BAREKENG: Journal of Mathematics and Its Applications
December 2026 Volume 20 Issue 4 Page 2769-2782
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol20iss4pp2769-2782

A NEW ONE-PARAMETER DISTRIBUTION FOR MODELING POSITIVE SKEWNESS DATA: THE GENERIC KRISON DISTRIBUTION

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Article Info

Article History:
Received: 4th September 2025
Revised: 12th May 2026
Accepted: 7th June 2026
Available online: 1st July 2026

Keywords:
continuous distribution;
Hyperbolic Trigonometry Identity;
Maximum Likelihood Estimation;
one-parameter distribution.

ABSTRACT

Inspired by the hyperbolic secant distribution and a specific hyperbolic trigonometric identity, we propose a new continuous distribution with one parameter, called the Generic Krison distribution. We derive its probability density function, survival function, cumulative distribution function, hazard function, quantile function, and mean excess loss function. Closed-form expressions for the moment generating function are also provided. Statistical measures including the raw moments, mean, variance, standard deviation, median, mode, skewness, kurtosis, coefficient of variation, and index of dispersion are obtained. The distribution exhibits constant moderate positive skewness and a constant coefficient of variation, making it a natural candidate for modeling positively skewed data. The parameter is estimated by the method of maximum likelihood; because the score equation is transcendental, the estimate is obtained numerically. We compare the Generic Krison distribution with the one-parameter Rayleigh, Lindley, and Bilal distributions across three datasets (income, survival time, mortality rate). The results demonstrate that the Generic Krison distribution performs well on all three datasets, offering a promising new tool for statistical modeling.

How to cite this article:

E. Krison, S. Nurrohmah, S. Devila and I. Fithriani., "A NEW ONE-PARAMETER DISTRIBUTION FOR MODELING POSITIVE SKEWNESS DATA: THE GENERIC KRISON DISTRIBUTION," *BAREKENG: J. Math. & App.*, vol. 20, no. 4, pp. 2769-2782, Dec, 2026.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*

Research Article · Open Access

1. INTRODUCTION

In this modern era, rapid advancements in technology, information, and science constantly present new uncertainties and unexpected situations. Probability theory, a significantly developed branch of mathematics, quantifies the likelihood of events on a scale from zero to one [1]. As the mathematical bedrock for statistics, it's vital for quantitative data analysis, decision-making, and prediction across many real-life activities [2]. Indeed, estimation and prediction are critical components of research investigations, highlighting probability theory's importance in various applications, from finance and economics to weather forecasting and healthcare [3].

To describe the spread of an event's probability, probability distributions are essential. These statistical tools are crucial in theory and application, modeling phenomena in social sciences, engineering, and natural sciences [4]. For instance, distributions have been applied to model diverse phenomena, such as the growth periods of guinea pigs treated with growth hormone, children undergoing growth hormone treatment, and tumor sizes in cancer patients, using a sine-modified Lindley distribution [5]. Another example of distributions being applied to real-world phenomena is the Lomax Exponential distribution, which has been used to model the fracture toughness of carbon fibers [6].

While traditional probability distributions are foundational, they sometimes struggle to capture the complex behaviors, unique shapes, or heavy tails seen in diverse modern datasets. This limitation has led to a prominent trend: modifying established distributions to enhance their flexibility and accuracy. Many new distributions also originate from various methods developed over time. Early notable approaches for generating univariate continuous distributions include those based on differential equations, translation methods, and quantile function-based methods [7]. A prominent trend is modifying established distributions to enhance their flexibility and accuracy. A recent example is a modified cosine-based distribution proposed by Wang et al. [8], which demonstrates strong performance in modeling real-world datasets in sports and reliability.

Based on the given fact, we introduce a new continuous distribution with support on the positive real numbers, based on the hyperbolic secant distribution. According to [9], the probability density function and the figure of the hyperbolic secant distribution can be written as follows:

$$f(x) = \frac{1}{2} sech\left(\frac{\pi x}{2}\right), x \in (-\infty, \infty). \tag{1}$$

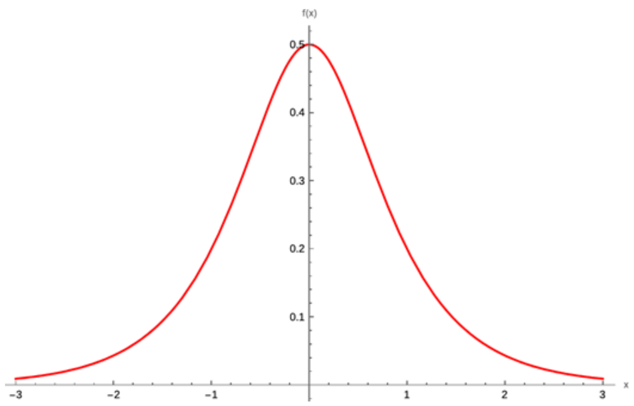

Figure 1. Probability Density Function of Hyperbolic Secant Distribution

Note that the hyperbolic secant distribution has no parameters, so there is only one form of the distribution. Additionally, when $x \in [0, \infty)$ the hyperbolic secant distribution is a right-continuous and non-increasing function that resembles a survival function for a continuous random variable X with domain $x \in [0, \infty)$. However, it is not a true survival function because the maximum value of the function is $\frac{1}{2}$ instead of 1. While the function $sech(x)$ raised to any power n satisfies all the properties of a survival function, we can introduce a parameter $r > 0$ within x to allow for greater shape flexibility in the distribution. The cumulative distribution function (CDF) of this modified distribution can be expressed as $1 - sech^n(rx)$. To derive a more specific form, we utilize the hyperbolic trigonometric identity $sech^2(x) + tanh^2(x) = 1$. By applying this identity, we define the survival function of the new distribution as $sech^2(rx)$ and the cumulative distribution function as $tanh^2(rx)$. This distribution, which we refer to as the "Generic Krison distribution," thus combines hyperbolic functions to provide a flexible and tractable model for statistical applications.

A remarkable consequence of this construction is that several fundamental shape characteristics such as the coefficient of variation, skewness, and kurtosis do not depend on the parameter r. This built-in profile of moderate positive skewness and tail weight makes the distribution a natural candidate for the type of asymmetrical data commonly encountered in actuarial, biomedical, and economic studies.

2. RESEARCH METHODS

2.1 Basic Property

If a random variable X follows a Generic Krison distribution, the notation would be $X \sim Krison(r)$. Following the methods of [1], we derive the fundamental functions of the Generic Krison distribution, including its survival, cumulative distribution, and probability density functions. The survival function represents the probability that a subject survives beyond a certain time x. As previously mentioned, for the Generic Krison distribution, the survival function $S_X(x)$ is expressed as:

$$S_X(x) = sech^2(rx), x > 0, r > 0. \tag{2}$$

The cumulative distribution function (CDF) represents the probability that a random variable is less than or equal to x. It is the complement of the survival function, and it illustrates how probabilities accumulate over time in lifetime distributions. The CDF is obtained by subtracting the survival function from 1. For the Generic Krison distribution, the CDF $F_X(x)$ is expressed as:

$$F_X(x) = tanh^2(rx), x > 0, r > 0. \tag{3}$$

The probability density function (PDF) describes how the probability of a random variable is distributed over a range of values. In this paper, the random variable is continuous, meaning it can take any value within a specified range or domain, and the total area under the PDF's curve across the entire range equals 1. The PDF is obtained by differentiating the cumulative distribution function with respect to x. The PDF $f_X(x)$ for the Generic Krison distribution is represented by the following expression and illustrated in the accompanying graph:

$$f_X(x) = 2rsech^2(rx)tanh(rx), x > 0, r > 0. \tag{4}$$

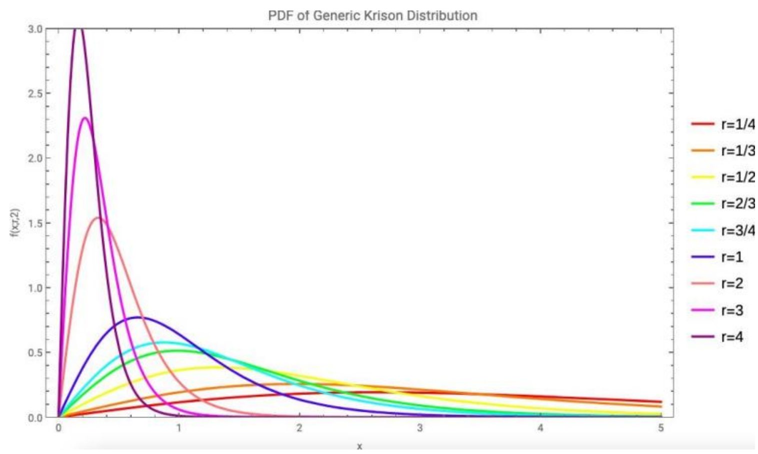

Figure 2. Probability Density Function of Generic Krison Distribution

Note that as the r parameter increases, the distribution becomes less compressed, meaning that scaling down a random variable by a fraction results in larger scaled-up values. Proof that the PDF integrates to 1:

$$\int_0^\infty f_X(x)dx = \int_0^\infty 2rsech^2(rx)tanh(rx)dx.$$

Use the substitution $u = tanh(rx)$. From this we can conclude that $\frac{du}{dx} = rsech^2(rx)$ and $2rsech^2(rx)tanh(rx)\, dx = 2u\, du$. As x goes from 0 to ∞, u goes from $tanh(0) = 0$ to $\lim_{x\to\infty} tanh(rx) = 1$. Therefore we can see that

$$\int_0^{\infty} f_X(x)\, dx = \int_0^1 2u\, du = [u^2]_0^1 = 1.$$

Hence the function is a valid probability density function.

2.2 Related Statistical Functions and Values

Building on the foundational functions derived above, we next present several key statistical measures, including the hazard function, cumulative hazard function, quantile function, and mean excess loss function, all derived from [1]. The hazard function measures the instantaneous failure rate given survival up to a time x. The hazard function is obtained by taking the ratio of the PDF $f_X(x)$ to the survival function $S_X(x)$. For the Generic Krison distribution, the hazard function $h_X(x)$ is expressed as:

$$h_X(x) = 2rtanh(rx), x > 0, r > 0. \tag{5}$$

The cumulative hazard function denoted as $H_X(x)$ represents the total accumulated risk up to time x. It can be obtained in two t ways. The first approach is to integrate the hazard function $h(t)$ from $t = 0$ to $t = x$. The second approach uses the relationship between the hazard function and the survival function: $h_X(x) - \frac{d}{dx} ln(S_X(x))$, integrating both sides immediately gives $H_X(x) = -ln(S_X(x))$ or $H_X(x) = ln(\frac{1}{S_X(x)})$. For the Generic Krison distribution, the cumulative hazard function is expressed as:

$$H_X(x) = 2ln(cosh(rx)), x > 0, r > 0. \tag{6}$$

The quantile function specifies the value below which a given proportion of observations fall. The p-th quantile function determines the value x such that the cumulative probability equals p, where $0 < p < 1$. For our distribution, the CDF $F_X(x) = tanh^2(rx)$ is right-continuous and depends on a single parameter r. In general, for a right-continuous CDF $F_X(x)$ or $F(x; r)$ [10] one can define the p-th quantile as $Q(p; r) = Inf\{x: F_X(x) \geq p\}, 0 < p < 1$, which is simply the inverse function when $F(x)$ is strictly increasing (as it is here). For the Generic Krison distribution, the quantile function $Q(p)$ is expressed as:

$$Q(p) = F_X^{-1}(p) = \frac{\operatorname{arctanh}(\sqrt{p})}{r}, 0 < p < 1, r > 0. \tag{7}$$

It is easy to verify that this quantile function satisfies $F_X(Q(p)) = p$ since:

$$F_X(Q(p)) = tanh^2(r \cdot \frac{\operatorname{arctanh}(\sqrt{p})}{r}) = tanh^2(arctanh(\sqrt{p})) = (\sqrt{p})^2 = p.$$

Additionally, we can derive closed-form expressions for the median and mode of the Generic Krison distribution. The median is a robust measure of central tendency, defined as the value of the random variable corresponding to the $\frac{1}{2}$ quantile of a distribution.

The median of a random variable X is the value x such that $F_X(x) = \frac{1}{2}$ or the solution of $S_X(x) = \frac{1}{2}$. Solving $F_X(x) = tanh^2(rx) = \frac{1}{2}$ gives the closed form shown below. For the Generic Krison distribution, the median is written as follows:

$$x_{median} = \frac{ln(1+\sqrt{2})}{r} \approx \frac{0.88137}{r}, r > 0. \tag{8}$$

The value that maximizes the PDF is known as the mode. If the mode is unique, the distribution is classified as unimodal. For a random variable X, the mode is determined by finding the positive root of the derivative of the PDF $f_X(x)$ with respect to x. For the Generic Krison distribution, the mode is written as follows:

$$x_{mode} = \frac{ln(2+\sqrt{3})}{2r} \approx \frac{0.6584789}{r}, r > 0. \tag{9}$$

2.3 Moments and Associated Measures

Because the random variable X is non-negative, its probability density function $f_X(x)$ and survival function $S_X(x) = Pr(X > x)$ are related by $f_X(x) = -\frac{d}{dx}S_X(x)$. Starting from the definition of the raw moment and using integration by parts, we obtain a convenient integral representation that avoids direct integration of the PDF:

$$E[X^n] = \int_0^\infty x^n f_X(x)dx = \int_0^\infty x^n\left(-\frac{d}{dx}S_X(x)\right)dx = [-x^n S_X(x)]_0^\infty + n\int_0^\infty x^{n-1}S_X(x)dx.$$

The boundary term vanishes because $S_X(0) = 1$ and $\lim_{x\to\infty} x^n S_X(x) = 0$ due to the exponential decay of $sech^2(rx)$ dominates any polynomial. Hence for every positive integer n,

$$E[X^n] = n\int_0^\infty \quad x^{n-1}S_X(x)dx. \tag{10}$$

Note that if the support were the whole real line, the boundary term at $-\infty$ would not vanish and this identity would not hold. Since the distribution is supported on $[0,\infty)$, the identity can be used. Substituting the survival function $S_X(x) = sech^2(rx)$ into Eq. (10) and changing the variable to $y = rx$ yields

$$E[X^n] = \frac{n}{r^n}\int_0^\infty y^{n-1}sech^2(y)dy. \tag{11}$$

To evaluate this integral, we expand the hyperbolic secant squared in an exponential series [11]:

$$sech^2(y) = \frac{4e^{-2y}}{(1+e^{-2y})^2} = 4\sum_{k=1}^{\infty}(-1)^{k-1}ke^{-2ky}, y > 0.$$

Now we can integrate term-by-term using the fact that it is justified by dominated convergence:

$$\begin{aligned}
\int_0^\infty \quad y^{n-1}sech^2(y)\,dy &= \int_0^\infty y^{n-1}\left[4\sum_{k=1}^{\infty}(-1)^{k-1}ke^{-2ky}\right]dy \\
&= 4\sum_{k=1}^{\infty}(-1)^{k-1}k\cdot\int_0^\infty y^{n-1}e^{-2ky}\,dy \\
&= 4\sum_{k=1}^{\infty}(-1)^{k-1}\,k\cdot\frac{(n-1)!}{(2k)^n} \\
&= 4\frac{(n-1)!}{2^n}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{n-1}}
\end{aligned}$$

The integral is the standard Gamma integral [1]. Substituting $z = 2ky$ yields

$$\int_0^\infty y^{n-1}e^{-2ky}\;dy = \frac{1}{(2k)^n}\int_0^\infty z^{n-1}e^{-z}\;dz.$$

For positive integer n, $\int_0^\infty z^{n-1}\,e^{-z}\,dz = \Gamma(n) = (n-1)!$; therefore the integral simplifies to $\frac{(n-1)!}{(2k)^n}$.

The sum that remains is exactly the Dirichlet eta function $\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}$ for $s > 0$. Special values include $\eta(1) = ln(2)$ and by analytic continuation, $\eta(0) = \frac{1}{2}$[12]. To see why $\eta(0) = \frac{1}{2}$, note that at $s = 0$ the series becomes Grandi's series $1 - 1 + 1 - 1 + ...$, which is not convergent in the ordinary sense but is Abel summable to $\frac{1}{2}$. By letting $S = 1 - 1 + 1 - 1 + \ldots$, then $1 - S = 1 - (1 - 1 + 1 - 1 + \ldots) = 1 - 1 + 1 - 1 + 1 - 1 + \ldots = S$ meaning that $1 - S = S$ and hence $S = \frac{1}{2}$. This value coincides with the analytic continuation of $\eta(s)$ at $s = 0$. Consequently,

$$\int_0^{\infty} y^{n-1}\, sech^2(y) dy = \frac{(n-1)!}{2^{n-2}} \eta(n-1).$$

Substituting this to Eq. (11) gives the closed form for all positive integers n:

$$E[X^n] = \frac{\eta(n-1) \cdot n!}{2^{n-2} r^n}, n \in \mathbb{N}, r > 0. \tag{12}$$

Additionally, the Dirichlet eta function is related to the Riemann zeta function through the identity $\eta(s) = (1 - 2^{1-s})\zeta(s)$ for all $s \neq 1$ (the case for $s = 1$ is understood via a limit). Thus, once one of these functions is evaluated, the other can be obtained directly, allowing the moments to be rewritten in terms of the Riemann zeta function $\zeta(s)$ [13]:

$$E[X^n] = \frac{\zeta(n-1) \cdot n!\,(2^{n-2} - 1)}{4^{n-2} r^n}, n \in \mathbb{N}\backslash 1, r > 0. \tag{13}$$

The first four raw moments follow immediately from Eq. (12):

$$E[X] = \frac{1!}{2^{-1} r} \eta(0) = \frac{2}{r}\frac{1}{2} = \frac{1}{r}, r > 0, \tag{14}$$

$$E[X^2] = \frac{2!}{2^0 r^2} \eta(1) = \frac{2}{r^2} ln(2) = \frac{\ln(4)}{r^2}, r > 0, \tag{15}$$

$$E[X^3] = \frac{3!}{2^1 r^3} \eta(2) = \frac{6}{2r^3}\frac{\pi^2}{12} = \frac{\pi^2}{4r^3}, r > 0, \tag{16}$$

$$E[X^4] = \frac{4!}{2^2 r^4} \eta(3) = \frac{24}{4r^4}\frac{3\zeta(3)}{4} = \frac{9\zeta(3)}{2r^4}, r > 0, \tag{17}$$

where $\eta(2) = \frac{\pi^2}{12}$ and $\eta(3) = \frac{3\zeta(3)}{4}$; $\zeta(3) \approx 1.20206$ is Apéry's constant [14]. Using Eqs. (14), (15), (16), and (17), the mean μ, variance σ^2, standard deviation σ, coefficient of variation CV, and index of dispersion ID are obtained as:

$$\mu = E[X] = \frac{1}{r}, r > 0, \tag{18}$$

$$\sigma^2 = E[X^2] - \mu^2 = \frac{\ln(4) - 1}{r^2} \approx \frac{0.386294}{r^2}, r > 0, \tag{19}$$

$$\sigma = \sqrt{Var[X]} = \frac{\sqrt{ln(4) - 1}}{r} \approx \frac{0.6215258}{r}, r > 0, \tag{20}$$

$$CV = \frac{\sigma}{\mu} = \sqrt{ln(4) - 1} \approx 0.6215258, \tag{21}$$

$$ID = \frac{\sigma^2}{\mu} = \frac{ln(4) - 1}{r} \approx \frac{0.386294}{r}, r > 0. \tag{22}$$

The coefficient of skewness $\gamma_1 = \frac{E[(X-\mu)^3]}{\sigma^3}$ and coefficient of kurtosis are also obtainable in closed form, and they are parameter-free constants, a distinctive property of this one-parameter distribution. The skewness is

$$\gamma_1 = \frac{E[X^3] - 3\mu E[X^2] + 2\mu^3}{\sigma^3} = \frac{\pi^2 - 12ln(4) + 8}{4(ln(4) - 1)^{1,5}} \approx 1.285. \tag{23}$$

For the kurtosis, two related quantities are commonly reported. The raw kurtosis, which is also called the standardized fourth central moment, is

$$\beta_2 = \frac{E[(X-\mu)^4]}{\sigma^4} = \frac{E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 3\mu^4}{\sigma^4} = \frac{9\zeta(3) - 2\pi^2 + 12ln(4) - 6}{2(ln(4) - 1)^2} \approx 5.74587. \tag{24}$$

Because the normal distribution has $\beta_2 = 3$, many authors prefer to use the excess kurtosis which is also called the coefficient of excess kurtosis, the formula is simply $\gamma_2 = \beta_2 - 3$. The excess kurtosis measures the heaviness of the tails compared with a normal distribution: a positive value indicates tails that are heavier than normal. For the Generic Krison distribution the excess kurtosis is

$$\gamma_2 = \frac{9\zeta(3) - 2\pi^2 + 12ln(4) - 6}{2(ln(4) - 1)^2} - \frac{3(\ln(4) - 1)^2}{(\ln(4) - 1)^2} = \frac{9\zeta(3) - 2\pi^2 + 12\ln(4) - 6 - 6(\ln(4) - 1)^2}{2(\ln(4) - 1)^2} \approx 2.74587. \tag{25}$$

The mean excess loss function (or mean residual life function) represents the expected remaining value of a variable X, given that it has exceeded a threshold d. In the context of payments, it is the expected amount paid, given that a claim exceeds the deductible d. When X represents age at death, it reflects the expected remaining lifetime for a person alive at age d. The mean excess loss function can be obtained by integrating the survival function $S(x)$ with respect to x from $x = d$ to $x = \infty$, then dividing the result by the survival function evaluated at $x = d$ respectively. For the Generic Krison distribution, the mean excess loss function $e_X(d)$ is expressed as:

$$e_X(d) = \frac{1}{S_X(d)} \cdot \int_d^\infty S_X(x)\,dx = \frac{1 + e^{-2rd}}{2r}, r > 0, d > 0. \tag{26}$$

2.4 Moment Generating Function

The moment generating function $M_X(t)$ or MGF of $X \sim Krison(r)$ is defined as

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} 2r sech^2(rx) tanh(rx) dx.$$

Because $f_X(x) = -\frac{d}{dx} S_X(x)$ with $S_X(x) = sech^2(rx)$, integration by parts gives a convenient form. For $t < 2r$,

$$M_X(t) = [-e^{tx} S_X(x)]_0^\infty + t\int_0^\infty e^{tx}\, sech^2(rx)dx = 1 + t\int_0^\infty e^{tx} sech^2(rx)dx. \tag{27}$$

The boundary term vanishes because $S_X(0) = 1$ and for large x, the survival function decays as $S_X(x) \sim 4e^{-2rx}$. Hence $e^{tx} sech^2(rx)$ behaves like $4e^{(t-2r)x}$, which tends to 0 as $x \to \infty$ provided that $t < 2r$. The remaining integral is evaluated with the substitution $u = e^{-2rx}$. Then $x = -\frac{ln(u)}{2r}, dx = -\frac{du}{2ru}$, and $sech^2(rx) = \frac{4u}{(1+u)^2}$. We obtain

$$\int_0^\infty e^{tx}\, sech^2(rx)\, dx = \frac{2}{r}\int_0^1 u^{-\frac{t}{2r}} (1+u)^{-2}\, du.$$

Now set $w = \frac{u}{1+u}$, this resulted in $u = \frac{w}{1-w}$, $du = \frac{dw}{(1-w)^2}$, and the integral becomes

$$\int_0^{\infty} e^{tx} sech^2(rx)\, dx = \frac{2}{r}\int_0^{\frac{1}{2}} w^{-\frac{t}{2r}}(1-w)^{\frac{t}{2r}}\, dw.$$

The integral on the right is the incomplete Beta function [15] with the formula shown below:

$$B(x; a, b) = \int_0^{x} s^{a-1}(1-s)^{b-1}\, ds, a, b > 0.$$

It is clear that $a = 1 - \frac{t}{2r}$ and $b = 1 + \frac{t}{2r}$ meaning that the integral final form is:

$$\int_0^{\infty} e^{tx}\, sech^2(rx)\, dx = \frac{2}{r} B(\tfrac{1}{2}; 1 - \tfrac{t}{2r}, 1 + \tfrac{t}{2r}), |t| < 2r. \tag{28}$$

Inserting Eq. (28) into Eq. (27) gives a compact closed-form MGF:

$$M_X(t) = 1 + \frac{2t}{r} B\left(\frac{1}{2}; 1 - \frac{t}{2r}, 1 + \frac{t}{2r}\right), |t| < 2r. \tag{29}$$

Eq. (29) is the primary expression for the MGF since it involves only the standard incomplete Beta function and can be evaluated straightforwardly in any statistical software. On a given side note, two equivalent representations may be useful in different contexts. Using the identity $B(x; a, b) = {}_2F_1(a, 1-b; a+1; x) \cdot \frac{x^a}{a}$, one obtains a hypergeometric form [16]:

$$M_X(t) = 1 + {}_2F_1\left(1 - \frac{t}{2r}, -\frac{t}{2r}; 2 - \frac{t}{2r}; \frac{1}{2}\right) \cdot \frac{4t}{2r - t}, |t| < 2r. \tag{30}$$

Alternatively, expanding $sech^2(rx)$ as an exponential series and summing yields an expression in terms of the digamma function $\psi(x)$ [17]:

$$M_X(t) = 1 + \frac{t}{r} + \frac{t^2}{2r^2}\left[\psi\left(\frac{t}{4r}\right) - \psi\left(\frac{1}{2} + \frac{t}{4r}\right)\right], |t| < 2r. \tag{31}$$

All three forms are mathematically equivalent, but the incomplete Beta representation from Eq. (29) is recommended for numerical work because it avoids the potential cancellation errors that can occur when evaluating the digamma function near its poles.

2.5 Maximum Likelihood Estimation

The r parameter from the generic Krison distribution will be estimated using the maximum likelihood estimation. Let $(x_1, x_2, \ldots, x_n)$ be a random sample from the Generic Krison distribution. The likelihood function $L(r)$ for the Generic Krison distribution can be written as follows:

$$L(r) = \prod_{i=1}^{n} f(x_i; r) = (2r)^n \prod_{i=1}^{n} sech^2\,(rx_i) tanh(rx_i). \tag{32}$$

To simplify the optimization process, we define the log-likelihood function $l(r) = ln(L(r))$:

$$l(r) = n(ln(2)) + n(ln(r)) + \sum_{i=1}^{n} [2ln(sech(rx_i) + ln(tanh(rx_i)]. \tag{33}$$

To find the maximum likelihood estimate $\hat{r}$, we derive the score function $U(r)$ by taking the first derivative of the log-likelihood with respect to r:

$$U(r) = \frac{n}{r} - \sum_{i=1}^{n} x_i tanh(rx_i) + \sum_{i=1}^{n} x_i \frac{sech^2(rx_i)}{\tanh(rx_i)}. \tag{34}$$

By applying the hyperbolic identity $\frac{sech^2(\theta)}{tanh(\theta)} = 2csch(2\theta)$, the score equation $U(r) = 0$ is expressed as:

$$\frac{n}{r} - \sum_{i=1}^{n} x_i\, tanh(rx_i) + 2\sum_{i=1}^{n} x_i\, csch(2rx_i) = 0. \tag{35}$$

The score equation is transcendental because the parameter r appears in both polynomial and hyperbolic forms meaning that this structure prevents the isolation of r to obtain a closed-form algebraic solution. Consequently, the estimate $\hat{r}$ must be determined through numerical optimization. All numerical work was performed in R version 4.3.2 (R Core Team, 2024) using the built-in optim() function via RStudio to maximize the log-likelihood. This numerical approach ensures that the resulting estimates provide the best fit for the analyzed datasets.

2.6 Competing Models

For our comparative analysis, we selected three one-parameter distributions as competing models: the Rayleigh, Lindley, and Bilal distributions. These models are well-established in statistical literature and are frequently used to analyze positively skewed data, making them suitable benchmarks for evaluating the performance of the Generic Krison distribution. The probability density functions of these distributions are given below:

1. Rayleigh distribution: $f(x;\theta) = 2\theta^2 x e^{-(\theta x)^2}$ with $x \geq 0$ and $\theta > 0$ [18].
2. Lindley distribution: $f(x;\lambda) = \frac{\lambda^2(1+x)e^{-\lambda x}}{\lambda+1}$ with $x > 0$ and $\lambda > 0$ [19].
3. Bilal distribution: $f(x;b) = \frac{6}{b}\left(e^{\frac{-2x}{b}} - e^{\frac{-3x}{b}}\right)$ with $x > 0$ and $b > 0$ [20].

2.7 Goodness-of-Fit Tests

To determine the most suitable distribution among the Generic Krison, Rayleigh, Lindley, and Bilal distributions for modeling each dataset, a goodness-of-fit test will be applied using the Kolmogorov-Smirnov (K-S) statistic. The K-S test is a non-parametric test used to compare a sample's empirical CDF $F_n(x)$ with the theoretical CDF $G(x)$ of a candidate distribution [1]. After estimating the parameters using maximum likelihood estimation for each distribution across the chosen datasets, the procedure will be as follows:

1. Sample Selection:

   For each dataset, let n represent the sample size. From the random sample $X_1, X_2, \ldots, X_n$, calculate the empirical CDF $F_n(x)$, where each data point is assigned a probability of $\frac{1}{n}$.

2. Hypotheses:

   H_0: The data follow one of the specified distributions (Rayleigh, Lindley, Bilal, or Generic Krison).

   H_1: The data do not follow one of the specified distributions (Rayleigh, Lindley, Bilal, or Generic Krison).

3. Significance Level:

   We will use 0.05 as the chosen significance level α.

4. Test Statistic:

   The test statistic T is calculated using the formula $T = max\,|F_n(x) - G(x)|$.

5. Decision Rule:

   For $\alpha = 0{,}05$ the calculated value of T is compared to the critical value $\frac{1{,}36}{\sqrt{n}}$. If T exceeds the critical value, H_0 is rejected, indicating that the data do not follow the distribution. Conversely, if $T \leq$ critical value, H_0 is not rejected, meaning the data follow the distribution.

Two goodness-of-fit tests that will be used are the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC). The model with the lowest AIC and BIC values is considered the bestfitting. While AIC penalizes complexity based on the number of parameters, BIC imposes a steeper penalty that also accounts for the sample size. This results in BIC favoring simpler models, especially as the number of observations increases [21]. The formulas for AIC and BIC are as follows:

$$AIC = -2log(L(\hat{r})) + 2k, \tag{36}$$

$$BIC = -2log\left(L(\hat{r})\right) + k\, log(n), \tag{37}$$

where k is the number of parameters in the candidate model which is 1 for all models considered, n is the sample size, $\hat{r}$ estimated the model parameter, and $L(\hat{r})$ is the likelihood of the candidate model given the data, evaluated at the maximum likelihood estimate of $\hat{r}$.

3. RESULTS AND DISCUSSION

3.1 Theoretical Properties and Novelty

The Generic Krison distribution possesses several advantageous properties that follow from its simple one-parameter form. Table 1 summarises the key closed-form expressions derived in Section 2, and these results constitute the main theoretical contribution of this work.

Table 1. Summary of Theoretical Properties of the Generic Krison Distribution

Property	Expression	Remark
PDF	$2rsech^2(rx)tanh(rx)$	Valid for $x > 0$
CDF	$tanh^2(rx)$	–
Survival Function	$sech^2(rx)$	–
Hazard Function	$2rtanh(rx)$	Increasing, bounded by $2r$
Mean Excess Loss	$\frac{1 + e^{-2rd}}{2r}$	$d > 0$
Quantile Function	$\frac{arctanh(\sqrt{p})}{r}$	$0 < p < 1$
Mean	$\frac{1}{r}$	–
Median	$\frac{ln(1+\sqrt{2})}{r} \approx \frac{0.88137}{r}$	–
Mode	$\frac{ln(2+\sqrt{3})}{2r} \approx \frac{0.6584789}{r}$	–
Variance	$\frac{ln(4)-1}{r^2} \approx \frac{0.386294}{r^2}$	–
Coefficient of Variation	$\sqrt{ln(4)-1} \approx 0.6215258$	Independent of r
Index of Dispersion	$\frac{ln(4)-1}{r} \approx \frac{0.386294}{r}$	Variance to mean ratio
Skewness (γ_1)	$\frac{\pi^2 - 12ln(4) + 8}{4(ln(4)-1)^{1,5}} \approx 1.285$	Independent of r
Raw Kurtosis (β_2)	$\frac{9\zeta(3) - 2\pi^2 + 12ln(4) - 6}{2(ln(4)-1)^2} \approx 5.74587$	Independent of r

The Generic Krison distribution, like pure scale distributions (e.g., Rayleigh, exponential), has the property that skewness, kurtosis, and the coefficient of variation do not depend on the parameter. The specific values (skewness ≈ 1.285, raw kurtosis ≈ 5.746, CV ≈ 0.6215) define a distinct profile not available from other common single-parameter competitors: the Rayleigh distribution has lower skewness (≈ 0.63) and lighter tails, while the exponential has higher skewness (2.0) and heavier tails. The Generic Krison therefore fills a useful gap among unimodal, positively skewed distributions. Because scale and shape are completely decoupled, the user never needs to worry that the estimate of r imposes an unrealistic skewness since the moderate positive skewness is built into the distribution. This clean separation contrasts with Weibull or gamma models, where a single shape parameter governs both skewness and hazard shape. The Generic Krison avoids that confounding, making it an interpretable benchmark for positively skewed data.

Moreover, the closed-form expressions for the hazard, quantile, mean excess loss, and moment generating function, unusual for a distribution defined through hyperbolic functions, make the model analytically tractable. This tractability, combined with the parameter-free shape characteristics, establishes the theoretical novelty of the Generic Krison distribution and distinguishes it from other one-parameter proposals.

3.2 Empirical Application and Comparison

To assess the practical usefulness of the Generic Krison distribution, it was fitted to three real-world datasets that exhibit pronounced positive skewness. Their descriptive statistics are shown in Table 2.

Table 2. Descriptive Statistics for the Three Datasets

Descriptive Statistics	Dataset 1	Dataset 2	Dataset 3
Mean	13.500	176.819	5.800
Median	10.600	149.500	5.200
Standard Deviation	8.051	103.455	3.254
Coefficient of Variation	0.585	0.597	0.562
Skewness	1.651	1.371	1.001
Kurtosis	5.600	5.225	3.772

The theoretical coefficient of variation (0.6215) is close to the empirical CVs of the three datasets (0.585, 0.597, 0.562), showing that the distribution's built-in shape fits these data well even without parameter estimation. The datasets are: monthly tax revenue in Egypt from January 2006 to November 2010, shown in Table 3; survival times of 72 guinea pigs infected with tuberculosis, shown in Table 4; and the crude mortality rate in Mexico from March to July 2020, shown in Table 5.

Table 3. Monthly Tax Revenue Data from Egypt

4.1	5.1	5.2	5.9	6.1	6.1	6.7	6.8	7.0	7.1
7.1	7.7	7.8	8.0	8.4	8.5	8.5	8.5	8.6	8.6
8.9	9.2	9.2	9.6	9.7	9.7	10.0	10.2	10.3	10.6
10.8	11.0	11.2	11.6	11.9	11.9	12.5	13.3	14.3	14.9
15.7	16.2	16.5	16.7	17.0	17.2	18.1	18.5	19.1	20.4
20.5	21.3	21.6	21.9	26.2	35.4	35.7	36.0	39.2	

***Data source**:* [22]

Table 4. Guinea Pig Survival Time Data

10	33	44	56	59	72	74	77	92
93	96	100	100	102	105	107	107	108
108	108	109	112	113	115	116	120	121
122	122	124	130	134	136	139	144	146
153	159	160	163	163	168	171	172	176
183	195	196	197	202	213	215	216	222
230	231	240	245	251	253	254	254	278
293	327	342	347	361	402	432	458	555

***Data source**:* [23]

Table 5. Crude Mortality Rate Data from Mexico

1.041	1.205	1.402	1.800	1.815	1.867	1.923	2.058	2.065
2.070	2.077	2.326	2.352	2.438	2.500	2.506	2.601	2.838
2.926	2.988	3.027	3.029	3.215	3.218	3.219	3.228	3.233
3.257	3.286	3.298	3.327	3.336	3.359	3.395	3.440	3.499
3.537	3.632	3.751	3.778	3.922	4.089	4.120	4.292	4.344
4.424	4.557	4.648	4.661	4.697	4.730	4.909	4.949	5.143

5.242	5.317	5.392	5.406	5.442	5.459	5.854	5.985	6.015
6.105	6.122	6.140	6.182	6.327	6.370	6.412	6.535	6.560
6.625	6.656	6.697	6.814	6.986	7.151	7.260	7.267	7.486
7.630	7.840	7.854	7.903	8.108	8.325	8.551	8.696	8.813
8.826	9.284	9.391	9.550	9.935	10.035	10.043	10.158	10.383
10.685	10.855	11.665	12.042	12.878	13.220	14.604	14.962	16.498

***Data source**:* [23]

Parameter estimates (obtained by maximum likelihood), Kolmogorov–Smirnov statistics, critical values, AIC, and BIC for all distributions are summarized in Table 6.

Table 6. Maximum Likelihood Estimates and Model Comparison Metrics

Data	Model	Parameter Estimation	K-S Value	Critical Value	AIC	BIC
Data 1	Rayleigh	0.06380	0.172	0.177	397.422	399.499
	Lindley	0.13922	0.192		403.259	405.336
	Bilal	16.2550	0.171		398.691	400.768
	G. Krison	0.07532	0.123		392.301	394.378
Data 2	Rayleigh	0.00489	0.109	0.161	857.889	860.166
	Lindley	0.01125	0.170		856.555	862.832
	Bilal	213.1500	0.171		860.844	863.120
	G. Krison	0.00570	0.120		854.273	856.550
Data 3	Rayleigh	0.15137	0.093	0.131	540.461	543.142
	Lindley	0.30659	0.153		562.916	565.598
	Bilal	6.94410	0.107		548.442	551.124
	G. Krison	0.17429	0.064		539.173	541.855

From Table 6, it can be observed that based solely on the Kolmogorov–Smirnov (K-S) test, the Lindley distribution fails for all three datasets, while the Bilal distribution fails for the second dataset, as its K-S value exceeds the critical value. The Generic Krison distribution, on the other hand, provides the best fit across all datasets and consistently achieves the lowest AIC and BIC values, confirming its superiority when penalizing for model complexity.

The success of the Generic Krison distribution can be attributed directly to its built-in shape characteristics: the fixed skewness (≈ 1.285) and raw kurtosis (≈ 5.746) align well with the empirical skewness and raw kurtosis of the three datasets (1.651/5.600, 1.371/5.225, 1.001/3.772). Moreover, the constant coefficient of variation naturally captures the relative dispersion without tuning. This demonstrates that the Generic Krison distribution is not merely another one-parameter model, but one whose theoretical properties are intrinsically suited for positive-skew phenomena.

4. CONCLUSION

To conclude, the proposed one-parameter distribution, called the Generic Krison distribution, represents a new paradigm in distribution theory. Its mathematical properties have been thoroughly explored, including the shape, moments, skewness, kurtosis, hazard rate function, cumulative hazard function, mean excess loss function, quantile function, and moment generating function. Additionally, closed-form expressions for the median and mode have been derived. The distribution's parameter is estimated by the method of maximum likelihood; because the score equation is transcendental, the estimate is obtained numerically via R's optim() function. Various dispersion measures, including the index of dispersion and coefficient of variation, have been discussed. The distribution has a fixed positive skewness and a constant coefficient of variation, making it especially suitable for positively skewed data in actuarial, biomedical, and economic studies.

The goodness-of-fit test results using the Kolmogorov-Smirnov statistics, along with comparisons to Rayleigh, Lindley, and Bilal distributions across three datasets, demonstrate the Generic Krison distribution's superior applicability and performance. With its strong statistical properties and ability to model diverse datasets effectively, the Generic Krison distribution offers a promising new tool for statistical modeling and analysis. Future work could explore a two-parameter extension of the Generic Krison distribution to allow the shape characteristics such as skewness and kurtosis to vary, thereby widening its applicability to data with different skewness profiles. Bivariate and regression versions of the distribution may also be of interest.

Author Contributions

Elbert Krison: Conceptualization, Methodology, Software, Formal Analysis, Data Curation, Investigation, Writing - Original Draft. Siti Nurrohmah: Conceptualization, Methodology, Validation, Supervision, Writing - Review and Editing. Sindy Devila: Data Curation, Formal Analysis, Writing - Review and Editing. Ida Fithriani: Validation, Supervision, Writing - Review and Editing. All authors discussed the results, contributed critical feedback, and finalized the final manuscript.

Funding Statement

This research was supported by Hibah Riset Penugasan FMIPA UI (Grant Number 017/UN2.F3.D/PPM.00.02/2022).

Acknowledgment

The authors thank the Actuarial Science Study Program, Faculty of Mathematics and Natural Sciences, University of Indonesia, for its guidance, resources, and facilities.

Declarations

The authors declare no conflict of interest.

Declaration of Generative AI and AI-assisted Technologies

During the preparation of this work, the authors used AI-assisted technology in order to improve the readability and grammar of the manuscript. After using this tool/service, the authors reviewed and edited the content as needed and took full responsibility for the publication's content.

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