of conformable fractional calculus, available studies are largely restricted to scalar equations, linear systems, or models without delay, and typically employ comparison or integral inequality techniques. In particular, none of the works cited in ([10], [16], [18], [19]) address nonlinear delay differential systems formulated via the conformable fractional derivative, nor do they employ equivalence transformations combined with Riccati-type substitutions for deriving oscillation criteria.

The present study fills this gap by considering a general class of nonlinear conformable fractional delay differential systems of order $0<\alpha\leq 1$. The novelty of the proposed approach is threefold. First, the inclusion of both fractional-order dynamics and delay effects within the conformable framework significantly broadens the class of systems previously analyzed. Second, the use of an equivalence transformation coupled with an associated Riccati substitution leads to new oscillation criteria that are not obtainable through standard comparison methods. Third, the derived conditions are sufficiently general to recover known oscillation results for integer-order and non-delay systems as special cases, thereby demonstrating that the present results constitute a genuine extension rather than a parallel formulation.

Consequently, the methodology adopted in this work not only produces new oscillation conditions for a previously unexplored class of systems, but also provides a unified analytical framework that connects and strengthens several earlier results in oscillation theory. This observation motivated us to propose the following system

$$\begin{aligned} D^\alpha x(s)+\lambda(s)x(s)&=a(s)f_2\Big(y\big(\delta(s)\big)\Big)\\ D^\alpha y(s)&=-b(s)f_3\big(z(s)\big)\\ D^\alpha z(s)&=c(s)f_1\Big(x\big(\sigma(s)\big)\Big), s\geq s_0, \end{aligned} \tag{1}$$

where D^α denotes the conformable fractional derivative of order α, where α lies between 0 and 1, inclusive. The following conditions are needed in the sequel:

(A_1) $a(s)\in C^{2\alpha}([r_0,\infty),\mathrm{R}^+), b(s)\in C^{\alpha}([r_0,\infty),\mathrm{R}^+), c(s)\in C([r_0,\infty),\mathrm{R}^+), a(s), b(s)$ and $c(s)$ are not identically zero on $[T_0,\infty)$, where $T_0\geq r_0$, b(s) and c(s) are positive and decreasing, $\lambda(s)\in C([r_0,\infty),\mathrm{R}^+)$; $(A_2) f_i\in C^\alpha(\mathrm{R},\mathrm{R}), uf_i(u)>0, i=1,2,3, D^\alpha f_2(y)\geq k>0, D^\alpha f_3(z)\geq l>0, f_1\in C(\mathrm{R},\mathrm{R})$ and $\frac{f_i(x)}{x}\geq\mu>0$ for $u\neq 0$; (A_3) $\delta(s)\leq s, \sigma(s)\leq s$ with $D^\alpha\delta(s)\geq m>0$ and satisfies $\lim_{s\to\infty}\sigma(s)=\infty$, $\lim_{s\to\infty}\delta(s)=\infty$.

(A_4) $\int_{r_0}^{\infty} \varrho^{\alpha-1}\frac{1}{p(\varrho)}d\varrho=\infty, \int_{r_0}^{\infty} \varrho^{\alpha-1}\frac{1}{q(\varrho)}d\varrho=\infty$, where $p(s)=\frac{1}{b(s)}, q(s)=\frac{1}{a(s)}, \omega(s)=klm^2c(s)$ and $p(s), q(s), \omega(s)$ are positive real valued continuous functions. A solution of system Eq. (1), we mean that a non-trivial vector valued function $(x(s), y(s), z(s)) \in C^\alpha([T_x,\infty)\times[T_y,\infty)\times[T_z,\infty),\mathrm{R})$ for some $T_x\geq r_0, T_y\geq r_0$ and $T_z\geq r_0$, which has the properties that

$$q(s)(D^\alpha x(s)+\lambda(s)x(s))\in C^{2\alpha}([T_x,\infty),\mathrm{R})$$
$$p(s)D^\alpha\left(q(s)\left(D^\alpha x(s)+\lambda(s)x(s)\right)\right)\in C^\alpha([T_x,\infty),\mathrm{R}),$$

and$(x(s), y(s), z(s))$ satisfies the system Eq. (1) on $[T_x,\infty)$. It will be assumed throughout the sequel that the system Eq. (1) solutions exist on some half-line $[T_x,\infty), T_x>r_0$. We consider only the nontrivial solutions of system Eq. (1), that is, the solutions $(x(s), y(s), z(s))$ satisfying sup $\{|x(\varrho)|+|y(\varrho)|+|z(\varrho)|, T\leq\varrho<\infty\}>0$ for any $T\geq T_x$. We consider that system (1) admits at least one such solution. A proper solution $(x(s), y(s), z(s))$ of system Eq. (1) is defined as oscillatory if each of its components eventually exhibits oscillatory behavior; otherwise, the solution is termed nonoscillatory. The system itself is classified as oscillatory if every proper solution behaves in this manner. In this work, we employ an equivalence transformation along with the Riccati transformation technique to establish new oscillation criteria for system Eq. (1). As a direct implication of the analysis, if $(x(s), y(s), z(s))$ is a nonoscillatory solution of system Eq. (1), then the component function $x(s)$ must also be nonoscillatory, as noted in [32].

These are the technical details of this paper. We reviewed several ideas related to the conformable fractional derivative and provided several necessary criteria for the oscillatory nature of the system's solutions Eq. (1) in section 3. The paper's conclusion includes examples demonstrating the effectiveness of the new theorems.

3. RESULTS AND DISCUSSION

In this section, we present the fundamental definitions of conformable fractional derivatives and integrals, which will be crucial for the entire study. Next, we investigate the system's oscillatory behavior described by Eq. (1), under the assumption(A_4).

Definition 1. [27] *Let a function $f:[0,\infty)\to R$ and $s>0$. Then the conformable α-fractional derivative of f is defined by*

$$D^\alpha(f)(s):=\lim_{\epsilon\to 0}\frac{f(s+\epsilon s^{1-\alpha})-f(s)}{\epsilon}\ \text{for all } s>0, \tag{2}$$

$\alpha\in(0,1)$. *If f is α-differentiable in some range $(0,a)$, and $\lim_{s\to 0^+} f^\alpha(s)$ exists, then define*

$$f^\alpha(0):=\lim_{s\to 0^+} f^\alpha(s).$$

Definition 2. [27] *The conformable α-fractional integral of f starting from $a\geq 0$ is defined as*

$$I_a^\alpha(f)(s):=\int_a^s \frac{f(x)}{x^{1-\alpha}}dx, \tag{3}$$

if the Riemann integral (improper) exists and $\alpha\in(0,1]$.

Lemma 1. [24] *Let $f:(a,b)\to R$ be differentiable and $0<\alpha\leq 1$. Then, for all $s>a$ we have*

$$I_a^\alpha D_a^\alpha(f)(s)=f(s)-f(a). \tag{4}$$

Lemma 2. [32] *If $\left(x(s), y(s), z(s)\right)$ non-oscillatory solution of Eq. (1), then the component function $x(s)$ non-oscillatory.*

Lemma 3. *Suppose that (A_1) and (A_4) hold. Then there exists a $r_1\geq r_0$ such that either*

1. $x(s)>0, D^\alpha x(s)>0, D^\alpha(q(s)D^\alpha x(s)+\lambda(s)x(s))>0$ *for $s\geq r_1$, or*
2. *(II) $x(s)>0, D^\alpha x(s)<0, D^\alpha(q(s)D^\alpha x(s)+\lambda(s)x(s))>0$fors $\geq r_1$ holds.*

Proof. Assume that $x(s)$ is a solution of system Eq. (1) that eventually stays positive for all s greater than some r_0. Based on this assumption, system Eq. (1) can be transformed into the following nonlinear delay differential inequality:

$$D^{\alpha}\left(\frac{1}{b(s)}D^{\alpha}\left(\frac{1}{a(s)}(D^{\alpha}x(s)+\lambda(s)x(s))\right)\right)+k\mathrm{lm}^{2}\omega(s)f(x(\sigma(\delta(s))))\leq 0, \tag{5}$$

which implies,

$$D^{\alpha}\left(p(s)D^{\alpha}\big(q(s)(D^{\alpha}x(s)+\lambda(s)x(s))\big)\right)+\omega(s)f(x(\sigma(\delta(s))))\leq 0, s\geq r_1. \tag{6}$$

From Eq. (6), we get

$$D^{\alpha}\left(p(s)D^{\alpha}\big(q(s)(D^{\alpha}x(s)+\lambda(s)x(s))\big)\right)\leq 0 \ \text{ for } s\geq r_0.$$

Then $p(s)D^{\alpha}\big(q(s)(D^{\alpha}x(s)+\lambda(s)x(s))\big)$ is decreasing on (r_0,∞). If we admit

$$D^{\alpha}\big(q(s)(D^{\alpha}x(s)+\lambda(s)x(s))\big)\leq 0,$$

then $q(s)(D^{\alpha}x(s)+\lambda(s)x(s))$ is decreasing and there exists a constant $\eta, r_2\geq r_0$ such that

$$p(s)D^{\alpha}\left(q(s)\big(D^{\alpha}x(s)+\lambda(s)x(s)\big)\right)\leq -\eta \text{ for } s\geq r_2.$$

Taking I_{α} from r_2 to s, by using the Definition (3.1) (see Khalil et al. [27]), we obtain

$$q(s)\big(D^{\alpha}x(s)+\lambda(s)x(s)\big)\leq q(r_2)\big(D^{\alpha}x(r_2)+\lambda(r_2)x(r_2)\big)-\eta\int_{r_2}^{s}\varrho^{\alpha-1}\frac{1}{p(\varrho)}d\varrho. \tag{7}$$

Letting $s\to\infty$ and using (A_4), we get $q(s)(D^{\alpha}x(s)+\lambda(s)x(s))\to-\infty$. Hence, there is an integer $r_3\geq r_2$ such that

$$q(s)\big(D^{\alpha}x(s)+\lambda(s)x(s)\big)\leq q(r_3)\big(D^{\alpha}x(r_3)+\lambda(r_3)x(r_3)\big)<0,$$

for $s\geq r_3$. Again taking I_{α} from r_3 to s , by using the Definition (3.1) (see Khalil et al. [27]), we get

$$x(s)\leq x(r_3)+q(r_3)\big(D^{\alpha}x(r_3)+\lambda(r_3)x(r_3)\big)\int_{r_3}^{s}\varrho^{\alpha-1}\frac{1}{q(\varrho)}d\varrho. \tag{8}$$

As $s\to\infty, x(s)\to-\infty$ by (A_4), which gives a contradiction to $x(s)>0$. Therefore, $D^{\alpha}\big(q(s)(D^{\alpha}x(s)+\lambda(s)x(s))\big)$ is positive and $q(s)(D^{\alpha}x(s)+\lambda(s)x(s))$ is increasing.

Lemma 4. *Suppose that $\sigma(s)\leq s, \delta(s)\leq s, D^{\alpha}u(s)>0, D^{2\alpha}u(s)>0$ on (r_0,∞). Then for each $\beta\in(0,1)$, there exists a $T_{\beta}\geq r_0$, such that*

$$\frac{u(\sigma(\delta(s)))}{(\sigma(\delta(s)))^{\alpha}}\geq\beta\frac{u(s)}{s^{\alpha}}. \tag{9}$$

Proof. From the Mean Value Theorem 2.4 (see Khalil et al. [27]), we have

$$u(s)-u\big(\sigma(\delta(s))\big)=D^{\alpha}u(\xi)\left(\frac{1}{\alpha}s^{\alpha}-\frac{1}{\alpha}(\sigma(\delta(s)))^{\alpha}\right), \xi\in\big(\sigma(\delta(s)),s\big).$$

Now, from the monotone properties of $D^{\alpha}u(s)$,

$$u(s)-u\big(\sigma(\delta(s))\big)\leq D^{\alpha}u\big(\sigma(\delta(s))\big)\left(\frac{1}{\alpha}s^{\alpha}-\frac{1}{\alpha}(\sigma(\delta(s)))^{\alpha}\right),$$

then

$$\frac{u(s)}{u(\sigma(\delta(s)))}\leq 1+\frac{D^{\alpha}u\big(\sigma(\delta(s))\big)}{u\big(\sigma(\delta(s))\big)}\left(\frac{1}{\alpha}s^{\alpha}-\frac{1}{\alpha}(\sigma(\delta(s)))^{\alpha}\right). \tag{10}$$

Again, using Mean Value Theorem 2.4 (see Khalil et al. [27]), we have

$$u\left(\sigma\left(\delta(s)\right)\right) \geq u\left(\sigma\left(\delta(s)\right)\right) - u(r_0) \geq D^\alpha u\left(\sigma\left(\delta(s)\right)\right)\left(\frac{1}{\alpha}(\sigma(\delta(s)))^\alpha - \frac{1}{\alpha}(r_0)^\alpha\right).$$

Here for each $\beta \in (0,1)$, there exists a $T_\beta \geq r_0$, such that

$$\frac{u(\sigma(\delta(s)))}{D^\alpha u(\sigma(\delta(s)))} \geq \frac{\beta}{\alpha}(\sigma(\delta(s)))^\alpha, s \geq T_\beta. \tag{11}$$

Substitute Eq. (10) into Eq. (11), we obtain that

$$\frac{u(s)}{u(\sigma(\delta(s)))} \leq 1 + \frac{\alpha}{\beta}\frac{1}{(\sigma(\delta(s)))^\alpha}\left(\frac{1}{\alpha}s^\alpha - \frac{1}{\alpha}(\sigma(\delta(s)))^\alpha\right) \leq \frac{s^\alpha}{\beta(\sigma(\delta(s)))^\alpha},$$

and hence the proof.

Lemma 5. *Suppose that* $x(s) > 0, D^\alpha x(s) > 0, D^{2\alpha}x(s) > 0, D^{3\alpha}x(s) \leq 0$ *on* (T_β, ∞). *Then*

$$\frac{x(s)}{D^\alpha x(s)} \geq \frac{s - T_\beta}{2}s^{\alpha-1}, s \in \left(T_\beta, \infty\right). \tag{12}$$

Proof. Define

$$X(s) := \left(s - T_\beta\right)s^{1-\alpha}x(s) - \frac{\left(s - T_\beta\right)^2}{2}D^\alpha x(s).$$

Then $X\left(T_\beta\right) = 0$, and

$$D^\alpha X(s) = s^{1-\alpha}\left(s^{1-\alpha}x(s) + (1-\alpha)\left(s - T_\beta\right)s^{-\alpha}x(s) - \frac{\left(s - T_\beta\right)^2}{2}\left(D^\alpha x(s)\right)'\right),$$

which implies

$$X'(s) \geq s^{1-\alpha}x(s) - \frac{\left(s - T_\beta\right)^2}{2}\left(D^\alpha x(s)\right)'.$$

Now, by Taylor's theorem,

$$s^{1-\alpha}x(s) \geq s^{1-\alpha}x\left(T_\beta\right) + \left(s - T_\beta\right)D^\alpha x\left(T_\beta\right) + \int_{T_\beta}^{s}(s-u)\left(D^\alpha x(u)\right)' du,$$

since $(D^\alpha x(u))'$ is nonincreasing. Therefore,

$$X'(s) \geq s^{1-\alpha}x\left(T_\beta\right) + \left(s - T_\beta\right)D^\alpha x\left(T_\beta\right) > 0.$$

Given that $X\left(T_\beta\right) = 0$, thus $X(s) > 0$ for $s \geq T_\beta$. Hence the proof is complete.

These preparations now yield our main theorem.

Theorem 1. *Let* $(A_1) - (A_4)$ *holds. Furthermore, assume that*

$$\limsup_{s\to\infty} \int_{r_0}^{s}\left(\frac{1}{4}\varrho^{1-\alpha}(G_\alpha(\varrho))^2 p(\varrho)\right) d\varrho = \infty, \tag{13}$$

where

$$G_\alpha(s) = \frac{\beta\mu}{2\gamma}\frac{\omega(s)}{p(s)}\frac{(\sigma(\delta(s)))^{2\alpha-1}}{s\phi\left(\sigma\left(\delta(s)\right)\right)}\left(\sigma\left(\delta(s)\right) - T_\beta\right).$$

Then every solution of system Eq. (1) is oscillatory.

Proof. Let Eq. (1) admits a non-oscillatory solution $(x(s), y(s), z(s))$on $[r_0, \infty)$. then$x(s)$ is always nonoscillatory, say there exists $r_1 \in [r_0, \infty)$, such that, for $x(s) > 0, x(\sigma(s)) > 0, x(\sigma(\delta(s))) > 0$ for $s \geq T \geq r_0$. Suppose that case (I) of Lemma 1 holds for $s \geq r_1$. Define the equivalence transformation

$$\phi(s) = \frac{q(s)D^\alpha\left(e^{I^\alpha\lambda(s)}x(s)\right)}{e^{I^\alpha\lambda(s)}x(s)}. \tag{14}$$

One can see that

$$\phi(s) = \frac{q(s)\big(D^\alpha x(s) + \lambda(s)x(s)\big)}{x(s)}, \qquad s \geq r_1. \tag{15}$$

Define the associated Riccati substitution

$$U(s) = \frac{p(s)D^\alpha\big(\phi(s)x(s)\big)}{\phi(s)x(s)}, \qquad s \geq r_1. \tag{16}$$

Thus $U(s) > 0$, differentiating Eq. (16) α-times with respect to s, using Eq. (14), and (A_2), we obtain

$$D^\alpha U(s) \leq -\mu\omega(s)\frac{x\big(\sigma(\delta(s))\big)}{\phi(s)x(s)} - \frac{U^2(s)}{p(s)}. \tag{17}$$

Take $U(s) = D^\alpha(\phi(s)x(s))$, consider $\sigma(\delta(s)) \leq \sigma(s) \leq s$ from Lemma 2, to choose a large $\gamma \geq 2$,

$$\gamma\frac{\phi(\sigma(\delta(s)))D^\alpha(x(\sigma(\delta(s))))}{(\sigma(\delta(s)))^\alpha} \geq \frac{\beta}{s^\alpha}D^\alpha\big(\phi(s)x(s)\big), \qquad s \geq T_\beta. \tag{18}$$

Using Eq. (18) in Eq. (17), we have

$$D^\alpha U(s) \leq -\mu\omega(s)\frac{U(s)}{p(s)}\frac{\beta}{\gamma}\left(\frac{\sigma(\delta(s))}{s}\right)^\alpha \frac{x(\sigma(\delta(s)))}{\phi(\sigma(\delta(s)))D^\alpha(x(\sigma(\delta(s))))} - \frac{U^2(s)}{p(s)}. \tag{19}$$

By Lemma 3, we get

$$D^\alpha U(s) \leq -\mu\omega(s)\frac{U(s)}{p(s)\phi(\sigma(\delta(s)))}\frac{\beta}{\gamma}\left(\frac{\sigma(\delta(s))}{s}\right)^\alpha \frac{\sigma(\delta(s)) - T_\beta}{2}(\sigma(\delta(s)))^{\alpha-1} - \frac{U^2(s)}{p(s)}, \tag{20}$$

which implies that

$$\begin{aligned} U'(s) &\leq -\frac{\beta\mu}{2\gamma}\frac{\omega(s)}{p(s)}\frac{(\sigma(\delta(s)))^{2\alpha-1}}{s\phi(\sigma(\delta(s)))}\big(\sigma(\delta(s)) - T_\beta\big)U(s) - s^{\alpha-1}\frac{U^2(s)}{p(s)} \\ &\leq G_\alpha(s)U(s) - s^{\alpha-1}\frac{U^2(s)}{p(s)}, \end{aligned}$$

by using the inequality $Bu - Au^2 \leq \frac{B^2}{4A}$ becomes

$$U'(s) \leq -\frac{1}{4}s^{1-\alpha}p(s)\big(G_\alpha(s)\big)^2. \tag{21}$$

Integrating from r_1 to s, we get that

$$\int_{r_1}^{s}\frac{1}{4}\varrho^{1-\alpha}p(\varrho)\big(G_\alpha(\varrho)\big)^2 ds \leq U(r_1),$$

which contradicts the hypothesis Eq. (13).

Theorem 2. *Let $(A_1) - (A_4)$ hold. Furthermore, there exist a function $\mathcal{H} \in C(D, s)$, where $D := \{(s, \varrho): s \geq \varrho \geq r_0\}$ such that*

1. *$\mathcal{H}(s, s) = 0$ for $s \geq r_0$.*
2. *$\mathcal{H}(s, \varrho) > 0$for $(s, \varrho) \in D_0$, where $D_0 := \{(s, \varrho): s > \varrho \geq r_0\}$, and $\mathcal{H}$ has a continuous and non-positive partial derivative $\mathcal{H}_s'(s, \varrho) = \frac{\partial\mathcal{H}(s,\varrho)}{\partial\varrho}$ on D_0 with respect to the second variable and satisfies*

$$\limsup_{s\to\infty} \frac{1}{4\mathcal{H}(s,r_1)} \int_{r_1}^{s} \mathcal{H}(s,\varrho)\varrho^{1-\alpha}\big(G_\alpha(\varrho)\big)^2 p(\varrho)d\varrho = \infty, \tag{22}$$

where $G_\alpha(\varrho)$ is defined as in Theorem 1. Then every solution of system Eq. (1) is oscillatory.

Proof. Let Eq. (1) admits a non-oscillatory solution $(x(s), y(s), z(s))$on $[r_0,\infty)$. then $x(s)$ is always nonoscillatory, say there exists $r_1 \in [r_0,\infty)$, such that, for $x(s) > 0, x(\sigma(s)) > 0, x(\sigma(\delta(s))) > 0$ for $s \geq T \geq r_0$. Suppose that Case (I) of Lemma 1 holds for $s \geq r_1$. Then proceding as in the proof of Theorem 1, we get

$$U'(s) \leq -\frac{1}{4}s^{1-\alpha}p(s)(G_\alpha(s))^2. \tag{23}$$

Multiplying by $\mathcal{H}(s,\varrho)$ and integrating both sides from r_1 to s for $s \in [r_1,\infty)$, we obtain

$$\frac{1}{4}\int_{r_1}^{s} \varrho^{1-\alpha}\mathcal{H}(s,\varrho)p(\varrho)(G_\alpha(\varrho))^2 ds \leq \int_{r_1}^{s} \mathcal{H}(s,\varrho)U'(\varrho)d\varrho$$
$$\leq \mathcal{H}(s,r_1)U(r_1) + \int_{r_1}^{s} \mathcal{H}'_s(s,\varrho)U(\varrho)d\varrho.$$

Therefore

$$\limsup_{s\to\infty} \frac{1}{4\mathcal{H}(s,r_1)} \int_{r_1}^{s} \mathcal{H}(s,\varrho)\varrho^{1-\alpha}\big(G_\alpha(\varrho)\big)^2 p(\varrho)d\varrho < U(r_1), \tag{24}$$

which contradicts to the hypothesis Eq. (22).

Theorem 3. *Let $(A_1)-(A_4)$ holds. Furthermore, assume that*

$$\int_{r_0}^{\infty} v^{\alpha-1}\lambda(v)dv < \infty \tag{25}$$

and

$$\int_{r_0}^{\infty}\left(v^{\alpha-1}\frac{1}{q(v)}\int_{v}^{\infty}\left(u^{\alpha-1}\frac{1}{p(u)}\int_{u}^{\infty}\varrho^{\alpha-1}\omega(\varrho)d\varrho\right)du\right)dv = \infty, \tag{26}$$

then $\lim_{s\to\infty} x(s) = 0$.

Proof. Let Eq. (1) admits a non-oscillatory solution $\big(x(s), y(s), z(s)\big)$ on $[r_0,\infty)$. Then $x(s)$ is always non-oscillatory, say there exists $r_1 \in [r_0,\infty)$, such that, for $x(s) > 0$ for $s \geq r_1 \geq r_0$, where r_1 is large enough, Lemma 1 and Lemma 2 hold.

Consider Case (II) of Lemma 1, that is, $D^\alpha x(s) < 0, D^\alpha(\omega(s)D^\alpha x(s)) > 0$for $s \geq r_1$. Since $x(s)$ is postive and decreasing, there exists a $\lim_{s\to\infty} x(s) = \eta \geq 0$. Suppose that $\eta > 0$.

Given that $x(\sigma(\delta(s))) \leq \sigma(s) \leq s$, then $x(\sigma(\delta(s))) \geq x(s) > \eta$ for $s \geq r_2$. Now, from Eq. (3) and (A_2) we get,

$$D^\alpha\Big(p(s)D^\alpha\big(q(s)(D^\alpha x(s) + \lambda(s)x(s))\big)\Big) \leq -\mu\omega(s)x\Big(\sigma\big(\delta(s)\big)\Big), \qquad s \geq r_2. \tag{27}$$

Integrating Eq. (27) from s to ∞, we get

$$I^\alpha D^\alpha\Big(p(s)D^\alpha\big(q(s)(D^\alpha x(s) + \lambda(s)x(s))\big)\Big) \leq -\int_{s}^{\infty} \varrho^{\alpha-1}\mu\omega(\varrho)x\Big(\sigma\big(\delta(\varrho)\big)\Big)d\varrho,$$

then,

$$p(s)D^\alpha\Big(q(s)\big(D^\alpha x(s) + \lambda(s)x(s)\big)\Big) \geq \mu\int_{s}^{\infty} \varrho^{\alpha-1}\omega(\varrho)x\Big(\sigma\big(\delta(\varrho)\big)\Big)d\varrho.$$

Again integrating,

$$-q(s)(D^{\alpha}x(s)+\lambda(s)x(s)) \geq \mu \int_{s}^{\infty}\left(u^{\alpha-1}\frac{1}{p(u)}\int_{u}^{\infty}\varrho^{\alpha-1}\omega(\varrho)x(\sigma(\delta(\varrho)))d\varrho\right)du$$
$$-D^{\alpha}x(s) \geq \lambda(s)x(s)+\mu\frac{1}{q(s)}\int_{s}^{\infty}\left(u^{\alpha-1}\frac{1}{p(u)}\int_{u}^{\infty}\varrho^{\alpha-1}\omega(\varrho)x\left(\sigma\left(\delta(\varrho)\right)\right)d\varrho\right)du,$$

then

$$-x'(s) \geq s^{\alpha-1}\lambda(s)x(s)+\mu s^{\alpha-1}\frac{1}{q(s)}\int_{s}^{\infty}\left(u^{\alpha-1}\frac{1}{p(u)}\int_{u}^{\infty}\varrho^{\alpha-1}\omega(\varrho)x\left(\sigma\left(\delta(\varrho)\right)\right)d\varrho\right)du.$$

Integrating the above inequality from r_2 to ∞, we get

$$x(r_2)\geq \int_{r_2}^{\infty}v^{\alpha-1}\lambda(v)x(v)dv$$
$$+\mu\int_{r_2}^{\infty}\left(v^{\alpha-1}\frac{1}{q(v)}\int_{v}^{\infty}\left(u^{\alpha-1}\frac{1}{p(u)}\int_{u}^{\infty}\varrho^{\alpha-1}\omega(\varrho)x(\sigma(\delta(\varrho)))d\varrho\right)du.\right.$$

Since $x(s)\geq\eta$, we get

$$x(r_2)\geq\eta\int_{r_2}^{\infty}v^{\alpha-1}\lambda(v)dv+\eta\mu\int_{r_2}^{\infty}\left(v^{\alpha-1}\frac{1}{q(v)}\int_{v}^{\infty}\left(u^{\alpha-1}\frac{1}{p(u)}\int_{u}^{\infty}\varrho^{\alpha-1}\omega(\varrho)d\varrho\right)du,\right.$$

then,

$$x(r_2)\geq\eta\mu\int_{r_2}^{\infty}\left(v^{\alpha-1}\frac{1}{q(v)}\int_{v}^{\infty}\left(u^{\alpha-1}\frac{1}{p(u)}\int_{u}^{\infty}\varrho^{\alpha-1}\omega(\varrho)d\varrho\right)du,\right.$$

which contradicts to Eq. (26). Hence $\eta=0$.

Next, we include two examples that highlight the practical relevance and effectiveness of the derived results.

Example 1. We focus on the conformable fractional delay differential system.

$$\begin{aligned} D^{\frac{1}{2}}(x(s))+\frac{1}{\sqrt{s}}x(s)&=\frac{1}{\sqrt{s}}f_2\left(y\left(\frac{s}{2}\right)\right)\\ D^{\frac{1}{2}}(z(s))&=\frac{1}{\sqrt{s}}f_1\left(x\left(\frac{s}{3}\right)\right), s\geq r_0 \end{aligned} \tag{28}$$

Here $\alpha=\frac{1}{2},\lambda(s)=a(s)=b(s)=c(s)=\frac{1}{\sqrt{s}},f_1(x)=C\sqrt{(1-x^2)}+Dx,f_2(y)=Ay+B\sqrt{(1-y^2)}$and $f_3(z)=z$, where $A=\cos(\ln 2)+\sin(\ln 2)$, $B=\cos(\ln 2)-\sin(\ln 2)$,$C=\cos(\ln 3)$, $D=\sin(\ln 3)$.

We can see that $D^{\frac{1}{2}}f_2(y)=\frac{1}{\sqrt{s}}(|B|-|A|)\geq\frac{(|B|-|A|)}{\epsilon}=k>0$, since $\sqrt{s}>\frac{1}{\epsilon}$ for some $\epsilon>0, D^{\frac{1}{2}}f_3(z)=\frac{1}{\sqrt{s}}(|\cos(\ln s)|)\geq\frac{1}{\epsilon}=l>0, f_1(x)/x=C\sqrt{\frac{1}{x^2}-1}+D\geq C+D=1.3475=\mu>0$, since $x^2<1$. $\sigma(s)=\frac{s}{3},\delta(s)=\frac{s}{2},\sigma(\delta(s))=\frac{s}{6}$and $D^{\frac{1}{2}}\delta(s)=\frac{\sqrt{s}}{3}\geq\frac{1}{3\epsilon}m>0,\omega(s)=\frac{|B|-|A|}{9\epsilon^4\sqrt{s}},\phi(s/6)\leq 2,\ \beta=1/2,\gamma=3,G_{\frac{1}{2}}(s)\geq\frac{1.3475}{216\epsilon^4}|B|-|A|\frac{\frac{s}{6}-T_\beta}{s^2}$.

Now consider,

$$\begin{aligned}\limsup_{s\to\infty}\int_{r_0}^{s}\frac{1}{4}\varrho^{1-\alpha}(G_\alpha(\varrho))^2p(\varrho)d\varrho &=\limsup_{s\to\infty}\int_{r_0}^{s}\left(\frac{1}{4}\varrho^{-\frac{1}{2}}\left(\frac{1.3475}{216\epsilon^4}|B|-|A|\frac{\varrho}{\varrho^2}\right)^2\sqrt{\varrho}\right)d\varrho\\ &=\limsup_{s\to\infty}\frac{1}{4}\left(\frac{1.3475}{216\epsilon^4}|B|-|A|\right)^2\int_{r_0}^{s}\frac{\left(\frac{\varrho}{6}-T_\beta\right)^2}{\varrho^3}d\varrho\\ &\to\infty \text{ as } s\to\infty.\end{aligned}$$

All the conditions of Theorem 1 are satisfied. Hence every solution of Eq. (28) is oscillatory. Thus $(x(s),y(s),z(s))=(\sin(\ln s),\cos(\ln s),\sin(\ln s))$ is one such solution.

Example 2. We focus on the conformable fractional delay differential system.

$$\begin{aligned} D^{\frac{1}{3}}(x(s)) + s^{\frac{2}{3}}x(s) &= \frac{s^{\frac{2}{3}}}{e^{\pi}} f_2(y(s-\pi)) \\ D^{\frac{1}{3}}(z(s)) &= \frac{s^{\frac{2}{3}}}{s-\pi} e^{2r-\pi} f_1\big(x(s-\pi)\big), \qquad s \geq r_0 \end{aligned} \tag{29}$$

Here $\alpha = \frac{1}{3}, \lambda(s) = s^{\frac{2}{3}}, a(s) = \frac{s^{\frac{2}{3}}}{e^{\pi}}, b(s) = \frac{s^{\frac{2}{3}}}{e^{2\pi}}, c(s) = \frac{s^{\frac{2}{3}}}{s-\pi} e^{2r-\pi}$, c(s) is positive but increasing and $f_i(u) = u (i = 1,2,3)$.

We can see that $D^{\frac{1}{3}} f_2(y) = D^{\frac{1}{3}}(y) = -\frac{y^{\frac{2}{3}}}{e^{s}} < 0, D^{\frac{1}{3}} f_3(z) = D^{\frac{1}{3}}(z) = z^{\frac{2}{3}} e^{s} = l > 0,\ f_1(x)/x = 1 = \mu > 0, \sigma(s) = \delta(s) = s - \pi$ and $D^{\frac{1}{3}}\delta(s) = s^{\frac{2}{3}} = m > 0$. Now consider,

$$\limsup_{s\to\infty} \int_{r_0}^{\infty} \varrho^{\alpha-1} \frac{1}{p(\varrho)} d\varrho = \limsup_{s\to\infty} \int_{r_0}^{s} e^{-2s} d\varrho \to 0 \text{ as } s \to \infty,$$

Some of the conditions in Theorem 3 are not satisfied. In fact, (A_2) and (A_4) fail to hold. Thus, the system Eq. (29) admits a non-oscillatory solution $(x(s), y(s), z(s)) = (te^{-s}, e^{-s}, e^{s})$.

Remark 1. A forced system of the following type can use the results from our study with a small change

$$\begin{aligned} D^{\alpha}x(s) + \lambda(s)x(s) &= a(s)F_2(y(\delta(s))) \\ D^{\alpha}y(s) &= -b(s)F_3(z(s)) \\ D^{\alpha}z(s) &= F_1\Big(s, x\big(\sigma(s)\big)\Big) + e(s), s \geq r_0. \end{aligned}$$

4. CONCLUSION

In this paper, the authors have established several new necessary and sufficient conditions for the oscillation of a specific class of nonlinear conformable fractional delay differential systems. This has been achieved by applying the Riccati transformation method in conjunction with an equivalence transformation, providing a robust framework for analyzing the oscillatory behavior of these systems. The results presented represent a significant advancement and can be regarded as a natural extension of the work in [33], albeit without consideration of impulsive effects. By omitting the impulse component, this study broadens the scope of oscillation criteria applicable to conformable fractional delay systems, thereby deepening the understanding of their dynamic properties and contributing to the existing body of research on fractional differential equations.

Author Contributions

M. Deepa: Investigation, Formal Analysis, Writing-Original Draft. M. Sathish Kumar: Conceptualization, Methodology, Validation, Supervision, Formal Analysis, Writing-Review and Editing. K. Karuppiah: Supervision, Methodology, Validation. S. Abhirami: Validation, Formal Analysis, Writing-Review and Editing. All authors discussed the results and contributed to the final manuscript.

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The authors declare no competing interest.

Declaration of Generative AI and AI-assisted technologies

The authors declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, including for writing, editing, data analysis, or the creation of tables and figures.

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