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BIFRAMES AND THEIR PROPERTIES IN QUATERNIONIC HILBERT SPACES

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ABSTRACT

In this paper, we focus on defining and analyzing biframes within quaternionic Hilbert spaces, thereby extending the study of biframes beyond the complex Hilbert space framework. We establish that frames can be viewed as a specific instance of biframes. Moreover, we introduce the concept of biframe operators and utilize it to derive a reconstruction formula. Several characterization results pertaining to biframes are also discussed. Additionally, the conditions under which the image of a biframe under a bounded linear operator continues to retain the biframe structure are examined.

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1. INTRODUCTION

During the investigation of non-harmonic Fourier series, Duffin and Schaeffer [1] introduced the concept of frames in the setting of Hilbert spaces. The full potential of frame theory was not widely recognized until the seminal work of Daubechies, Grossmann, and Meyer [2]. They highlighted the usefulness and importance of frame theory, with applications in various related areas such as image and signal processing, sigma–delta schemes, filter bank structures, sampling techniques, and wireless communication systems. Christensen [3] explained the general properties of frames and their interconnections with Riesz bases, emphasizing the mathematical significance.

In response to evolving mathematical and real-world applications, many generalizations of frames (classical) have emerged in recent years. Notable among these are fusion frames, studied by Cazassa and Kutyniok [4], and K-frames introduced by Găvruţa [5]. Each generalization [6] provided a more flexible framework for different analytical scenarios.

Another notable extension is the theory of biframes, formulated by Parizi, Alijani, and Dehghan [7]. Biframes involve the interplay between two distinct sequences or families of vectors that collectively facilitate stable and redundant signal representations within Hilbert spaces. The flexibility of biframes makes them particularly useful for tackling complex problems in multi-channel signal processing, operator-driven decompositions, and multi-resolution analysis. Despite their promising applications, biframes remain a relatively underdeveloped topic—especially in the realm of quaternionic Hilbert spaces, where the non-commutative nature of quaternions introduces unique theoretical challenges. For foundational concepts and detailed background in functional analysis, one may refer to [8], [9].

Quaternions, a generalization of the complex number system, were introduced by William Rowan Hamilton in 1843. They form a four-dimensional non-commutative division algebra over the field of real numbers. One may observe that quaternions [10] are widely applied in analyzing rotations within higher-dimensional Euclidean spaces and hold importance in Newtonian mechanics, quantum mechanics, and relativity theory. Specifically, in general relativity, they provide a framework for expressing Lorentz transformations.

Frames in finite-dimensional quaternionic Hilbert spaces were first introduced by Khokulan et al. [11], while Sharma and Goel [12] extended the study to separable right quaternionic Hilbert spaces. Ellouz [13] investigated K-frames in this setting, and Bhardwaj et al. [14] proposed the concept of OPV-frames. Further generalizations, including fusion frames [15] and K-fusion frames [16], [17], have recently drawn considerable interest among researchers. Additionally, Ghiloni et al. [18] broadened the notion of continuous functional calculus to quaternionic Hilbert spaces.

Biframes have been well-studied in complex Hilbert spaces (e.g., Parizi et al. [7]), but their extension to quaternionic Hilbert spaces has received little attention. In this work, we fill this gap by developing the theory of biframes specifically for quaternionic Hilbert spaces [18]. We show that classical frames are a special case of biframes, providing a unified framework that builds on existing results. In contrast to the complex Hilbert space setting, the quaternionic case brings in additional difficulties mainly due to the non-commutativity of scalars. This affects how various associated operators are defined and handled. As a result, several results from the complex theory do not carry over directly and need to be suitably reformulated in the quaternionic setting. We define and study the biframe operator in this context and derive a reconstruction formula adapted to the non-commutative setting. In addition, we present new characterization results specific to quaternionic Hilbert spaces and examine operator-stability by identifying conditions under which bounded linear operators preserve the biframe property. These contributions go beyond a straightforward extension of prior work, introducing new methods and insights that advance quaternionic frame theory and open possibilities for applications in signal processing and harmonic analysis.

2. RESEARCH METHODS

Biframes were originally introduced and investigated by Parizi et al. [7]. This study extends frame theory from complex to quaternionic Hilbert spaces, carefully addressing the complexities of non-commutative quaternionic multiplication. The methodology begins by defining biframes in quaternionic Hilbert spaces, highlighting their connection to classical frames as a specific instance. A biframe operator is introduced as a key tool to examine biframes' structure and properties. Using this operator, a reconstruction formula is developed for signal representation and recovery in these spaces. The study also provides several characterization results to enhance understanding of biframes' structural attributes. Lastly, an operator-theoretic analysis establishes conditions ensuring that the image of a biframe under a bounded linear operator retains the biframe property.

Throughout the manuscript, $\mathfrak{Q}$ denotes the non-commutative field of quaternions, $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{R}$ denotes the set of real numbers. The symbols $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$, represent separable right quaternionic Hilbert spaces. The set of all bounded (right $\mathfrak{Q}$ -linear) operators on $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ is denoted by $\mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$. A separable right quaternionic Hilbert space will be abbreviated as QHS.

In the study of quaternionic Hilbert spaces, recall from [18] the definition of a right quaternionic pre-Hilbert space, denoted by $V_R(\mathfrak{Q})$.

Definition 1. [12] *The space $V_R(\mathfrak{Q})$ is a right quaternionic vector space together with the binary mapping $\langle .|.\rangle : V_R(\mathfrak{Q}) \times V_R(\mathfrak{Q}) \to \mathfrak{Q}$ (called the Hermitian quaternionic inner product) satisfying:*

1. $\overline{\langle a_1|a_2\rangle} = \langle a_2|a_1\rangle$ *for all* $a_1, a_2 \in V_R(\mathfrak{Q})$.
2. $\langle a|a\rangle > 0$ *if* $a \neq 0$.
3. $\langle a|a_1 + a_2\rangle = \langle a|a_1\rangle + \langle a|a_2\rangle$ *for all* $a, a_1, a_2 \in V_R(\mathfrak{Q})$.
4. $\langle a|uq\rangle = \langle a|u\rangle q$ *for all* $a, u \in V_R(\mathfrak{Q})$ *and* $q \in \mathfrak{Q}$.

From this inner product, we define a norm on $V_R(\mathfrak{Q})$ by $\|a\| = \sqrt{\langle a|a\rangle}$, for all $a \in V_R(\mathfrak{Q})$. The space $V_R(\mathfrak{Q})$ is complete under the above norm and is called the right quaternionic Hilbert space, denoted by $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. Within this framework, we can talk about frames, which are generalizations of orthonormal bases.

Definition 2. [12] *Let $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. The sequence $\{u_i\}_{i=1}^{\infty}$ is defined to be a frame for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$, if there exist two real numbers with $0 < r_1 \leq r_2 < \infty$ (termed as lower and upper bounds) satisfying the following:*

$$r_1\|v\|^2 \leq \sum_{i=1}^{\infty} |\langle u_i|v\rangle|^2 \leq r_2\|v\|^2, \quad \textit{for all } v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q}).$$

Operators also play a crucial role in quaternionic Hilbert spaces. An operator $T \in \mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$ is called a positive [18] operator if $\langle Tv|v\rangle = \langle v|Tv\rangle \geq 0$, for all $v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. A fundamental result is that every positive operator $\mathrm{T} \in \mathfrak{B}\big(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})\big)$ is also a self-adjoint operator. In a right quaternionic Hilbert space, the Cauchy-Schwarz Inequality [18] states that $|\langle v|w\rangle|^2 \leq \langle v|v\rangle \langle w|w\rangle$ for all $v, w \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. Finally, we consider the concept of a Riesz basis for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$.

Definition 3. [19] *Let $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ be a right quaternionic Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Then, a Riesz basis for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$, is of the form $\{Se_i\}_{i=1}^{\infty}$ where $S \in \mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$ is a bijective right linear operator.*

Definition 4. [10] *Let $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ be a right quaternionic Hilbert space and $T \in \mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$ be a right linear operator, then the adjoint operator T^* of T is defined as follows:*

$$< v|Tw > = < T^*v|w > \forall\, v, w \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q}).$$

3. RESULTS AND DISCUSSION

In this work, we extend the study of biframes to the setting of quaternionic Hilbert spaces, which are natural generalizations of complex Hilbert spaces. This section is devoted to the definition and existence of biframes in quaternionic Hilbert spaces through various examples and counterexamples. Also, we demonstrate that every frame is a special case of a biframe, showing that biframes provide a more flexible and encompassing structure for analysis. To deepen the theoretical framework, we introduce the notion of biframe operators, a central tool in our study. Using this operator, we derive a reconstruction formula analogous to those in classical frame theory, ensuring that any element in the space can be stably and accurately reconstructed using the biframe components. We also present a range of characterization theorems that offer insight into the structural aspects of biframes in the quaternionic setting. Finally, we explore the stability of biframes under bounded linear operators, identifying specific conditions under which the image of a biframe remains a biframe. This analysis is crucial for practical applications where data transformations are common.

3.1 Biframes in QHS

Definition 5. *A pair of sequences* $(F,G) = (\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ *in* $\mathbb{H}^{R}(\mathfrak{Q})$ *is termed a biframe for* $\mathbb{H}^{R}(\mathfrak{Q})$*, if there exist* $0 < r_1 \leq r_2 < \infty$ *satisfying*

$$r_1 ||v||^2 \leq \sum_{i=1}^{\infty} \langle v \mid f_i \rangle \langle g_i \mid v \rangle \leq r_2 ||v||^2, \qquad for\ all\ \ v \in \mathbb{H}^{R}(\mathfrak{Q}).$$

The real numbers r_1 *and* r_2 *are called the lower and upper biframes bound, respectively. Moreover, if* $r_1 = r_2$ *then biframe is called tight biframe and if* $r_1 = r_2 = 1$*, the biframe is termed as Parseval biframe.*

Example 1. Let $\{u_i\}_{i=1}^{\infty}$ be any frame in $\mathbb{H}^{R}(\mathfrak{Q})$, with lower and upper bounds as r_1 and r_2 respectively. Consider $f_i = g_i = u_i$, for all $i \in \mathbb{N}$. Then the pair $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ is a biframe with biframe bounds same as frame bounds.

The above example shows that the frames are a particular case of biframes, but it is not always true that any two frames form a biframe for $\mathbb{H}^{R}(\mathfrak{Q})$,

Example 2. Let $\mathbb{H}^{R}(\mathfrak{Q})$ be any three-dimensional QHS over $\mathfrak{Q}$ with an orthonormal basis $\{e_1, e_2, e_3\}$. Then the families $\{f_i\}_{i=1}^{\infty} = \{e_1, e_1, e_2, e_3, 0,0, \ldots\}$ and $\{g_i\}_{i=1}^{\infty} = \{e_2, e_3, e_1, e_1, 0,0, \ldots\}$ are frames for $\mathbb{H}^{R}(\mathfrak{Q})$. But the pair $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ does not form a biframe as

$$\textstyle\sum_{i=1}^{\infty} \langle v|f_i\rangle \langle g_i|v\rangle = \langle v|e_1\rangle \langle e_2|v\rangle + \langle v|e_1\rangle \langle e_3|v\rangle + \langle v|e_2\rangle \langle e_1|v\rangle + \langle v|e_3\rangle \langle e_1|v\rangle.$$

For $v = e_3$ the lower bound condition does not hold.

Next, we gave an example of biframe such that none of the family members form a frame for $\mathbb{H}^{R}(\mathfrak{Q})$.

Example 3. Let $\mathbb{H}^{R}(\mathfrak{Q})$ be any infinite dimensional QHS over $\mathfrak{Q}$ with an orthonormal basis

$\{e_i\}_{i=1}^{\infty}$. Define $\{f_i\}_{i=1}^{\infty} = \{e_1, \frac{e_2}{2}, \frac{e_3}{3}, \cdots\}$ and $\{g_i\}_{i=1}^{\infty} = \{e_1, 2e_2, 3e_3, \cdots\}$. Clearly, the families $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ do not form a frame for $\mathbb{H}^{R}(\mathfrak{Q})$. But the pair $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ forms a biframe for $\mathbb{H}^{R}(\mathfrak{Q})$. For any $v \in \mathbb{H}^{R}(\mathfrak{Q})$,

$$\begin{aligned}\sum_{i=1}^{\infty} \langle v|f_i\rangle \langle g_i|v\rangle &= \sum_{i=1}^{\infty} \left\langle v \middle| \frac{e_i}{i} \right\rangle \langle ie_i|v\rangle \\ &= \sum_{i=1}^{\infty} \langle v|e_i\rangle \ \langle e_i|v\rangle \\ &= \sum_{i=1}^{\infty} \overline{\langle e_i|v\rangle} \ \langle e_i|v\rangle \\ &= \sum_{i=1}^{\infty} |\langle e_i|v\rangle|^2 \\ &= ||v||^2.\end{aligned}$$

One important property of biframes is that if we swap the two sequences, they still form a biframe, as shown in the following result.

Lemma 1. *If* $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ *is a biframe for* $\mathbb{H}^{R}(\mathfrak{Q})$*, then* $(\{g_i\}_{i=1}^{\infty}, \{f_i\}_{i=1}^{\infty})$ *also forms a biframe for* $\mathbb{H}^{R}(\mathfrak{Q})$.

Proof. Let r_1 and r_2 be lower and upper bounds such that

$$r_1\|v\|^2 \leq \sum_{i=1}^{\infty}\langle v|f_i\rangle\ \langle g_i|v\rangle \leq\ r_2\ \|v\|^2, \quad \text{for all}\ \ v\ \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q}).$$

Since

$$\sum_{i=1}^{\infty}\langle v|f_i\rangle\,\langle g_i|v\rangle = \sum_{i=1}^{\infty}\overline{\langle v|f_i\rangle\,\langle g_i|v\rangle} = \sum_{i=1}^{\infty}\langle v|g_i\rangle\,\langle f_i|v\rangle,$$

we obtain $r_1\|v\|^2 \leq \sum_{i=1}^{\infty}\langle v|g_i\rangle\,\langle f_i|v\rangle \leq r_2\|v\|^2$, for all $v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. Hence $(\{g_i\}_{i=1}^{\infty}, \{f_i\}_{i=1}^{\infty})$ is a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. ■

Definition 6. *Let $(F,G) = (\{f_i\}_{i=1}^{\infty}\ ,\{g_i\}_{i=1}^{\infty})$ be a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. The biframe operator $S_{F,G} : \mathbb{H}^{\mathrm{R}}(\mathfrak{Q}) \rightarrow \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ is defined as*

$$S_{F,G}(v) = \sum_{i=1}^{\infty} f_i\langle g_i|v\rangle,\ \text{for all}\ \ v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q}).$$

Next, the equivalent conditions for an operator to be invertible are examined. These conditions are useful in deriving various properties of the biframe operators.

Lemma 2. *Let $T \in \mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$ be a right linear operator, then the following statements are equivalent:*

1. *T is invertible.*
2. *T and T^* are bounded below.*
3. *T and T^* are injective operators with closed range.*

Proof. $(i) \Rightarrow (ii)$. Let T be an invertible operator. For $v\ \in\ \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$, we have

$$\|v\| = \|T^{-1}Tv\| \leq \|T^{-1}\|\|Tv\|; \qquad \|T^{-1}\|^{-1}\|v\|\ \leq\ \|Tv\|.$$

Thus, T is bounded below. By Remark 2.16 [18], $(T^*)^{-1}$ exists and T^* is bounded below.

$(ii)\ \Rightarrow (iii)$. Let the operators T and T^* be bounded below. Thus, both are injective operators. As every bounded below right linear operator has a closed range, we have the operators T and T^* have closed range.

$(iii)\ \Rightarrow (i)$. By Proposition 2.14 [18], we have $R(T)\ =\ (N(T^*))^{\perp}\ =\ \{0\}^{\perp}\ =\ \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. Hence, T is an invertible linear operator. ■

Theorem 1. *The biframe operator $S_{F,G}$ is a well defined right linear operator that is bounded, positive and invertible.*

Proof. The linearity and positivity of $S_{F,G}$ follow directly from the definition. For $n \in \mathbb{N}$, define

$$S_n(v) = \sum_{k=1}^{n} f_k\langle g_k|v\rangle, \qquad v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q}).$$

Then, $\{S_n\}_{n\,\in\mathbb{N}}$ is a bounded right linear operator on $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. Consider $i,j\ \in \mathbb{N}, i > j$

$$\begin{aligned}\left|\langle S_i v|v\rangle - \langle S_j v|v\rangle\right| &= \left|\langle (S_i\ - S_j)v|v\rangle\right| \\ &\leq \left|\langle \sum_{k=j+1}^{i}\ f_k\langle g_k|v\rangle|v\rangle\right| \\ &\leq \sum_{k=j+1}^{i}\left|\overline{\langle g_k|v\rangle}\,\langle f_k|v\rangle\right| \\ &= \sum_{k=j+1}^{i}|\langle v|g_k\rangle\,\langle f_k|v\rangle|.\end{aligned}$$

The series $\sum_{k=1}^{\infty}\langle v|g_k\rangle\,\langle f_k|v\rangle$ converges in $\mathbb{R}$. Thus, the sequence $\{\langle S_n v|v\rangle\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Hence there exists a right linear operator $S\ \in\ \mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$ such that $\{S_n\}_{n=1}^{\infty}$ weakly convergent to S.

$$\langle Sv|v\rangle =\ \lim_{n\rightarrow\infty}\langle S_n v|v\rangle$$

$$= \sum_{i=1}^{\infty} f_i \langle g_i | v \rangle | v \rangle$$
$$= \langle S_{F,G} v | v \rangle, \quad v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q}).$$

This shows that $S = S_{F,G}$ and hence, $S_{F,G}$ is a well defined right linear operator on $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. For $v, w \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$, we write

$$\begin{aligned}\langle S_{F,G} v | w \rangle &= \langle \sum_{i=1}^{\infty} f_i \langle g_i | v \rangle | w \rangle \\ &= \sum_{i=1}^{\infty} \overline{\langle g_i | v \rangle} \langle f_i | w \rangle \\ &= \sum_{i=1}^{\infty} \langle v | g_i \rangle \langle f_i | w \rangle \\ &= \langle v | \sum_{i=1}^{\infty} g_i \langle f_i | w \rangle \rangle.\end{aligned}$$

Therefore, the adjoint of the operator $S_{F,G}$ is given by:

$$S_{F,G}^* (v) = \sum_{i=1}^{\infty} g_i \langle f_i | v \rangle.$$

Using Lemma 1, we have $S_{F,G}^* = S_{G,F}$ and by the definition of biframe the operators $S_{F,G}$ and $S_{G,F}$ are injective. To prove $S_{F,G}$ is invertible it is sufficient to prove that $R(S_{F,G})$ is closed (Lemma 2). Let $\{v_n\}_{n=1}^{\infty} \in R(S_{F,G}))$ be a sequence converges to $v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. Let $w_n = S_{F,G} v_n$, where $v_n \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. For any $n, m \in \mathbb{N}$, $n > m$, by biframe lower inequality and Theorem 2, we have

$$\begin{aligned} r_1 \|v_n - v_m\|^2 &\le \langle v_n - v_m | S_{F,G}(v_n - v_m) \rangle \\ &\le \|S_{F,G}(v_n - v_m)\| \, \|v_n - v_m\| \\ &= \|w_n - w_m\| \|v_n - v_m\|. \end{aligned}$$

Since $\|v_n - v_m\| \to 0$ as $n, m \to \infty$, we get that $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. Thus, there exists an element $v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ such that $\|v_n - v\| \to 0$ as $n \to \infty$. Again as $S_{F,G}$ is bounded and it's a continuous linear operator. Therefore, we have $S_{F,G}\, v = w$. Similarly, we can prove $R(S_{F,G}^*)$ is a closed set. ■

Remark 1. *For the biframe (F, G), the operator $S_{F,G}$ is a self-adjoint operator* (Theorem 1). Thus,

$$S_{F,G} = S_{F,G}^* = S_{G,F}\,. \tag{1}$$

Next, we have studied the reconstruction theorem by using biframes. It shows that any arbitrary element of the right quaternionic Hilbert space $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ can be rewritten in terms of a biframe operator.

Theorem 2. (Reconstruction Formula) *Let $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ be a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ with the corresponding biframe operator $S_{F,G}$. Then, any $v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ can be expressed as*:

$$v = \sum_{i=1}^{\infty} f_i \langle S_{G,F}^{-1}\, g_i | v \rangle \tag{2}$$

$$= \sum_{i=1}^{\infty} S_{F,G}^{-1}\, f_i \langle g_i | v \rangle \tag{3}$$

$$= \sum_{i=1}^{\infty} S_{G,F}^{-1}\, f_i \langle g_i | v \rangle.$$

Proof. Let $v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. To derive Eq. (3), we use the fact that the biframe operator $S_{F,G}$ is invertible and write

$$\begin{aligned} v &= S_{F,G}\, S_{F,G}^{-1}\, v \\ &= \sum_{i=1}^{\infty} f_i \, \langle g_i | S_{F,G}^{-1}\, (v) \rangle \end{aligned}$$

$$= \sum_{i=1}^{\infty} f_i \langle (S_{F,G}^{-1})^* g_i | v \rangle$$
$$= \sum_{i=1}^{\infty} f_i \langle (S_{F,G}^*)^{-1} g_i | v \rangle$$
$$= \sum_{i=1}^{\infty} f_i \langle S_{G,F}^{-1} g_i | v \rangle.$$

Again, using the invertibility of the biframe operator, we express

$$v = S_{F,G}^{-1} S_{F,G} v$$
$$= S_{F,G}^{-1} \left(\sum_{i=1}^{\infty} f_i \langle g_i | v \rangle \right)$$
$$= \sum_{i=1}^{\infty} S_{F,G}^{-1} f_i \langle g_i | v \rangle,$$

which proves Eq. (3). Using Eq. (1), we obtain

$$v = \sum_{i=1}^{\infty} S_{F,G}^{-1} f_i \langle g_i | v \rangle = \sum_{i=1}^{\infty} S_{G,F}^{-1} f_i \langle g_i | v \rangle. \blacksquare$$

Remark 2. *For* $v \in \mathbb{H}^{R}(\mathfrak{Q})$, *the sequence* $\{\langle S_{G,F}^{-1} g_i | v \rangle\}_{i=1}^{\infty})$ *is called as the biframe coefficient of* v.

Theorem 3. *Let* $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ *be a biframe for* $\mathbb{H}^{R}(\mathfrak{Q})$. *Then, the sequence* $\{f_i\}_{i=1}^{\infty}$ *is a Riesz basis for* $\mathbb{H}^{R}(\mathfrak{Q})$ *if and only if the sequence* $\{g_i\}_{i=1}^{\infty}$ *is a Riesz basis for* $\mathbb{H}^{R}(\mathfrak{Q})$.

Proof. Let $\{g_i\}_{i=1}^{\infty}$ be a Riesz basis. Then, there exists an invertible right linear operator $T \in \mathfrak{B}(\mathbb{H}^{R}(\mathfrak{Q}))$ and an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for $\mathbb{H}^{R}(\mathfrak{Q})$ such that $g_i = T(e_i), i \in \mathbb{N}$. Define a right linear operator $U : \mathbb{H}^{R}(\mathfrak{Q}) \to \mathbb{H}^{R}(\mathfrak{Q})$ by $U(v) = S_{F,G}(T^*)^{-1} v$, where $v \in \mathbb{H}^{R}(\mathfrak{Q})$. Then, U is a bounded invertible right linear operator on $\mathbb{H}^{R}(\mathfrak{Q})$. For any $j \in \mathbb{N}$, we write

$$U(e_j) = S_{F,G}(T^*)^{-1}(e_j)$$
$$= \sum_{i=1}^{\infty} f_i \langle g_i | (T^*)^{-1}(e_j) \rangle$$
$$= \sum_{i=1}^{\infty} f_i \langle T(e_i) | (T^*)^{-1}(e_j) \rangle$$
$$= \sum_{i=1}^{\infty} f_i \langle e_i | e_j \rangle$$
$$= f_j.$$

Thus, $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis for $\mathbb{H}^{R}(\mathfrak{Q})$. The converse can be proved by employing the Eq. (1) and retracing the above steps. ■

3.2 Characterization of Biframes in QHS

In this section, we study various characterization of biframes in QHS. It is shown that the bounded image of a biframe in $\mathbb{H}^{R}(\mathfrak{Q})$ need not be a biframe. The necessary conditions under which a bounded image of a biframe is again a biframe for $\mathbb{H}^{R}(\mathfrak{Q})$ explored.

The next result presents a characterization for a pair of sequences to form biframes in terms of their corresponding biframe operator. The next lemma plays a crucial role in the proof of subsequent results.

Lemma 3. *Let* $T \in \mathfrak{B}(\mathbb{H}^{R}(\mathfrak{Q}))$ *be a bounded, positive and self-adjoint right linear operator. Then, for any* u *and* $v \in \mathbb{H}^{R}(\mathfrak{Q})$,

$$|\langle u | Tv \rangle| \leq \langle u | Tu \rangle^{\frac{1}{2}} \langle v | Tv \rangle^{\frac{1}{2}}.$$

Hence,

$$\|Tv\| \leq \|T\|^{\frac{1}{2}} \langle v|Tv\rangle^{\frac{1}{2}}, \qquad \text{for all } v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q}).$$

Proof. By Theorem 2.18 [18], the square root operator $\sqrt{T}$ of T exists and is a bounded, positive and self-adjoint operator. Consider,

$$\begin{aligned}\langle u|Tv\rangle &= \langle \sqrt{T}u|\sqrt{T}v\rangle \\ &\leq \|\sqrt{T}u\|\|\sqrt{T}v\| \\ &= \sqrt{\langle \sqrt{T}u|\sqrt{T}u\rangle}\ \sqrt{\langle \sqrt{T}v|\sqrt{T}v\rangle} \\ &= \sqrt{\langle u|Tu\rangle}\ \sqrt{\langle v|Tv\rangle}.\end{aligned}$$

For $u = Tv$ in the above inequality, we have

$$\begin{aligned}\|Tv\|^2 &= \langle Tv|Tv\rangle \\ &\leq \sqrt{\langle Tv|T^2v\rangle}\ \sqrt{\langle v|Tv\rangle}.\end{aligned} \tag{4}$$

since

$$\langle Tv|T^2v\rangle \leq \|Tv\|\ \|T^2v\| \leq \|T\|\|Tv\|^2.$$

Therefore, from the Eq. (4), we get

$$\|Tv\| \leq \langle v|Tv\rangle^{\frac{1}{2}}\ \|T\|^{\frac{1}{2}}. \ \blacksquare$$

Theorem 4. *Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be any two sequences in $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. Then the pair $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ forms a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ if and only if the operator $S_{F,G}$ is a positive and bounded below right linear operator on $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$.*

Proof. Let $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ be a biframes in $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. By Theorem 1, the corresponding biframe operator $S_{F,G}$ is positive and bounded below.

Conversely, let $S_{F,G}$ is a positive and bounded below right linear operator. For any $v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$,

$$\begin{aligned}\textstyle\sum_{i=1}^{\infty} \langle v|f_i\rangle \langle g_i|v\rangle &= \langle v|S_{F,G}v\rangle \\ &= |\langle v|S_{F,G}v\rangle| \\ &\leq \|v\|\|S_{F,G}v\| \\ &\leq \|S_{F,G}\|\|v\|^2.\end{aligned}$$

Therefore, $\|S_{F,G}\|$ is an upper biframe bound for the pair of sequences $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$.

Again, as $S_{F,G}$ is bounded below, there exists $\alpha > 0$ such that $\alpha\|v\| \leq \|S_{F,G}v\|$ for all $v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. Using Lemma 3, we get that $\|S_{F,G}v\|^2 \leq \|S_{F,G}\|\langle v|S_{F,G}v\rangle$. Thus, from these two inequalities, we deduce that

$$\alpha^2\|v\|^2 \leq \|S_{F,G}v\|^2 \leq \|S_{F,G}\| \sum_{i=1}^{\infty} \langle v|f_i\rangle \langle g_i|v\rangle. \ \blacksquare$$

The next example shows that the bounded image of a biframe may not be a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$.

Example 4. Let $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ be an infinite dimensional QHS with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$.

Take $f_i = g_i = e_i$, for all $i \in \mathbb{N}$, then the pair $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ is a Parseval biframe. Define $T : \mathbb{H}^{\mathrm{R}}(\mathfrak{Q}) \to \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ *by* $T(e_i) = \frac{e_i}{i}$. Clearly, T is a bounded right linear operator. As,

$$\textstyle\sum_{i=1}^{\infty} \langle v|T(f_i)\rangle \langle T(g_i)|v\rangle = \sum_{i=1}^{\infty} \left\langle v\middle|\frac{e_i}{i}\right\rangle \left\langle \frac{e_i}{i}\middle|v\right\rangle = \sum_{i=1}^{\infty} \frac{1}{i^2} |\langle e_i|v\rangle|^2.$$

For $v = e_n$, for any $n \in \mathbb{N}$, we have $\sum_{i=1}^{\infty}\langle v|T(f_i)\rangle \langle T(g_i)|v\rangle = \frac{1}{n^2}$. Since, there does not exist any positive real number r_1 with $r_1 \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$, we have $(\{Tf_i\}_{i=1}^{\infty}, \{Tg_i\}_{i=1}^{\infty})$ is not a biframe.

Theorem 5. *Let $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ be a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. If $T \in \mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$ is an operator with its adjoint T^* as a bounded below right linear operator, then the pair $(\{Tf_i\}_{i=1}^{\infty}, \{Tg_i\}_{i=1}^{\infty})$ forms a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$.*

Proof. Let $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ be a biframe, then there exist positive real numbers, r_1 and r_2 satisfying the following:

$$r_1\|v\|^2 \leq \sum_{i=1}^{\infty}\langle v|f_i\rangle \langle g_i|v\rangle \leq r_2\|v\|^2, \text{ for all } v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q}).$$

Let $v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$ be an arbitrary element, then

$$\sum_{i=1}^{\infty}\langle v|Tf_i\rangle \langle Tg_i|v\rangle = \sum_{i=1}^{\infty}\langle T^*v|f_i\rangle \langle g_i|T^*v\rangle \leq r_2\|T^*v\|^2 \leq r_2\|T^*\|^2 \|v\|^2. \tag{5}$$

Again, as T^* is bounded below, there exists $\alpha > 0$ such that $\alpha\|v\| \leq \|T^*v\|$, $\forall\, v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$.

Thus,

$$\alpha^2 r_1\|v\|^2 \leq r_1\|T^*v\|^2 \leq \sum_{i=1}^{\infty}\langle v|Tf_i\rangle\langle Tg_i|v\rangle. \tag{6}$$

From Eqs. (5) and (6), we derive that the pair $(\{Tf_i\}_{i=1}^{\infty}, \{Tg_i\}_{i=1}^{\infty})$ forms a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. ■

Next theorem proves that an invertible operator maps a biframe to a biframe in $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$.

Theorem 6. *Let $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ be a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$, and $T \in \mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$ be an invertible right linear operator. Then the pair $(\{Tf_i\}_{i=1}^{\infty}, \{Tg_i\}_{i=1}^{\infty})$ forms a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$.*

Proof. Let r_1 and r_2 be the lower and upper biframe bounds for $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$, respectively. The upper bound condition for the pair $(\{Tf_i\}_{i=1}^{\infty}, \{Tg_i\}_{i=1}^{\infty})$ follows from the Eq. (5). By remark 2.16 [18], we have $(T^*)^{-1}$ exists and is a bounded right linear operator on $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. For any $v \in \mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$, we can write it as $v = (T^*)^{-1}T^*v$. Hence,

$$\begin{aligned} r_1\|(T^*)^{-2}\|^{-1}\|v\|^2 &\leq r_1\|T^*v\|^2 \\ &\leq \sum_{i=1}^{\infty}\langle v|Tf_i\rangle\langle Tg_i|v\rangle. \end{aligned}$$

Example 4 shows that if $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ is a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$, and $T_1, T_2 \in \mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$ are bounded right linear operators, then the pair $(\{T_1f_i\}_{i=1}^{\infty}, \{T_2g_i\}_{i=1}^{\infty})$ may not form a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. But if one of them is invertible, then by proceeding as in Theorem 6 one can show that the pair $(\{T_1f_i\}_{i=1}^{\infty}, \{T_2\, g_i\}_{i=1}^{\infty})$ forms a biframe for $\mathbb{H}^{\mathrm{R}}(\mathfrak{Q})$. ■

Next, we extend the Proposition 3.1 [20] for the quaternionic Hilbert spaces.

Lemma 4. *Let $S_1, S_2, T, U \in \mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$ be bounded right linear operators. Further, let S_1, S_2 be positive and invertible right linear operators. Then $US_1T^* = S_2$ if and only if $T = S_2^tWS_1^{-q}$ and $U = S_2^rVS_1^{-p}$ where $V, W \in \mathfrak{B}(\mathbb{H}^{\mathrm{R}}(\mathfrak{Q}))$; $\mathrm{p, q, r, t} \in \mathbb{R}$ satisfying $VW^* = I$ and $p + q = 1, t + r = 1$.*

Proof. Let $US_1T^* = S_2$ and $\mathrm{p, q, r, t} \in \mathbb{R}$ be arbitrary real numbers satisfying $p + q = 1, t + r = 1$.

Define the right linear operators by $V := S_2^{-r}US_1^p$ and $W := S_2^{-t}TS_1^q$. Then, V and W are bounded right linear operators, satisfying the given conditions. The converse follows easily. ■

To study operators that preserve the biframe property, we consider the action of two distinct operators on the sequence pair defining a biframe. The following theorem characterizes classes of such operators that maintain the biframe structure.

Theorem 7. *Let* $(\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty})$ *be a biframe for* $\mathbb{H}^R(\mathfrak{Q})$ *with biframe operator* $S_{F,G}$. *Further, let* $U, V \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$. *Then* $(\{Uf_i\}_{i=1}^{\infty}, \{Vg_i\}_{i=1}^{\infty})$ *is a biframe for* $\mathbb{H}^R(\mathfrak{Q})$ *if and only if there exist right linear operators* $Q, T, W \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$, *where* Q *is a positive, bounded below right linear operator such that* $TW^* = I$ *and* U= $Q^r W S_{F,G}^{-p}$; $V = Q^t T S_{F,G}^{-q}$ *with* $\mathrm{p,q,r,t} \in \mathbb{R}$ *satisfying* $p + q = 1, t + r = 1$.

Proof. Suppose $(\{Uf_i\}_{i=1}^{\infty}, \{Vg_i\}_{i=1}^{\infty})$ is a biframe for $\mathbb{H}^R(\mathfrak{Q})$ with the corresponding biframe operator $S_{UF,VG}$. For any $v \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned} S_{UF,VG}(v) &= \sum_{i=1}^{\infty} Uf_i \langle Vg_i | v \rangle \\ &= U S_{F,G} V^*(v). \end{aligned}$$

By Theorem 1 and Theorem 4, $S_{F,G}$ and $S_{UF,VG}$ are positive, invertible and bounded below right linear operators. Further, by Lemma 4 there exist $T, W \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ such that $TW^* = I$ and $U = S_{UF,VG}^r W S_{F,G}^{-p}$, $V = S_{UF,VG}^t T S_{F,G}^{-q}$, *where* $\mathrm{p,q,r,t} \in \mathbb{R}$ *satisfying* $p + q = 1, t + r = 1$.

Then, the operator $Q = S_{UF,VG}$ satisfies all the required conditions. Conversely, using Lemma 4, we obtain $Q = V\, S_{F,G}\, U^*$. For any $v \in \mathbb{H}^R(\mathfrak{Q})$, we write

$$\begin{aligned} Qv &= V\, S_{F,G}\, U^*(v) \\ &= \sum_{i=1}^{\infty} V\, f_i \langle Ug_i | v \rangle. \end{aligned}$$

Using Theorem 4, we can conclude that the pair $(\{Uf_i\}_{i=1}^{\infty}, \{Vg_i\}_{i=1}^{\infty})$ is a biframe for $\mathbb{H}^R(\mathfrak{Q})$. ■

4. CONCLUSION

In the present work, we have extended the theory of biframes to quaternionic Hilbert spaces, thereby generalizing the existing framework that was previously confined to complex Hilbert spaces. Our study provides a robust foundation for understanding biframes in the non-commutative setting of quaternions, which is both mathematically rich and applicable to areas such as quantum physics and signal analysis. It is observed that traditional frames can be viewed as a specific instance of biframes, underscoring the generality and flexibility of the biframe structure. The introduction of the biframe operator proved useful in deriving a reconstruction formula. In addition to developing the foundational theory, we explored several characterization results that deepen our understanding of the internal structure of biframes. Further, the work provides sufficient conditions under which a bounded linear transformation preserves the biframe property. The results will add to the theoretical landscape of quaternionic Hilbert spaces and pave the way for future investigations.

Author Contributions

Nitin Sharma: Conceptualization, Writing - Original Draft. Amita Aggarwal: Writing - Review and Editing. Raksha Sharma: Validation. All authors discussed the results, reviewed, and approved the final version of the manuscript prior to submission.

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Declaration of Generative AI and AI-assisted Technologies

Generative AI tools were used solely for language refinement (grammar, spelling, and clarity). The scientific content, analyses, interpretations, and conclusions were developed entirely by the authors. The authors reviewed and approved the final manuscript and take full responsibility for its content.

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