BAREKENG: Journal of Mathematics and Its Applications
December 2026 Volume 20 Issue 4 Page 2949-2966
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol20iss4pp2949-2966

COORDINATE-WISE INTEGRAL CONSTRAINT PURSUIT GAME OF STATE-TRANSITION MODELLED BY INFINITE SYSTEM OF TWO-COUPLED DIFFERENTIAL EQUATIONS

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Article Info

Article History:

Received: 19th September 2025
Revised: 2nd April 2026
Accepted: 12th May 2026
Available online: 1st July 2026

Keywords:

Coordinate-Wise Integral Constraints;
Infinite system;
Pursuit Differential Game;
State Transfer;
Two-Coupled system of ODEs.

ABSTRACT

Pursuit-evasion games are pivotal in control theory, robotics, and autonomous systems, modelling adversarial interactions where agents operate under resource constraints. This paper investigates a two-player pursuit game governed by a two-coupled infinite system of first-order ODEs. The pursuer in the game seeks to steer the state trajectory precisely to a prescribed target state in finite time, while the evader works to prevent this outcome. Unlike most existing works, which impose global geometric or integral resource constraints on players' controls, this study adopts coordinate-wise integral constraints. Under this formulation, for each coordinate, the pursuer's and evader's control inputs satisfy per-coordinate energy bounds. It is shown that if, for every coordinate, the pursuer's energy bound exceeds that of the evader, and if the scaled initial target mismatches are uniformly bounded so that the coordinate-wise pursuit completion times are well-defined and their supremum is finite, then the pursuer possesses a guaranteed winning strategy. An explicit strategy is constructed for the pursuer and its admissibility under the per coordinate integral constraints is proved. Moreover, for every admissible evader control, it is shown that the resulting trajectory reaches the target state exactly at the guaranteed pursuit time, ensuring pursuit completion. An illustrative example is provided to demonstrate the applicability of the obtained results.

How to cite this article:

C. S. Odiliobi, R. M. Hasim, and G. Ibragimov, "COORDINATE-WISE INTEGRAL CONSTRAINT PURSUIT GAME OF STATE-TRANSITION MODELLED BY INFINITE SYSTEM OF TWO-COUPLED DIFFERENTIAL EQUATIONS", *BAREKENG: J. Math. & App.*, vol. 20, no. 4, pp. 2949-2966, Dec. 2026.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*

Research Article · Open Access

1. INTRODUCTION

Pursuit-evasion games are fundamental in applied mathematics, control theory, and game theory. They provide robust frameworks for modelling dynamic interactions between competing agents. These games have diverse applications, including robotics [1]- [4], economic and competitive markets [5], [6], military strategy [7], [8], [9], and biological processes [10], [11], where the actions of pursuing and evading entities are analyzed and optimized under various constraints. Among the different formulations of pursuit-evasion (PE) games, differential games are particularly notable for capturing the continuous-time dynamics of player interactions. In PE differential games, two teams compete: the pursuer(s) and the evader(s). For a game of two players, the pursuer's intent is typically to drive the system's state from an initial point z^0 to a prescribed target point z^1 at some finite time, while the evader seeks to prevent this. The state z^1 could either be the origin or a nonzero state. Pursuit succeeds when the pursuer achieves this objective. Examples of studies focusing on z^1 as the origin include [12], [13], whereas works allowing z^1 to be a nonzero state include [14], [15].

Integral constraints restrict players' control functions in differential games. They typically bound the total control effort exerted over a given duration, preventing players from applying arbitrarily large inputs. This makes games more realistic and applicable to resource-limited real-world scenarios. A significant body of research has addressed differential games where control functions adhere to integral constraints (see, e.g., [16]- [23]). Within control theory, geometric constraints — which are restrictions on the instantaneous magnitude of the control inputs — are extensively developed in foundational texts [25], [26]. These works establish the principle of bounded controls, a foundation that naturally extends to more structured formulations. One such extension is the per-coordinate integral constraint, where each component of a player's control is subject to an individual integral constraint—a formulation studied in [14], [19], [27].

It is well established (see, e.g., [28], [29], [30]) that decomposing distributed parameter systems of the evolution type reduces them to an infinite system of ODEs. The study of differential games modelled by infinite systems of ODEs has gained significant interest among researchers [12], [13], [14], [32]. In [12], the authors investigated a two-player pursuit game where the system dynamics are described by an infinite system of ODEs (see Eq. (2)). In this game, the pursuer and the evader control the system's state (Eq. (2)) using their own control functions. The players' control functions obey per-coordinate integral constraints, where the pursuer's control resources exceed the evader's own at every coordinate (These resources are denoted by ϱ_k^2 and σ_k^2 for the pursuer and the evader, respectively, and are formally defined in § 2.1 (Definition 1 and Definition 2). Pursuit is considered complete when the pursuer steers the system's state to the zeroth state after some time. The study established sufficient conditions for pursuit completion. The work in [13] extended these results by deriving the optimal time of pursuit and constructing the players' optimal strategies. The study in [14] further extended [12] by considering a two-player pursuit game modelled by the infinite system (Eq. (2)), where the pursuer's intent was generalized to steering the system's state to z^1 within finite time against the evader's actions.

Motivated by these studies, this article investigates a pursuit differential game governed by an infinite system of two-coupled first-order ODEs. The players' control functions adhere to per-coordinate integral constraints. Specifically, the problem addressed in this work is to construct a control strategy for the pursuer, dependent on the evader's actions, that guarantees driving the state of the system from z^0 to z^1 within a finite time, regardless of the admissible control applied by the evader. The time at which this occurs is termed the guaranteed pursuit time. Notably, the existence and uniqueness results for infinite systems of ODEs in ℓ_2 (the Hilbert space of square-summable sequences) do not apply to this paper's framework, unlike in [23]. Those results require $z^0 \in \ell_2$ (i.e., $\sum_{k=1}^{\infty} \| z_k^0 \| < \infty$) and impose a global integral constraint on the control input $\omega(t)$ to ensure $z(t)$ remains in ℓ_2. By contrast, our formulation imposes neither square-summability on z^0 and $z(t)$ nor a global constraint on $\omega(t)$. The only conditions required are the per-coordinate energy bounds in Eqs. (3) and (4), and the uniform boundedness condition in § 3.4. Consequently, $\sum_{k=1}^{\infty} \|z_k(t)\|^2$ may diverge, placing the problem outside the ℓ_2 setting. Furthermore, recent works [15], [24], [25], [32] examined pursuit games for infinite systems under global integral or geometric constraints. In those formulations, players may freely allocate control resources across coordinates at any time, provided the total control effort does not exceed a predefined budget. This total effort may be measured either instantaneously (geometric constraints) or cumulatively (integral constraints). However, in the per-coordinate integral constraints setting of this research, no transfer of unused control resources (energy) from one coordinate to another is permitted; each coordinate has its own energy budget, which the players must stick to.

Consequently, the primary novelty of this work lies in the analysis and solution of a pursuit game explicitly formulated outside the classical ℓ_2 Hilbert space framework. Unlike prior studies that rely on square-summability for well-posedness, our infinite-dimensional system permits trajectories with unbounded total energy, thereby introducing significant analytical challenges. To address this, we move beyond Hilbert space methods and instead develop a per-coordinate constructive strategy that operates independently on each subsystem. The main contributions of this paper are therefore threefold: (i) constructing an admissible pursuer strategy for an infinite-dimensional system without requiring square-summability of the state; (ii) identifying sufficient conditions on per-coordinate control resources that guarantee pursuit completion in this non-ℓ_2 setting; and (iii) deriving an explicit formula for the guaranteed pursuit time, even when the full state trajectory may diverge in the ℓ_2-norm.

The structure of the remainder of this article is as follows. Section 2 formulates the infinite-dimensional pursuit-evasion game model and introduces all relevant definitions. Section 3 is subdivided into Subsections 3.1, 3.2, and 3.3. In Subsection 3.1, an admissible strategy is established for the pursuer, and it is shown that this strategy leads to completing the game. Subsection 3.2 presents an illustrative example concretizing the findings of this research, while Subsection 3.3 discusses the findings and some of their implications. Section 4 summarizes the research and offers concluding remarks.

2. RESEARCH METHODS

This section describes the entire research framework, including the mathematical model (materials), the formal definitions and constraints that govern the game, and the step-by-step analytical procedure used to construct the solution.

2.1 Problem Formulation and Mathematical Model

This research explores a two-player pursuit game problem with system dynamics modelled by an infinite two-coupled ODE system:

$$\begin{cases} \dot{x} + \alpha_k x_k - \beta_k y_k = u_k^1 - v_k^1, & x_k(0) = x_k^0, \\ \dot{y} + \beta_k x_k + \alpha_k y_k = u_k^2 - v_k^2, & y_k(0) = y_k^0, \end{cases} \quad k = 1, 2, \dots, \tag{1}$$

where $x_k, y_k, x_k^0, y_k^0, u_k^1, u_k^2, v_k^1, v_k^2 \in \mathbb{R}$, $\alpha_k \geq 0$, and $\beta_k \in \mathbb{R}$. The k-th coordinate of the initial state $(x^0, y^0)^\top$ is given by $\left(x_k^0, y_k^0\right)^\top$. The pursuer and the evader utilize the respective control parameters $u = (u_1, u_2, \dots)$ and $v = (v_1, v_2, \dots)$ where $u_k = \left(u_k^1, u_k^2\right)^\top \in \mathbb{R}^2$ and $v_k = \left(v_k^1, v_k^2\right)^\top \in \mathbb{R}^2$ to control the state of Eq. (1). These control parameters depend on time, so they are denoted using $u(t), v(t)$, $t \in [0, \infty)$. Let $x_k^1, y_k^1 \in \mathbb{R}$, where $\left(x_k^1, y_k^1\right)^\top$ is the k-th coordinate of a predefined target state $(x^1, y^1)^\top = z^1$ distinct from the initial state $(x^0, y^0)^\top = z^0$. The pursuer's objective is to achieve a state transition, meaning to transfer every k-th coordinate of the system's state in Eq. (1) from $\left(x_k^0, y_k^0\right)^\top$ to its corresponding target $\left(x_k^1, y_k^1\right)^\top$ at some finite time, while the evader actively works to prevent this outcome.

Remark 1. *For comparison, consider the infinite system of ODEs modelling the differential game problems studied in [12] and [14]:*

$$\dot{z}_k + \lambda_k z_k = u_k - v_k, \qquad z_k(0) = z_k^0, \qquad k = 1, 2, \dots, \tag{2}$$

whereby $z_k, u_k, v_k \in \mathbb{R}^{n_k}$, $n_k \in \mathbb{N}$, and $\lambda_k > 0$ represents the medium's elasticity.

1. *By defining the variables $n_k, z_k, u_k, v_k, \lambda_k$ in Eq. (2) appropriately—specifically, setting $n_k = 1$, taking $z_k, u_k, v_k, \lambda_k \in \mathbb{C}$, with $z_k = x_k + iy_k$, $u_k = u_k^1 + iu_k^2$, $v_k = v_k^1 + iv_k^2$, $\lambda_k = \alpha_k + i\beta_k$, for $k = 1, 2, \dots$— Eq. (2) reduces precisely to the coupled system in Eq. (1). Consequently, the study aligns with and builds upon the work in [14].*
2. *Conversely, Eq. (1) reduces to Eq. (2) under specific parameter regimes. Setting $\alpha_k = \lambda_k$, $\beta_k = 0$, together with $x_k = z_k$, $y_k = 0$, $u_k^1 = u_k$, $v_k^1 = v_k$, and $u_k^2 = v_k^2 = 0$, the coupled system (Eq. (1)) collapses exactly to the infinite system studied in [14]. If, in addition, the target state is $z^1 = 0$ (the origin), the formulation further encapsulates the pursuit problem investigated in [12]. Thus, the models in [12], [14] emerge as special real (decoupled) cases of the more general*

coupled dynamics considered in this paper, where $\beta_k \not\equiv 0$ introduces rotational coupling between coordinates.

Let τ be a fixed positive number which is sufficiently large. Moreover, let $L_2([0,\tau],\mathbb{R}^2)$ denote the space consisting of all functions $f\colon [0,\tau] \to \mathbb{R}^2$ that are measurable and square-integrable in the sense that the integral $\int_0^\tau \|f(s)\|^2 ds$ is finite. The following definitions, adapted from literature (see, e.g., [14]), act as a foundation for this study.

Definition 1. A vector function $u(t) = (u_1(t), u_2(t), \dots)$, $t \in [0,\tau]$, is defined as the pursuer's admissible control if its components $u_k(t) \in L_2([0,\tau],\mathbb{R}^2)$ for each $k = 1,2,\dots$, and satisfy the coordinate-wise constraint

$$\int_0^\tau \|u_k(s)\|^2 ds \le \varrho_k^2, \qquad k = 1,2,\dots, \tag{3}$$

where $\varrho_k > 0$ are given. The parameter ϱ_k^2 is the maximum energy the pursuer can expend on the k-th subsystem of Eq. (1).

Definition 2. A vector function $v(t) = (v_1(t), v_2(t), \dots)$, $t \in [0,\tau]$, is referred to as the evader's admissible control if its components $v_k(t) \in L_2([0,\tau],\mathbb{R}^2)$ for each $k = 1,2,\dots$, and satisfy the coordinate-wise constraint

$$\int_0^\tau \|v_k(s)\|^2 ds \le \sigma_k^2, \qquad k = 1,2,\dots, \tag{4}$$

where $\sigma_k > 0$ are given. The parameter σ_k^2 is the maximum energy the evader can expend on the k-*th* subsystem of Eq. (1).

Remark 2. *Most prior studies impose global integral constraints [15], [23] (respectively, geometric constraints [24], [25]) taking the form*

$$\int_0^\tau \|\omega(t)\|^2 dt \le \varsigma^2, \qquad (\textit{respectively}, \ \|\omega(t)\| \le \varsigma,\ t \in [0,\tau])$$

where $\tau > 0$ is a fixed duration, $\omega(t) = (\omega_1(t), \omega_2(t), \dots)$ denotes a player's control input, and ς^2 represents the player's total available control resources. In contrast, in our per-coordinate formulation, each coordinate k has dedicated budgets: ρ_k^2 for the pursuer and σ_k^2 for the evader. Consequently, the pursuer possesses the control resource (energy) parameter $\varrho = (\varrho_1, \varrho_2, \dots)$ while the evader has $\sigma = (\sigma_1, \sigma_2, \dots)$. This structure models systems where control resources are allocated a priori to specific coordinates.

Define $z = (x,y)^\top$, $z = (z_1, z_2, \dots)$, where $z_k = (x_k, y_k)^\top$. Then, the infinite system in Eq. (1) can be transformed into the initial value problem (IVP)

$$\dot z_k + A_k z_k = u_k - v_k, \qquad z_k(0) = z_k^0, \qquad k = 1,2,\dots, \tag{5}$$

where $z_k^0 = \left(x_k^0, y_k^0\right)^\top$, and the coefficient matrix

$$A_k = \begin{pmatrix} \alpha_k & -\beta_k \\ \beta_k & \alpha_k \end{pmatrix}, \qquad k = 1,2,\dots .$$

One can easily show that the matrix exponential of the system in Eq. (5), or equivalently in Eq. (1), is

$$\Psi_k(t) := e^{-A_k t} = e^{-\alpha_k t}\begin{pmatrix} \cos\beta_k t & \sin\beta_k t \\ -\sin\beta_k t & \cos\beta_k t \end{pmatrix}, \qquad k = 1,2,\dots . \tag{6}$$

The following concerns the properties of the matrix in Eq. (6).

Property 1. *The matrix $\Psi_k(t)$ satisfies the following:*

1. $\Psi_k(0) = I_2$,
2. $\Psi_k(t+s) = \Psi_k(t)\Psi_k(s) = \Psi_k(s)\Psi_k(t)$,
3. $\left\|\Psi_k^\top(t) z_k(s)\right\| = \|\Psi_k(t) z_k(s)\| = e^{-\alpha_k t}\|z_k(s)\|$,

where I_2 is the order-2 identity matrix and $\Psi_k^\top(t)$ denotes the transpose of $\Psi_k(t)$.

Proof. Observe from Eq. (6) that $\Psi_k(\mathrm{t}) = e^{-\alpha_k t} R(\beta_k t)$, where

$$R(\beta_k t) = \begin{pmatrix} \cos \beta_k t & \sin \beta_k t \\ -\sin \beta_k t & \cos \beta_k t \end{pmatrix}$$

is a rotation matrix (since $R^\top R = I_2$ and $\det(R) = 1$).

1. At $t = 0$, $\mathrm{e}^{-\alpha_k \cdot 0} = 1$ and $R(0) = I_2$; hence, $\Psi_k(0) = I_2$.
2. Using the exponential property and commutativity of rotations:

$$\Psi_k(t)\Psi_k(s) = e^{-\alpha_k t} R(\beta_k t)\, e^{-\alpha_k s} R(\beta_k s) = e^{-\alpha_k (t+s)} R(\beta_k t) R(\beta_k s).$$

Since rotations in the plane commute, $R(\beta_k t)R(\beta_k s) = R(\beta_k(t+s))$. Therefore

$$\Psi_k(t)\Psi_k(s) = e^{-\alpha_k (t+s)} R(\beta_k(t+s)) = \Psi_k(t+s).$$

The same holds for the reverse order, proving $\Psi_k(t+s) = \Psi_k(t)\Psi_k(s) = \Psi_k(s)\Psi_k(t)$.

3. Since the rotation matrix $R(\cdot)$ is orthogonal: $R(\theta)^\top R(\theta) = I_2$ and so, $R(\theta)^\top = R(-\theta)$. Hence, for any vector $z_k \in \mathbb{R}^2$,

$$\|R(\theta) z_k\| = \|z_k\|, \qquad \|R(\theta)^\top z_k\| = \|z_k\|.$$

Now,

$$\|\Psi_k(t) z_k\| = \|e^{-\alpha_k t} R(\beta_k t) z_k\| = e^{-\alpha_k t} \|R(\beta_k t) z_k\| = e^{-\alpha_k t} \|z_k\|,$$

and similarly

$$\left\|\Psi_k^\top(t) z_k\right\| = e^{-\alpha_k t} \|R(\beta_k t)^\top z_k\| = e^{-\alpha_k t} \|z_k\|.$$

Thus $\left\|\Psi_k^\top(t) z_k\right\| = \|\Psi_k(t) z_k\| = e^{-\alpha_k t} \|z_k\|$. ■

In the game, the pursuit starts from the initial state $z^0 = (z_1^0, z_2^0, \dots)$, where $z_k^0 \in \mathbb{R}^2$, $k = 1, 2, \dots$. Replacing the control parameters u_k and v_k in Eq. (6) with the admissible controls $u_k(t)$ and $v_k(t)$, $t \in [0, \tau]$, the theory of ODE assures a unique solution $z(t) = (z_1(t), z_2(t), \dots)$ to the IVP in Eq. (5) on $[0, \tau]$. Applying the variation-of-parameters formula to Eq. (5) using the fundamental matrix $\Psi_k(t)$ from Eq. (6) yields the following explicit expression for the k-th component of the solution:

$$z_k(t) = \Psi_k(t) z_k^0 + \int_0^{\mathrm{t}} \Psi_k(t-s)\big(u_k(s) - v_k(s)\big)\, ds. \tag{7}$$

Define $AC([0, \tau], \mathbb{R}^2)$ to be the space of absolutely continuous functions defined on $[0, \tau]$ and taking values in $\mathbb{R}^2$. The solution $z(t)$, is analyzed in the function space $f(t) = (f_1(t), f_2(t), \dots)$ where each component $f_k(t)$, $k = 1, 2, \dots$, lies in $AC([0, \tau], \mathbb{R}^2)$. Thus, $z_k(\cdot) \in AC([0, \tau], \mathbb{R}^2)$ and the full trajectory satisfies

$$z(\cdot) \in \prod_{k=1}^{\infty} AC([0, \tau], \mathbb{R}^2).$$

Definition 3. An admissible pursuer strategy is defined as a function

$$U(t, v(t)) = \omega(t) + v(t) = (\omega_1(t) + v_1(t), \omega_2(t) + v_2(t), \dots), \qquad t \in [0, \tau],$$

where the kth coordinate $U_k(t, v_k(t)) = \omega_k(t) + v_k(t) \in \mathbb{R}^2$, $k = 1, 2, \dots$, satisfies the integral constraint

$$\int_0^{\tau} \left\|U_k(s, v_k(s))\right\|^2 ds \le \varrho_k^2, \qquad k = 1, 2, \dots , \tag{8}$$

$v = v(t)$ is an admissible evader's control, and $\varrho_k > 0$ is given.

Following this, what constitutes completion of the pursuit in the game is defined forthwith.

Definition 4. For a given initial state $z^0 = (z_1^0, z_2^0, \dots)$, the pursuit is said to be completed at time $\tau > 0$ if there exists an admissible pursuer strategy $U(t, v)$ such that for every admissible evader control $v(t)$, $t \in [0, \tau]$, the unique solution of the IVP

$$\dot{z}_k + A_k z_k = U_k\big(t, v_k(t)\big) - v_k(t), \qquad z_k(0) = z_k^0, \qquad t \in [0,\tau], \qquad k = 1,2,\dots, \tag{9}$$

satisfies $z(\theta) = z^1$ at some instant $\theta \in (0,\tau]$. That is, the state vector coincides exactly with the target point z^1 at time θ, meaning $z_k(\theta) = z_k^1$ for every coordinate $k = 1,2,\dots$. In this case, the strategy $U(t,v)$ is said to guarantee the completion of the pursuit, and τ is identified as a guaranteed time of pursuit.

At this point, the research problem that directs this study is formally presented.

Problem 1. *Find a time $\tau' \in (0,\tau]$ and construct an admissible pursuer strategy $U(t,v(t)) = (U_1(t,v_1(t)), U_2(t,v_2(t)),\dots)$ such that for any admissible evader control $v(\cdot) = (v_1(\cdot), v_2(\cdot),\dots)$, the unique solution $z(t) = (z_1(t), z_2(t),\dots)$ of the IVP (Eq. (9)) satisfies $z(\tau') = z^1$, i.e., $z_k(\tau') = z_k^1$ for all $k = 1,2,\dots$.*

2.2 Research Procedure and Analytical Framework

The procedure to solve **Problem 1** involves the following steps:

1. *Strategy Design*: The methodology centers on constructing a candidate pursuer strategy. For times τ_k and τ' (defined in Section 3), we introduce the variable $\Delta_k(\tau') = z_k^0 - \Psi_k(-\tau')z_k^1$, which quantifies the mismatch between the k-th coordinate of the initial state and the k-th coordinate of the target state transformed by the system dynamics. The pursuer's strategy is defined coordinate-wise, with two distinct cases based on whether the mismatch $\Delta_k(\tau')$ is zero or non-zero:
   a. *Case 1:* $\Delta_k(\tau') = 0$. For such coordinates, free motion alone would carry the state from z_k^0 to z_k^1 at time τ' in the absence of controls. Thus, to cancel out the evader's interference, the pursuer simply matches the evader's control: $U_k\big(t, v_k(t)\big) = v_k(t)$, $0 \le t \le \tau'$.
   b. *Case 2:* $\Delta_k(\tau') \neq 0$. For these coordinates, the pursuer strategy is designed to operate in two phases (see Eq. (11)): active steering phase (on the time interval $[0,\tau_k]$) and passive phase (on the time interval $(\tau_k,\tau]$).
2. *Proof of Admissibility (Lemma 1)*: The procedure requires proving that the constructed strategy is admissible, i.e., it does not violate the per-coordinate energy constraints ϱ_k^2 for any k. This was done by substituting the strategy into the energy integral, applying the Cauchy-Schwarz inequality, and using the definition of τ_k and the constraints on $v_k(t)$ to show that Eq. (8) is satisfied.
3. *Proof of Pursuit Completion (Theorem 1)*: The final step is to prove the strategy's effectiveness. By substituting the admissible strategy $U(t,v(t))$ into the solution $z_k(\tau')$ and simplifying using the properties of $\Psi_k(t)$ and the definition of τ_k, it was shown that $z_k(\tau') = z_k^1$ for all k. This proves that the pursuit is completed by time τ'.

The sufficient conditions for this method to work are that the pursuer has an energy advantage in every coordinate ($\varrho_k > \sigma_k$) and that the scaled initial-target mismatch is uniformly bounded across all coordinates ($\sup\limits_k \frac{\|\Delta_k(\tau')\|}{\varrho_k - \sigma_k} < \infty$).

3. RESULTS AND DISCUSSION

This section presents the core constructive and analytical results of the paper. A per-coordinate pursuer strategy tailored to the two-coupled infinite system is proposed and shown to satisfy the coordinate-wise integral constraints (Lemma 1). It is then demonstrated that the strategy guarantees successful pursuit completion, and an equation for the guaranteed pursuit time is established (Theorem 1). A concrete example illustrating the results follows. The section concludes with a brief discussion that explains the strategy's key mechanisms and situates the findings within the existing literature.

Define the functions $B_k(t) = \Psi_k(-t)\Psi_k^{\top}(-t)$ and $C(\tau_k) = \int_0^{\tau_k} B_k(s)\,ds$ for all $k = 1,2,\dots$. Then,

$$B_k(t) = \Psi_k(-t)\Psi_k^{\top}(-t) = e^{2\alpha_k t}\begin{pmatrix} \cos\beta_k t & -\sin\beta_k t \\ \sin\beta_k t & \cos\beta_k t \end{pmatrix}\begin{pmatrix} \cos\beta_k t & \sin\beta_k t \\ -\sin\beta_k t & \cos\beta_k t \end{pmatrix}$$

$$= e^{2\alpha_k t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e^{2\alpha_k t} I_2.$$

Hence, $C(\tau_k) = \int_0^{\tau_k} e^{2\alpha_k s} I_2\, ds = \varphi_k^2(0,\tau_k) I_2$, where

$$\varphi_k(0,t) = \sqrt{\int_0^t e^{2\alpha_k s}\, ds}.$$

For brevity's sake, set

$$\Delta_k(t) = z_k^0 - \Psi_k(-t) z_k^1 .$$

Consider the equation

$$\frac{\|\Delta_k(\tau')\|}{\varrho_k - \sigma_k} = \varphi_k(0,t) \tag{10}$$

where $\tau' > 0$ is arbitrary and $\varrho_k > \sigma_k$ for all $k = 1, 2, \ldots$. For the subsequent analysis, it is assumed that $\sup\limits_k \frac{\|\Delta_k(\tau')\|}{\varrho_k - \sigma_k} < \infty$.

For each fixed k, set

$$C_k = \frac{\|\Delta_k(\tau')\|}{\varrho_k - \sigma_k}.$$

Thus, Eq. (10) becomes

$$\varphi_k(0,t) = C_k.$$

Note that $\varphi_k(0,0) = 0$ and, for $\alpha_k \geq 0$, the integrand $e^{2\alpha_k s}$ is strictly positive for $t > 0$. Hence, $\varphi_k(0,t)$ is continuous, strictly increasing, and $\lim_{t\to\infty} \varphi_k(0,t) = +\infty$. Moreover, the range of $\varphi_k(0,t)$ for $t \in [0,\tau']$, is $[0, \varphi_k(0,\tau')]$. Thus, by the Intermediate Value Theorem, a unique root $t = \tau_k \in [0,\tau']$ exists, satisfying Eq. (10).

Observe that the assumption $\sup\limits_k \frac{\|\Delta_k(\tau')\|}{\varrho_k - \sigma_k} < \infty$ implies that $\sup\limits_{k\in\mathbb{N}} \tau_k$ is finite. From this point forward, let $\sup\limits_{k\in\mathbb{N}} \tau_k = \tau'$, where $\tau' \leq \tau$.

Remark 3. *If $\Delta_k(\tau') \neq 0$, then the left-hand side of Eq. (10) is positive, and because $\varphi_k(0,\cdot)$ is strictly increasing from zero, we obtain $\tau_k > 0$. Conversely, if $\Delta_k(\tau') = 0$ for some index k, then the left-hand side of Eq. (10) vanishes, so the unique root is $\tau_k = 0$. Such coordinates contribute $\tau_k = 0$ to the supremum $\tau' = \sup_k \tau_k$ and therefore do not affect the guaranteed pursuit time. Moreover, since $z^1 \neq z^0$, we must have $\tau' > 0$. Indeed, if $\tau' = 0$ then Eq. (10) would imply $\Delta_k(0) = 0$ for all k, i.e., $z_k^0 = z_k^1$ for every k, contradicting $z^1 \neq z^0$. Hence, under the hypotheses of Theorem 1, the value of τ' is determined exclusively by those coordinates for which $\Delta_k(\tau') \neq 0$ and is necessarily positive.*

3.1 Constructing the Admissible Strategy of the Pursuer

Lemma 1. *Let $\varrho_k > \sigma_k$ for all $k = 1, 2, \ldots$ and $v(t) = (v_1(t), v_2(t), \ldots)$ be an arbitrary admissible evader control. Define the pursuer strategy $U(t, v(t)) = (U_1(t, v_1(t)), U_2(t, v_2(t)), \ldots)$ coordinate-wise as follows, depending on whether $\Delta_k(\tau') = 0$ or $\Delta_k(\tau') \neq 0$:*

a. *If $\Delta_k(\tau') = 0$:*

$$U_k\big(t, v_k(t)\big) = v_k(t), \qquad 0 \leq t \leq \tau'.$$

b. *If $\Delta_k(\tau') \neq 0$:*

$$U_k\big(t, v_k(t)\big) = \begin{cases} v_k(t) - \dfrac{\varrho_k - \sigma_k}{\|\Delta_k(\tau')\| \varphi_k(0,\tau_k)} \Psi_k^\top(-t)\, \Delta_k(\tau'), & 0 \leq t \leq \tau_k, \\ v_k(t), & \tau_k < t \leq \tau', \end{cases} \tag{11}$$

Then, $U\big(t, v(t)\big)$ is admissible in the game.

Proof. Here, the admissibility of the strategy $U(t,\ v(t))$ is established by demonstrating its adherence to per-coordinate integral constraints. Two cases are considered for each fixed coordinate k.

Case 1: $\Delta_k(\tau') = 0$.

The strategy is simply $U_k(t, v_k(t)) = v_k(t)$ for all $t \in [0, \tau']$. Since v_k is admissible, we have

$$\int_0^{\tau'} \left\| U_k\big(s, v_k(s)\big) \right\|^2 ds = \int_0^{\tau'} \|v_k(s)\|^2\, ds \le \sigma_k^2.$$

Because $\varrho_k > \sigma_k$, it follows that $\sigma_k^2 < \varrho_k^2$, and therefore $\int_0^{\tau'} \left\| U_k\big(s, v_k(s)\big) \right\|^2 ds \le \varrho_k^2$. Hence, the strategy is admissible for this coordinate.

Case 2: $\Delta_k(\tau') \ne 0$.

For this case, the two-phase strategy defined in Eq. (11) above is used. For a fixed but arbitrary coordinate k in this regime, computing its total pursuer's control energy expenditure over $[0, \tau']$ gives

$$\int_0^{\tau'} \left\| U_k\big(s, v_k(s)\big) \right\|^2 ds = \underbrace{\int_0^{\tau_k} \left\| v_k(s) - \frac{\varrho_k - \sigma_k}{\|\Delta_k(\tau')\| \varphi_k(0, \tau_k)} \Psi_k^{\mathsf{T}}(-s)\, \Delta_k(\tau') \right\|^2 ds}_{Active\ Steering\ Phase} + \underbrace{\int_{\tau_k}^{\tau'} \|v_k(s)\|^2\, ds}_{Passive\ Phase}.$$

Next, consider the square in the integrand of the active steering phase. For brevity, define the positive constant $\mu_k = \frac{\varrho_k - \sigma_k}{\|\Delta_k(\tau')\| \varphi_k(0,\tau_k)}$. By expanding the square, the active steering phase integral becomes:

$$\int_0^{\tau_k} \left(\|v_k(s)\|^2 - 2\mu_k \langle v_k(s), \Psi_k^{\mathsf{T}}(-s)\, \Delta_k(\tau') \rangle + \mu_k^2 \left\| \Psi_k^{\mathsf{T}}(-s)\, \Delta_k(\tau') \right\|^2 \right) ds.$$

Therefore, the pursuer's total energy expenditure is:

$$\int_0^{\tau'} \left\| U_k\big(s, v_k(s)\big) \right\|^2 ds = \int_0^{\tau_k} \|v_k(s)\|^2\, ds + \int_{\tau_k}^{\tau'} \|v_k(s)\|^2\, ds - 2\mu_k \int_0^{\tau_k} \langle v_k(s), \Psi_k^{\mathsf{T}}(-s)\, \Delta_k(\tau') \rangle\, ds$$
$$+ \mu_k^2 \int_0^{\tau_k} \left\| \Psi_k^{\mathsf{T}}(-s)\, \Delta_k(\tau') \right\|^2 ds.$$

The first two integrals in the above sum combine to give the evader's total energy on $[0,\ \tau']$, which is $\int_0^{\tau'} \|v_k(s)\|^2\, ds$. Using Property 1 (c), it follows that $\left\| \Psi_k^{\mathsf{T}}(-s)\, \Delta_k(\tau') \right\| = e^{\alpha_k s} \|\Delta_k(\tau')\|$. Substituting these into the last integral yields:

$$\int_0^{\tau'} \left\| U_k\big(s, v_k(s)\big) \right\|^2 ds = \int_0^{\tau'} \|v_k(s)\|^2 ds - 2\mu_k \int_0^{\tau_k} \langle v_k(s), \Psi_k^{\mathsf{T}}(-s)\, \Delta_k(\tau') \rangle ds + \mu_k^2 \|\Delta_k(\tau')\|^2 \int_0^{\tau_k} e^{2\alpha_k s} ds.$$

By definition, $\varphi_k^2(0, \tau_k) = \int_0^{\tau_k} e^{2\alpha_k s} ds$. Substituting the value of μ_k, the last term simplifies to:

$$\mu_k^2 \|\Delta_k(\tau')\|^2 \varphi_k^2(0, \tau_k) = \left(\frac{(\varrho_k - \sigma_k)^2}{\|\Delta_k(\tau')\|^2 \varphi_k^2(0, \tau_k)} \right) \|\Delta_k(\tau')\|^2 \varphi_k^2(0, \tau_k) = (\varrho_k - \sigma_k)^2.$$

Thus, the integral simplifies to:

$$\int_0^{\tau'} \left\| U_k\big(s, v_k(s)\big) \right\|^2 ds = \int_0^{\tau'} \|v_k(s)\|^2 ds + (\varrho_k - \sigma_k)^2 - 2\mu_k \int_0^{\tau_k} \langle v_k(s), \Psi_k^{\mathsf{T}}(-s)\, \Delta_k(\tau') \rangle ds.$$

Since $v(t)$ is an admissible evader's control, we have $\int_0^{\tau'} \|v_k(s)\|^2 ds \le \sigma_k^2$. Applying this bound, the integral is estimated as

$$\int_0^{\tau'} \left\| U_k\big(s, v_k(s)\big) \right\|^2 ds \le \sigma_k^2 + (\varrho_k - \sigma_k)^2 - 2\mu_k \int_0^{\tau_k} \langle v_k(s), \Psi_k^{\mathsf{T}}(-s)\, \Delta_k(\tau') \rangle ds. \tag{12}$$

Applying the Cauchy-Schwarz inequality yields the following bound on the magnitude of the inner product integrals:

$$\left|\int_0^{\tau_k} \langle v_k(s), \Psi_k^{\top}(-s)\, \Delta_k(\tau')\rangle ds\right| \le \left(\int_0^{\tau_k} \|v_k(s)\|^2\, ds\right)^{1/2} \left(\int_0^{\tau_k} \left\|\Psi_k^{\top}(-s)\, \Delta_k(\tau')\right\|^2 ds\right)^{1/2}.$$

Again, using $\left\|\Psi_k^{\top}(-s)\, \Delta_k(\tau')\right\| = e^{\alpha_k s}\|\Delta_k(\tau')\|$, the second integral on the RHS becomes

$$\|\Delta_k(\tau')\|^2 \int_0^{\tau_k} e^{2\alpha_k s} ds = \|\Delta_k(\tau')\|^2 \varphi_k^2(0, \tau_k).$$

Therefore:

$$\left|\int_0^{\tau_k} \langle v_k(s), \Psi_k^{\top}(-s)\, \Delta_k(\tau')\rangle ds\right| \le \varphi_k(0, \tau_k)\|\Delta_k(\tau')\| \left(\int_0^{\tau_k} \|v_k(s)\|^2\, ds\right)^{1/2}.$$

Since the integral on the RHS is a part of the evader's total energy, it follows that $\left(\int_0^{\tau_k}\|v_k(s)\|^2\, ds\right)^{1/2} \le \sigma_k$. Consequently:

$$-\int_0^{\tau_k} \langle v_k(s), \Psi_k^{\top}(-s)\, \Delta_k(\tau')\rangle ds \le \left|\int_0^{\tau_k} \langle v_k(s), \Psi_k^{\top}(-s)\, \Delta_k(\tau')\rangle ds\right| \le \sigma_k \varphi_k(0, \tau_k)\|\Delta_k(\tau')\|.$$

Substituting this inequality and the definition of μ_k back into Eq. (12):

$$\begin{aligned}\int_0^{\tau'} \left\|U_k\big(s, v_k(s)\big)\right\|^2 ds &\le \sigma_k^2 + (\varrho_k - \sigma_k)^2 + 2\left(\frac{\varrho_k - \sigma_k}{\|\Delta_k(\tau')\|\varphi_k(0, \tau_k)}\right)\left(-\int_0^{\tau_k} \langle v_k(s), \Psi_k^{\top}(-s)\, \Delta_k(\tau')\rangle ds\right) \\ &\le \sigma_k^2 + (\varrho_k - \sigma_k)^2 + 2\left(\frac{\varrho_k - \sigma_k}{\|\Delta_k(\tau')\|\varphi_k(0, \tau_k)}\right)\big(\sigma_k\varphi_k(0, \tau_k)\|\Delta_k(\tau')\|\big).\end{aligned}$$

The terms $\|\Delta_k(\tau')\|$ and $\varphi_k(0, \tau_k)$ cancel out completely, leaving:

$$\int_0^{\tau'} \left\|U_k\big(s, v_k(s)\big)\right\|^2 ds \le \sigma_k^2 + (\varrho_k - \sigma_k)^2 + 2(\varrho_k - \sigma_k)\sigma_k.$$

Recognizing the right-hand side as a perfect square trinomial, we simplify:

$$\int_0^{\tau'} \left\|U_k\big(s, v_k(s)\big)\right\|^2 ds \le [\sigma_k + (\varrho_k - \sigma_k)]^2 = \varrho_k^2.$$

Therefore, for every coordinate k with $\Delta_k(\tau') \neq 0$, the strategy (Eq. (11)) also satisfies the required energy bound.

Both cases yield $\int_0^{\tau'}\left\|U_k\big(s, v_k(s)\big)\right\|^2 ds \le \varrho_k^2$ for all $k = 1,2, \ldots$. Hence, the pursuer's strategy $U(t, v(t))$ is admissible. ∎

3.2 Showing that Pursuit can be Completed

Theorem 1. *Let $\varrho_k > \sigma_k$ and $\sup\limits_k \frac{\|\Delta_k(\tau')\|}{\varrho_k - \sigma_k} < \infty$, for all $k = 1, 2, \ldots$, where τ_k is defined as the root of Eq. (10). For the game (Eq. (1)) with per-coordinate integral constraints, the instant $\sup\limits_{k\in\mathbb{N}} \tau_k = \tau'$ is a guaranteed pursuit time. This means that the pursuer can force the system to achieve the exact transition from z^0 to z^1 by time τ'.*

Proof. The pursuer is equipped with the strategy $U(t, v(t))$ defined coordinate-wise in Lemma 1, which handles two cases:

1. *If* $\Delta_k(\tau') = 0$**:** $U_k(t, v_k(t)) = v_k(t)$ for all $t \in [0, \tau']$.
2. *If* $\Delta_k(\tau') \neq 0$**:** $U_k(t, v_k(t))$ is given by the two-phase strategy in Eq. (11).

Lemma 1 establishes that this strategy is admissible for every coordinate k. For any fixed $k \in \mathbb{N}$, substitute $u_k(s) = U_k(s, v_k(s))$ into the solution formula in Eq. (7):

$$z_k(t) = \Psi_k(t) z_k^0 \int_0^t \Psi_k(t-s) \Big(U_k\big(s, v_k(s)\big) - v_k(s) \Big)\, ds \tag{13}$$

It is now shown that for every k, the state reaches the target at time τ', i.e., $z_k(\tau') = z_k^1$.

Case 1: $\Delta_k(\tau') = 0$.

From the definition $\Delta_k(\tau') = z_k^0 - \Psi_k(-\tau') z_k^1 = 0$, it follows that $\Psi_k(\tau') z_k^0 = z_k^1$. Substituting the strategy $U_k(t, v_k(t)) = v_k(t)$ into the solution formula (Eq. (7)):

$$z_k(\tau') = \Psi_k(\tau') z_k^0 + \int_0^{\tau'} \Psi_k\,(\tau' - s)(v_k(s) - v_k(s))\, ds = \Psi_k(\tau') z_k^0 = z_k^1.$$

Thus, for the coordinates with $\Delta_k(\tau') = 0$, the pursuit objective is achieved at time τ' regardless of the evader's control.

Case 2: $\Delta_k(\tau') \neq 0$.

For these coordinates, we use the two-phase strategy given in Eq. (11). To facilitate comprehension and improve verifiability, the proof is decomposed into five steps.

Step 1: Strategy Substitution and Setup

For $0 \leq t \leq \tau_k$ (the active steering phase), we use the first case of Eq. (11) in Eq. (13):

$$\begin{aligned} z_k(t) &= \Psi_k(t) z_k^0 - \int_0^t \Psi_k(t-s) \left[\frac{\varrho_k - \sigma_k}{\|\Delta_k(\tau')\| \varphi_k(0, \tau_k)} \Psi_k^{\top}(-s)\, \Delta_k(\tau') \right] ds \\ &= \Psi_k(t) z_k^0 - \frac{\varrho_k - \sigma_k}{\|\Delta_k(\tau')\| \varphi_k(0, \tau_k)} \int_0^t \Psi_k(t-s) \Psi_k^{\top}(-s) \Delta_k(\tau')\, ds. \end{aligned}$$

The evader's control cancels out, isolating the pursuer's steering term.

Step 2: Simplification Using Matrix Properties (Crucial Algebraic Step)

Adapting and using the semigroup property in Property 1 (b), namely $\Psi_k(t-s) = \Psi_k(t)\Psi_k(-s)$, we factor $\Psi_k(t)$:

$$z_k(t) = \Psi_k(t) \left[z_k^0 - \frac{\varrho_k - \sigma_k}{\|\Delta_k(\tau')\| \varphi_k(0, \tau_k)} \int_0^t \Psi_k(-s) \Psi_k^{\top}(-s) \Delta_k(\tau')\, ds \right].$$

Since $B_k(s) = \Psi_k(-s)\Psi_k^{\top}(-s) = e^{2\alpha_k s} I_2$, the integral above simplifies dramatically:

$$\int_0^t \Psi_k(t-s) \Psi_k^{\top}(-s) \Delta_k(\tau')\, ds = \left(\int_0^t e^{2\alpha_k s} ds \right) \Delta_k(\tau').$$

Recall that $\varphi_k(0, t) = \sqrt{\int_0^t e^{2\alpha_k s}\, ds}$, hence $\int_0^t e^{2\alpha_k s}\, ds = \varphi_k^2(0, t)$. Substituting yields:

$$z_k(t) = \Psi_k(t) \left[z_k^0 - \frac{(\varrho_k - \sigma_k)\, \varphi_k^2(0, t)}{\|\Delta_k(\tau')\| \varphi_k(0, \tau_k)} \Delta_k(\tau') \right]. \tag{14}$$

Step 3: Evaluation at $t = \tau_k$ and Application of Eq. (10) (Critical Transition)

Set $t = \tau_k$ in Eq. (14). Then $\varphi_k(0, \tau_k)$ appears both in the denominator and numerator:

$$z_k(\tau_k) = \Psi_k(\tau_k) \left[z_k^0 - \frac{(\varrho_k - \sigma_k)\, \varphi_k^2(0, \tau_k)}{\|\Delta_k(\tau')\| \varphi_k(0, \tau_k)} \Delta_k(\tau') \right] = \Psi_k(\tau_k) \left[z_k^0 - \frac{(\varrho_k - \sigma_k)\, \varphi_k(0, \tau_k)}{\|\Delta_k(\tau')\|} \Delta_k(\tau') \right].$$

Now invoke Eq. (10) at $t = \tau_k$: $\frac{\|\Delta_k(\tau')\|}{\varrho_k - \sigma_k} = \varphi_k(0, \tau_k)$ which is equivalent to $(\varrho_k - \sigma_k)\varphi_k(0, \tau_k) = \|\Delta_k(\tau')\|$. Substituting this into the bracket eliminates the fraction entirely:

$$z_k(\tau_k) = \Psi_k(\tau_k)\big[z_k^0 - \Delta_k(\tau')\big].$$

Step 4: Connecting to the Target State (Geometric Interpretation)

Recall that $\Delta_k(\tau') = z_k^0 - \Psi_k(-\tau')z_k^1$. Substituting:

$$z_k(\tau_k) = \Psi_k(\tau_k)\left[z_k^0 - (z_k^0 - \Psi_k(-\tau')z_k^1)\right] = \Psi_k(\tau_k)\Psi_k(-\tau')z_k^1.$$

Using the semigroup property in Property 1 (b) again, $\Psi_k(\tau_k)\Psi_k(-\tau') = \Psi_k(\tau_k - \tau')$. Hence:

$$z_k(\tau_k) = \Psi_k(\tau_k - \tau')z_k^1. \quad (15)$$

This shows that at time τ_k, the state has been steered to a position that, if left to evolve freely, will reach the target at time τ'.

Step 5: Passive Phase and Pursuit Completion

For $t \in (\tau_k, \tau']$, the strategy in Eq. (11) switches to the second case: $U_k(s, v_k(s)) = v_k(s)$. Thus, for any t in this interval:

$$z_k(t) = \Psi_k(t - \tau_k)z_k(\tau_k) + \int_{\tau_k}^{t} \Psi_k\,(t - s)(v_k(s) - v_k(s))ds = \Psi_k(t - \tau_k)z_k(\tau_k).$$

In particular, at $t = \tau'$ and using Property 1 (a):

$$z_k(\tau') = \Psi_k(\tau' - \tau_k)z_k(\tau_k) = \Psi_k(\tau' - \tau_k)\Psi_k(\tau_k - \tau')z_k^1 = \Psi_k(0)z_k^1 = z_k^1.$$

Thus, for every coordinate with $\Delta_k(\tau') \neq 0$, we also have $z_k(\tau') = z_k^1$.

Both cases yield $z_k(\tau') = z_k^1$ for all $k \in \mathbb{N}$. Hence, $z(\tau') = z^1$, confirming that the exact transition to the target point has been achieved and $\tau' = \sup_k \tau_k$ is a guaranteed pursuit time. ■

3.3 Example

The following example demonstrates the application of our results. For clarity of exposition and to maintain analytical tractability, we examine a case in which z^0 and z^1 vary periodically with k. This periodic structure enables explicit computation of the guaranteed pursuit time while preserving the essential features of the problem.

Example 1. *Consider a pursuit game modelled by Eq. (1) with per-coordinate integral constraints. For all $k = 1, 2, \dots$, let $\alpha_k = \frac{1}{2}$, $\beta_k = 0$, $\varrho_k = 3$, $\sigma_k = 1$. Thus, $\varrho_k > \sigma_k$ holds for every coordinate. The initial state is given by $z^0 = (x^0, y^0)^\top$, where $x^0 = (x_1^0, x_2^0, \dots)$, $y^0 = (y_1^0, y_2^0, \dots)$, with*

$$x_k^0 = \begin{cases} 1, & k \text{ is odd,} \\ 0, & k \text{ is even,} \end{cases} \qquad y_k^0 = \begin{cases} 0, & k \equiv 0 (\mathrm{mod}\, 4) \text{ or } k \equiv 1 (\mathrm{mod}\, 4), \\ 1, & k \equiv 2 (\mathrm{mod}\, 4) \text{ or } k \equiv 3 (\mathrm{mod}\, 4). \end{cases}$$

The target state $z^1 = (x^1, y^1)^\top$, where $x^1 = (x_1^1, x_2^1, \dots)$, $y^1 = (y_1^1, y_2^1, \dots)$, are given such that

$$x_k^1 = \begin{cases} 0, & k \text{ is odd,} \\ 1, & k \text{ is even,} \end{cases} \qquad y_k^1 = \begin{cases} 1, & k \equiv 0 (\mathrm{mod}\, 4) \text{ or } k \equiv 1 (\mathrm{mod}\, 4), \\ 0, & k \equiv 2 (\mathrm{mod}\, 4) \text{ or } k \equiv 3 (\mathrm{mod}\, 4). \end{cases}$$

Specifically, consider a pursuit game where the pursuer and the evader control the state of the system:

$$\begin{cases} \dot{x} + \frac{1}{2}x_k = u_k^1 - v_k^1, & x_k(0) = x_k^0, \\ \dot{y} + \frac{1}{2}y_k = u_k^2 - v_k^2, & y_k(0) = y_k^0, \end{cases} \qquad k = 1, 2, \dots,$$

with the control inputs $u = u(t)$ and $v = v(t)$ respectively, and they satisfy the constraints $\int_0^{\tau'} \left\|u_j(s)\right\|^2 ds \leq 9$ and $\int_0^{\tau'} \left\|v_j(s)\right\|^2 ds \leq 1$. The system has the initial configuration:

$$z^0 = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \dots\right)$$

and the pursuer's intent is to steer the system's state from z^0 to the target state

$$z^1 = \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \dots\right)$$

within a finite time, while the evader aims to prevent the state from reaching z^1. The pursuit is deemed complete if and only if the pursuer achieves this objective. Because $\beta_k = 0$, the transition matrix simplifies to $\Psi_k(-\tau') = e^{\alpha_k \tau'} I_2 = e^{\tau'/2} I_2$. For each k,

$$\Delta_k(\tau') = z_k^0 - e^{\tau'/2} z_k^1.$$

The squared norm of $\Delta_k(\tau')$ depends on the residue class of k modulo 4, as summarised in Table 1.

Table 1. Computation of $\|\Delta_k(\tau^1)\|$

k **mod 4**	z_k^0	z_k^1	$\Delta_k(\tau')$	$\|\Delta_k(\tau')\|^2$
1	$(1,0)^\top$	$(0,1)^\top$	$\left(1, -e^{\tau'/2}\right)^\top$	$1 + e^{\tau'}$
2	$(0,1)^\top$	$(1,0)^\top$	$\left(-e^{\tau'/2}, 1\right)^\top$	$1 + e^{\tau'}$
3	$(1,1)^\top$	$(0,0)^\top$	$(1,1)^\top$	2
0	$(0,0)^\top$	$(1,1)^\top$	$\left(-e^{\tau'/2}, -e^{\tau'/2}\right)^\top$	$2e^{\tau'}$

Observe that $\|\Delta_k(\tau')\| > 0$ for all k and all finite τ'. Hence, $\Delta_k(\tau') \neq 0$ for every coordinate, placing this in Case 2 of Lemma 1 and Theorem 1. Since $\varrho_k - \sigma_k = 2$ for all k, Eq. (10) becomes

$$\frac{\|\Delta_k(\tau^1)\|}{2} = \varphi_k(0, \tau_k),$$

where $\varphi_k(0,t) = \left(\int_0^t e^{2\alpha_k s} ds\right)^{1/2} = \sqrt{e^t - 1}$ (because $\alpha_k = \frac{1}{2}$). Squaring gives

$$\|\Delta_k(\tau')\|^2 = 4(e^{\tau_k} - 1), \qquad k = 1, 2, \dots .$$

Substituting the values from Table 1 and solving the above equation for τ_k, we find that

$$\tau_k = \begin{cases} \ln\left(\frac{5+e^{\tau'}}{4}\right), & k \equiv 1 (\mathrm{mod}\, 4) \text{ or } 2 (\mathrm{mod}\, 4), \\ \ln\left(\frac{3}{2}\right), & k \equiv 3 (\mathrm{mod}\, 4), \\ \ln\left(1 + \frac{e^{\tau'}}{2}\right), & k \equiv 0 (\mathrm{mod}\, 4). \end{cases}$$

Now, τ' must be chosen so that

$$\sup_k \tau_k = \tau'.$$

But then,

$$\sup_k \tau_k = \max\left\{\ln\left(\frac{5+e^{\tau'}}{4}\right), \ln\left(\frac{3}{2}\right), \ln\left(1 + \frac{e^{\tau'}}{2}\right)\right\}.$$

Thus, τ' must satisfy

$$\max\left\{\ln\left(\frac{5+e^{\tau'}}{4}\right), \ln\frac{3}{2}, \ln\left(1 + \frac{e^{\tau'}}{2}\right)\right\} = \tau'.$$

A straightforward comparison shows that the maximal value occurs precisely when $\tau' = \ln 2$. Thus,

$$\tau_k = \begin{cases} \ln\frac{7}{4}, & k \equiv 1 (\mathrm{mod}\, 4) \text{ or } 2 (\mathrm{mod}\, 4), \\ \ln\frac{3}{2}, & k \equiv 3 (\mathrm{mod}\, 4), \\ \ln 2, & k \equiv 0 (\mathrm{mod}\, 4), \end{cases}$$

and

$$\tau' = \sup_k \tau_k = \ln 2.$$

Moreover, from Table 1:

$$\|\Delta_k(\tau')\| = \|\Delta_k(\ln 2)\| = \begin{cases} \sqrt{3}, & k \equiv 1(\text{mod}\, 4) \text{ or } 2(\text{mod}\, 4), \\ \sqrt{2}, & k \equiv 3(\text{mod}\, 4), \\ 2, & k \equiv 0(\text{mod}\, 4). \end{cases}$$

Since $\|\Delta_k(\tau')\| \neq 0$ for all k, the pursuer is equipped with $U\big(t, v(t)\big)$ defined by Eq. (11). Furthermore, the uniform boundedness condition gives

$$\sup_k \frac{\|\Delta_k(\tau')\|}{\varrho_k - \sigma_k} = \sup_k \frac{\|\Delta_k(\ln 2)\|}{2} = 1 < \infty.$$

Therefore, by Theorem 1, the pursuer can guarantee completion of the pursuit by time $\tau' = \ln\, 2$ using the strategy defined coordinate-wise by Eq. (11).

In the example, the parameters $\alpha_k = \frac{1}{2}$ and $\beta_k = 0$ were chosen to simplify calculations. This example illustrates that even though the mismatches vary across coordinates, the conditions $\varrho_k > \sigma_k$ together with the uniform boundedness of the scaled initial-target mismatches ensure a finite guaranteed pursuit time τ'. This time τ' is determined by the coordinate requiring the longest time to reach the target.

3.4 Discussion

The primary contribution of this work is the formulation of an admissible pursuer strategy that guarantees pursuit completion at τ' for the infinite-dimensional pursuit game governed by Eq. (1) under per-coordinate integral constraints. The strategy, $U(t, v(t))$ defined coordinate-wise in Lemma 1, operates in two distinct modes depending on whether the mismatch $\Delta_k(\tau')$ is zero or non-zero:

1. If $\Delta_k(\tau') = 0$**:** The pursuer simply matches the evader's control, i.e., $\mathrm{U}_k\big(\mathrm{t}, \mathrm{v}(\mathrm{t})\big) = \mathrm{v}_k(\mathrm{t})$ for all $t \in [0, \tau']$. This neutralizes the evader's influence, allowing the free dynamics—which by virtue of $\Delta_k(\tau') = 0$ satisfy $\Psi_k(\tau')z_k^0 = z_k^1$—to carry the state to the target at time τ'.
2. If $\Delta_k(\tau') \neq 0$**:** The strategy employs a two-phase structure for each coordinate k:
   a. *Active steering phase* ($[0, \tau_k]$): The pursuer neutralizes the net effect of the evader's input $v_k(t)$ and injects a steering term
   $$\omega_k(t) = -\frac{\varrho_k - \sigma_k}{\|\Delta_k(\tau')\|\varphi_k(0, \tau_k)} \Psi_k^\top(-t)\, \Delta_k(\tau')$$
   to drive the state to the intermediate target $z_k(\tau_k) = \Psi_k(\tau_k - \tau')z_k^1$, where τ_k satisfies Eq. (10).
   b. *Passive phase* ($(\tau_k, \tau']$): The pursuer matches the evader's input $v_k(t)$, resulting in free evolution that propagates $z_k(\tau_k)$ to $z_k(\tau') = z_k^1$.

The per-coordinate formulation fundamentally shapes our solution approach: since energy cannot be transferred across coordinates, the strategy must independently satisfy each coordinate's integral constraint. This explains why Lemma 1's proof treats each k separately, applying the Cauchy-Schwarz inequality coordinate-wise rather than globally. The condition $\varrho_k > \sigma_k$ emerges naturally from this per-coordinate framework—global constraints would only require $\sum_{k=1}^{\infty} \varrho_k^2 > \sum_{k=1}^{\infty} \sigma_k^2$, a weaker condition that would allow resource reallocation across modes.

The pursuer strategy exploits the structure of A_k and a property of $\Psi_k(t)$, in particular $\left\|\Psi_k^\top(-s)\Delta_k(\tau')\right\| = e^{\alpha_k s}\|\Delta_k(\tau')\|$, to decouple the system, enabling per-coordinate control design. The diagonal form of $B_k(t) = e^{2\alpha_k t} I_2$ is essential for simplifying energy calculations.

Observe that the effectiveness of the strategy hinges on the following sufficient conditions:

1. *Resource parameter advantage*: The condition $\varrho_k > \sigma_k$ for all k is essential, as it guarantees the pursuer retains sufficient steering energy after neutralizing the evader's input $v_k(t)$.

2. *Uniform boundedness*: The requirement $\sup\limits_{k} \frac{\|\Delta_k(\tau')\|}{\varrho_k - \sigma_k} < \infty$ ensures that $\tau_k \leq \tau'$ for all coordinates, enabling simultaneous pursuit completion.

While prior work (e.g., [23]) examined a similar pursuit game in ℓ_2 space, the current game problem does not assume that z^0 belongs to ℓ_2, nor does it require the trajectory $z(t)$ to satisfy $\sum_{k=1}^{\infty} \|z_k(t)\|^2 < \infty$ for any $t > 0$. Furthermore, unlike the global integral constraint setting where a player's total available energy ς^2 satisfies $\sum_{k=1}^{\infty} \int_0^{\tau} \|\omega_k(s)\|^2 \leq \varsigma^2$, our formulation imposes independent integral constraints $\int_0^{\tau} \|\omega_k(s)\|^2 \leq \varsigma_k^2$ for each coordinate k, with no requirement that $\sum_{k=1}^{\infty} \varsigma_k^2$ converges. This absence of square-summability assumptions—both on the state trajectory and on the control resources—places the problem outside the ℓ_2 framework and necessitated the per-coordinate approach for constructing the pursuer strategy and analyzing the game.

Moreover, the authors in [14] considered the pursuit game (Eq. (2)) under the assumption that each λ_k is real and positive (see Remark 1). In the current paper, we allow $\lambda_k = \alpha_k + i\beta_k$ with $\beta_k \not\equiv 0$, which produces two-dimensional rotational (coupled) dynamics for each coordinate pair $(x_k, y_k)^\top$. This generalization is significant because it introduces oscillatory behaviour into the state evolution—the trajectories now follow spiral paths in the phase plane rather than monotonic decay along coordinate axes. Specifically, the matrix exponential $\Psi_k(t) = e^{-\alpha_k t} R(\beta_k t)$ reveals that while the Euclidean norm decays exponentially at a rate $e^{-\alpha_k t}$ (ensuring asymptotic stability whenever $\alpha_k > 0$), the coupling between x_k and y_k creates rotational motion governed by β_k. This distinction is physically meaningful: purely real coefficients ($\beta_k = 0$) capture only overdamped or critically damped behaviour, whereas complex parameters ($\beta_k \neq 0$) accommodate systems where damping and natural frequencies coexist. Importantly, our pursuit strategy successfully handles this additional complexity because the norm-preserving property $\left\|\Psi_k^\top(-s)\Delta_k(\tau')\right\| = e^{\alpha_k s}\|\Delta_k(\tau')\|$ holds for all β_k, rendering the energy calculations β_k-independent. This framework simultaneously captures damping and rotation at every coordinate and therefore strictly extends the setting of [14]. Consequently, their model (Eq. (2)) and its pursuit guarantees appear as the special case $\beta_k = 0$ of our system (Eq. (1)).

Consequently, their model (Eq. (2)) and its pursuit guarantee appear as the special case, $\beta_k = 0$, of our system (Eq. (1)).

The introduction of non-zero rotation parameters β_k (i.e., complex $\lambda_k = \alpha_k + i\beta_k$) introduces significant mathematical difficulties absent in the decoupled model of [14], which are resolved in this work:

1. *Coupled Dynamics:* In [14], the equations for each coordinate are independent. In our system, the equations for x_k and y_k are coupled through the β_k terms, creating rotational dynamics where the state vector $(x_k, y_k)^\top$ rotates as it evolves. This coupling prevents the direct application of scalar strategies and necessitates a matrix-based approach.

2. *Non-Trivial Matrix Exponential:* The coupling leads to a matrix exponential $\Psi_k(t)$ (see Eq. (6)) which combines both damping (through α_k) and rotation (through β_k). Deriving the key property $\left\|\Psi_k^\top(-s)\Delta_k(\tau')\right\| = e^{\alpha_k s}\|\Delta_k(\tau')\|$ (Property 1 (c)) required careful verification that the rotational component preserves norms. In [14], the analogous factor is simply the scalar $e^{\lambda_k s}$, making such properties immediate.

3. *Modified Steering Term:* The pursuer's steering term in Eq. (11) incorporates $\Psi_k^\top(-t)$ to actively counteract the rotation induced by β_k. This matrix-valued feedback represents a novel control law specifically designed for coupled dynamics. In contrast, the steering term in [14] involves only scalar multiplication of vectors, with no need to account for rotational effects.

4. *Rotation-dependent Pursuit Time:* The guaranteed pursuit time τ' in our game depends on both the damping parameters α_k and the rotation parameters β_k through the definition of $\Delta_k(\tau') = z_k^0 - \Psi_k(-\tau')z_k^1$ and the function $\phi_k(0,t) = \int_0^t e^{2\alpha_k s}\, ds$. This adds considerable complexity to the analysis of Eq. (10) compared to [14], where the pursuit time depends only on the damping parameter λ_k through simple exponential functions.

This framework therefore not only extends [14] parametrically (by allowing complex λ_k with non-zero imaginary parts) but also resolves the additional analytical challenges arising from the rotational coupling.

4. CONCLUSION

This study resolved the pursuit game problem involving two players whose motion dynamics are modelled by an infinite two-coupled system of first-order ODEs. The players' controls are restricted by per-coordinate integral constraints, with the pursuer's control resources surpassing those of the evader at each coordinate. The pursuer aims to guide the system's state from z^0 to the target z^1 within a finite duration, while the evader's objective is to prevent this. The pursuit is considered completed when the pursuer's objective is achieved. The work developed the pursuer's admissible strategy that ensures pursuit completion, identified sufficient conditions on control resources, initial and target states for pursuit completion, and derived the equation for the guaranteed pursuit time.

The evasion game problem—the natural dual of the pursuit game considered here—remains unexplored, to the best of our knowledge, for system (Eq. (1)) under per-coordinate integral constraints. In this formulation, the evader would aim to indefinitely avoid reaching z^1 (or the origin) while facing multiple pursuers. The pursuit game problem for higher-coupled ODE infinite systems with per-coordinate integral constraints also merits exploration.

Author Contributions

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Funding Statement

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Acknowledgment

The authors express their sincere appreciation to the anonymous reviewers for their meticulous and insightful comments, which contributed significantly to improving the clarity and precision of this work.

Declarations

The authors declare no conflicts of interest related to the study.

Declaration of Generative AI and AI-assisted technologies

The authors declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, including writing, editing, data analysis, or the creation of tables and figures.

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