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ALGEBRAIC PROPERTIES OF JORDAN TRIPLE DERIVATIONS ON RING *R*

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ABSTRACT

Polynomial rings play a fundamental role in algebra, with wide-ranging applications in number theory, algebraic geometry, and analysis. A key method for studying ring structures is through derivations. Among these, Jordan triple derivations generalize Jordan derivations and exhibit more intricate structural properties. While Jordan derivations have been extensively studied, their triple counterparts remain relatively underdeveloped, particularly in the context of polynomial rings. This paper adopts a theoretical algebraic approach to investigate the properties and structural aspects of Jordan triple derivations on rings and the polynomial ring R[x]. By extending existing results and constructing illustrative examples, we establish several new properties. In particular, we show that the set of Jordan triple derivations is closed under finite addition, finite linear combinations, and finite direct sums. Furthermore, every inner derivation is shown to be a Jordan triple derivation. Moreover, any Jordan triple derivation on the ring R induces a Jordan triple derivation on the polynomial ring $R[x]$. *These results contribute to a deeper understanding of the structure of Jordan triple derivations and provide a foundation for further research in this area.*

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1. INTRODUCTION

Polynomial rings are fundamental structures in algebra, with numerous applications across various fields, including number theory, algebraic geometry, and analysis. Among the key tools for studying the structure and behavior of rings are derivations, which help analyze change and capture essential algebraic properties. A particularly interesting class of these is the Jordan derivation, known for its unique characteristics in non-associative and associative rings. An extension of this concept, the Jordan triple derivation, provides a deeper framework for examining ring structures, especially in the context of polynomial rings. While Jordan derivations have been extensively studied, research on Jordan triple derivations remains relatively scarce. This presents an opportunity to explore new properties and potential applications of such derivations within polynomial rings. Several prior studies have contributed to the broader understanding of derivations and their variants. For example, Bresar revisits the structure of Jordan derivations [1], while Helmi et al. investigate Lie ideals and Jordan triple derivations on rings [2]. Research in [3] explores differential polynomial rings as a generalization of Asano prime rings, and Rehman [4] examines $(\alpha\,,\beta)$ $*$-Jordan triple derivations on semiprime rings with involutions. Further contributions include [5], which offers insights on Jordan derivations in semiprime rings, and Zhao and Qi [6], which characterizes $*$-Jordan derivations based on local actions in rings with involutions. Work in [7] addresses Jordan triple derivations in alternative rings, while Sayin and Kuzucuoglu [8] focus on Jordan derivations of special subrings within matrix rings.

Recent developments also highlight the increasing relevance of derivations in broader applications. For instance, Bagheri and Emami [9] discuss the role of polynomial rings in satellite coding and cryptography, and Darvish et al. [10] explore $*$-Jordan triple derivations on $*$-prime algebras. Furthermore, Ma et al. [11] examine Jordan semi-triple derivations and centralizers on generalized quaternion algebras, while Huang et al. [12] cover derivations on Murray–von Neumann algebras and various other ring structures. Additionally, Rahnaward et al. [13] investigate nonlinear λ-Jordan triple derivations on prime algebras, and Fitriani et al. [14] consider f-derivations on polynomial modules and continuing their research on commuting and centralizing on modules [15]. In 2025, Sitompul et al. [16] investigated the properties of Jordan derivation on the polynomial ring, while Mursyidah et al. [17] investigated nil derivations on the polynomial ring. Furthermore, in the same year, Faisol and Fitriani [18] introduced derivations on skew generalized modules, and Waluyo et al. studied (σ,τ)-derivation in a group ring [19].

The novelty of this study lies in providing new insights into the behavior of Jordan triple derivations in rings, a topic that has received limited direct attention in the existing literature [4]. In contrast to earlier works that mainly address classical derivations or related algebraic operators, this research uncovers and characterizes structural properties that are unique to Jordan triple derivations. We prove that the set of Jordan triple derivations is closed under finite addition, finite linear combinations, and finite direct sums, and that every inner derivation is a Jordan triple derivation. Moreover, any Jordan triple derivation on the ring R induces a Jordan triple derivation on the polynomial ring $R[x]$. These results not only bridge a gap in the current theoretical framework but also offer perspectives that conventional algebraic methods have not revealed. Consequently, the study makes an original theoretical contribution and paves the way for further developments in algebra and its applications.

2. RESEARCH METHODS

This study adopts a qualitative, theoretical research approach rooted in literature analysis and algebraic reasoning. The primary objective is to investigate Jordan triple derivations on polynomial rings, particularly the ring $R[x]$, where R is a ring with identity. Given the abstract and structural nature of the topic, the research is conducted entirely through analytical methods, supported by existing literature in the fields of ring theory, derivations, and polynomial algebras. The methodology consists of the following key stages:

1. Theoretical Analysis. The first stage explores the algebraic properties of Jordan triple derivations in rings. This involves examining their behavior under ring operations, analyzing conditions such as linearity and additivity, and clarifying their connections with other forms of derivations. An important focus is to determine whether, under specific restrictions, these derivations reduce to Jordan derivations or full derivations. The discussion is supported by lemmas, propositions, and theorems developed throughout the study.

2. Illustrative Examples. To reinforce the theoretical results, illustrative examples are presented. These examples provide concrete cases that highlight how the general properties operate within particular algebraic settings.
3. Extension to Polynomial Rings. The final stage constructs Jordan triple derivations on polynomial rings. This step extends the scope of the analysis, showing how the concepts apply within a broader and more complex algebraic framework.

3. RESULTS AND DISCUSSION

We recall the definition of Jordan triple derivation on the ring R as follows.

Definition 1 [14]. A mapping $\vartheta: R \to R$ is Jordan triple derivation if $\vartheta(vzv) = \vartheta(v)zv + v\vartheta(z)v + vz\vartheta(v)$, for all $v, z \in R$.

Before discussing the concept of Jordan triple derivations in polynomial rings, we will first construct the general form of Jordan triple derivations, which will serve as the basis for further proofs. Understanding Jordan triple derivations in ring R opens up opportunities to further explore the relationships between the structures involved. Therefore, in this section, we will develop a structure or operation that is an extension of the previous definition. Let $d_i: R \to R$, be a Jordan triple derivation on R, for every for each $i = 1, \dots, n$. We define $\sum_{i=1}^{n} \vartheta_i$ by $(\sum_{i=1}^{n} \vartheta_i)(v) = \sum_{i=1}^{n} \vartheta_i\,(v)$, for every $v \in R$. With this definition, the following discussion will show that the sum of Jordan triple derivations is also a Jordan triple derivation on R.

Lemma 1. *Given a ring R. If $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ are Jordan triple derivations on the ring R, then $\sum_{i=1}^{n} \vartheta_i = \vartheta_1 + \vartheta_2 + \cdots + \vartheta_n$ is Jordan triple derivation on R.*

Proof. Given a ring R. Let $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ be Jordan triple derivations on ring R.

We define $\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n: R \to R$, where $(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(v) = \vartheta_1(v) + \vartheta_2(v) + \cdots + \vartheta_n(v)$, for every $v \in R$. Given any $v, z \in R$. We have:

$$\begin{aligned}
\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n(v+z) &= \vartheta_1(v+z) + \vartheta_2(v+z) + \ldots + \vartheta_n(v+z) \\
&= \vartheta_1(v) + \vartheta_1(z) + \vartheta_2(v) + \vartheta_2(z) + \ldots + \vartheta_n(v) + \vartheta_n(z) \\
&= \vartheta_1(v) + \vartheta_2(v) + \ldots + \vartheta_n(v) + \vartheta_1(z) + \vartheta_2(z) + \ldots + \vartheta_n(z) \\
&= (\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(v) + (\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(z).
\end{aligned}$$

It follows that $\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n$ is additive. Then, we will show that

$(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(vzv) = (\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(v)zv + v(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(z)v + vz(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(v)$ as follows:

$$\begin{aligned}
(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(vzv) &= \vartheta_1(vzv) + \vartheta_2(vzv) + \cdots + \vartheta_n(vzv) \\
&= \vartheta_1(v)zv + v\vartheta_1(z)v + vz\vartheta_1(v) + \vartheta_2(v)zv + v\vartheta_2(z) + vz\vartheta_2(v) + \cdots \\
&\quad + \vartheta_n(v)zv + v\vartheta_n(z) + vz\vartheta_n(v) \\
&= [\vartheta_1(v)zv + \vartheta_2(v)zv + \cdots + \vartheta_n(v)zv] + [v\vartheta_1(z)v + v\vartheta_2(z)v + \cdots \\
&\quad + v\vartheta_n(z)v] + [vz\vartheta_1(v) + vz\vartheta_2(v) + \ldots + vz\vartheta_n(v)] \\
&= [\vartheta_1(v) + \vartheta_2(v) + \cdots + \vartheta_n(v)]zv + v[\vartheta_1(z) + \vartheta_2(z) + \cdots + \vartheta_n(z)]v \\
&\quad + vz[\vartheta_1(v) + \vartheta_2(v) + \cdots + \vartheta_n(v)] \\
&= [(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(v)]zv + v[(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(z)]v + \\
&\quad vz[(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(v)] \\
&= (\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(v)zv + v(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(z)v + \\
&\quad vz(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n)(v).
\end{aligned}$$

Hence, $(\vartheta_1+\vartheta_2+\cdots+\vartheta_n)(vzv)=(\vartheta_1+\vartheta_2+\cdots+\vartheta_n)(v)zv+v(\vartheta_1+\vartheta_2+\cdots+\vartheta_n)(z)v+vz(\vartheta_1+\vartheta_2+\cdots+\vartheta_n)(v)$, for every $v,z\in R$. Therefore, it is proven that $\vartheta_1+\vartheta_2+\cdots+\vartheta_n$ is a Jordan triple derivation. ∎

Corollary 1. *Given a ring R. If define $\vartheta_1,\vartheta_2: R\to R$ and both ϑ_1 and ϑ_2 are Jordan triple derivations on the ring R, then the sum ϑ_1 and ϑ_2 is also Jordan triple derivation on R.*

After establishing that $\vartheta_1+\vartheta_2+\cdots+\vartheta_n$ is a Jordan triple derivation on the ring, we proceed to demonstrate that the additive inverse of a Jordan triple derivation, defined by $-\vartheta: R\to R$ with $(-\vartheta)(v)=-\vartheta(v)$ for every $v\in R$, also satisfies the conditions of a Jordan triple derivation. The following section presents the detailed discussion.

Lemma 2. *Given a ring R. If $\vartheta: R\to R$ is a Jordan triple derivation on the ring R, then $-\vartheta$ is a Jordan triple derivation on R.*

Proof. Given a ring R. We define $-\vartheta(v)=-(\vartheta(v))$. Since $-\vartheta(v+z)=-(\vartheta(v+z))=-(\vartheta(v)+\vartheta(z))=-\vartheta(v)+(-\vartheta(z))$, for all $v,z\in R$, then $-\vartheta$ is additive. Now, we will show that $-\vartheta(vzv)=-\vartheta(v)zv-v\vartheta(z)v-vz\vartheta(v)$.

$$\begin{aligned}-\vartheta(vzv)&=-\big(\vartheta(vzv)\big)\\&=-\big(\vartheta(v)zv+v\vartheta(z)v+vz\vartheta(v)\big)\\&=-\vartheta(v)zv-v\vartheta(z)v-vz\vartheta(v).\end{aligned}$$

Hence, $-\vartheta(vzv)=-\vartheta(v)zv-v\vartheta(z)v-vz\vartheta(v)$, for every $v,z\in R$. It has been proven that $-\vartheta$ is a Jordan triple derivation. ∎

Example 1. Given $R=\mathbb{Z}[x]$. If we define a map $\vartheta: R\to R$, *with* $\vartheta\big(h(x)\big)=h'(x)$ and ϑ is Jordan triple derivation on the ring R. By Lemma 2, the map $-\vartheta: R\to R$, $(-\vartheta)\big(h(x)\big)=-h'(x)$ is also Jordan triple derivation on R.

We now present another example illustrating the sum of Jordan triple derivations on the ring $M_{2\times 2}[\mathbb{Z}]$.

Example 2. Given $M=\left\{\begin{bmatrix}\alpha_1&0\\0&\alpha_2\end{bmatrix}\middle|\alpha_1,\alpha_2\in\mathbb{Z}\right\}$. Choose $D=\begin{bmatrix}d_1&0\\0&d_2\end{bmatrix}, T=\begin{bmatrix}t_1&0\\0&t_2\end{bmatrix}\in M$. We define the map $\vartheta_1,\vartheta_2: M\to M$, with $\vartheta_1(L)=Dl-LD$ and $\vartheta_2(L)=TL-LT$. We can show that ϑ_1 and ϑ_2 are Jordan triple derivations on M. Now, we will show that $\vartheta_1+\vartheta_2$ is a Jordan triple derivation on M, that is, $(\vartheta_1+\vartheta_2)(LKL)=(\vartheta_1+\vartheta_2)(L)KL+L(\vartheta_1+\vartheta_2)(K)L+LK(\vartheta_1+\vartheta_2)(L)$, for every $L,K\in M$. Given any $L=\begin{bmatrix}l_1&0\\0&l_2\end{bmatrix}, K=\begin{bmatrix}k_1&0\\0&k_2\end{bmatrix}\in M$. We have:

$$\begin{aligned}(\vartheta_1+\vartheta_2)(LKL)&=(\vartheta_1+\vartheta_2)\left(\begin{bmatrix}l_1&0\\0&l_2\end{bmatrix}\begin{bmatrix}k_1&0\\0&k_2\end{bmatrix}\begin{bmatrix}l_1&0\\0&l_2\end{bmatrix}\right)\\&=(\vartheta_1+\vartheta_2)\left(\begin{bmatrix}l_1k_1l_1&0\\0&l_2k_2l_2\end{bmatrix}\right)\\&=\vartheta_1\left(\begin{bmatrix}l_1k_1l_1&0\\0&l_2k_2l_2\end{bmatrix}\right)+\vartheta_2\left(\begin{bmatrix}l_1k_1l_1&0\\0&l_2k_2l_2\end{bmatrix}\right).\end{aligned}$$

Then,

$$\begin{aligned}\vartheta_1\left(\begin{bmatrix}l_1k_1l_1&0\\0&l_2k_2l_2\end{bmatrix}\right)&=\begin{bmatrix}d_1&0\\0&d_2\end{bmatrix}\begin{bmatrix}l_1k_1l_1&0\\0&l_2k_2l_2\end{bmatrix}-\begin{bmatrix}l_1k_1l_1&0\\0&l_2k_2l_2\end{bmatrix}\begin{bmatrix}d_1&0\\0&d_2\end{bmatrix}\\&=\begin{bmatrix}d_1l_1k_1l_1&0\\0&d_2l_2k_2l_2\end{bmatrix}-\begin{bmatrix}l_1k_1l_1d_1&0\\0&l_2k_2l_2d_2\end{bmatrix}\\&=\begin{bmatrix}d_1l_1k_1l_1-l_1k_1l_1d_1&0\\0&d_2l_2k_2l_2-l_2k_2l_2d_2\end{bmatrix}.\end{aligned}$$

Similarly, we obtain

$$\vartheta_2\left(\begin{bmatrix}l_1k_1l_1&0\\0&l_2k_2l_2\end{bmatrix}\right)=\begin{bmatrix}t_1l_1k_1l_1-l_1k_1l_1t_1&0\\0&t_2l_2k_2l_2-l_2k_2l_2t_2\end{bmatrix}.$$

Hence,

$$\vartheta_1\left(\begin{bmatrix} l_1k_1l_1 & 0 \\ 0 & l_2k_2l_2 \end{bmatrix}\right) + \vartheta_2\left(\begin{bmatrix} l_1k_1l_1 & 0 \\ 0 & l_2k_2l_2 \end{bmatrix}\right) = \begin{bmatrix} (d_1+t_1)l_1k_1l_1 - l_1k_1l_1(d_1+t_1) & 0 \\ 0 & (d_2+t_2)l_1k_1l_1 - l_1k_1l_1(d_2+t_2) \end{bmatrix}. \tag{1}$$

Furthermore, we have

$$\begin{aligned}(\vartheta_1+\vartheta_2)(L)KL &+ L(\vartheta_1+\vartheta_2)(K)L + LK(\vartheta_1+\vartheta_2)(L) \\ &= \big(\vartheta_1(L)+\vartheta_2(L)\big)KL + L\big(\vartheta_1(K)+\vartheta_2(K)\big)L + LK\big(\vartheta_1(L)+\vartheta_2(L)\big) \\ &= \begin{bmatrix} (d_1+t_1)l_1k_1l_1 - l_1k_1l_1(d_1+t_1) & 0 \\ 0 & (d_2+t_2)l_1k_1l_1 - l_1k_1l_1(d_2+t_2) \end{bmatrix}. \end{aligned} \tag{2}$$

Based on Eqs. (1) and (2), we have $(\vartheta_1+\vartheta_2)(LKL) = (\vartheta_1+\vartheta_2)(L)KL + L(\vartheta_1+\vartheta_2)(K)L + LK(\vartheta_1+\vartheta_2)(L)$. So, $\vartheta_1+\vartheta_2$ is a Jordan triple derivation.

As an alternative approach to forming a new structure of Jordan triple derivations, linear combinations become one of the methods that can be examined. Define $l_i\vartheta_i: R \to R$, for each $i = 1,\dots,n$ and for each constant $l_i \in R$, where $(\sum_{i=1}^{n} l_i\vartheta_i)(v) = \sum_{i=1}^{n} l_i\vartheta_i(v)$, for every $v \in R$. With this definition, the following discussion will show the relationship between linear combinations and the properties of Jordan triple derivations.

Proposition 1. *Given a ring R. If $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ are Jordan triple derivations on the ring R and $l_1, l_2, \dots, l_n \in R$, then $\sum_{i=1}^{n} l_i\vartheta_i = l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n$ is Jordan triple derivation on R.*

Proof. Given a ring R. Let $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ are Jordan triple derivations on ring R. We will show that $l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n$ is a Jordan triple derivation.

We define:

$(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n): R \to R$ by $(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(v) = l_1\vartheta_1(v) + l_2\vartheta_2(v) + \cdots + l_1\vartheta_1(v)$,

for every $v \in R$. Given any $v, z \in R$, since $(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(v+z) = (l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(v) + (l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(z)$. It follows that $l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n$ is additive. Then, we will show that $(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(vzv) = (l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(v)zv + v(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(z)v + vz(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(v)$.

$$\begin{aligned}
&(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(vzv) \\
&= l_1\vartheta_1(vzv) + l_2\vartheta_2(vzv) + \cdots + l_n\vartheta_n(vzv) \\
&= l_1\vartheta_1(v)zv + vl_1\vartheta_1(z)v + vzl_1\vartheta_1(v) + l_2\vartheta_2(v)zv + vl_2\vartheta_2(z)v + vzl_2\vartheta_2(v) + \cdots + l_n\vartheta_n(v)zv + \\
&\quad vl_n\vartheta_n(z)v + vzl_n\vartheta_n(v) \\
&= [l_1\vartheta_1(v)zv + l_2\vartheta_2(v)zv + \cdots + l_n\vartheta_n(v)zv] + [vl_1\vartheta_1(z)v + vl_2\vartheta_2(z)v + \cdots + vl_n\vartheta_n(z)v] \\
&\quad + [vzl_1\vartheta_1(v) + vzl_2\vartheta_2(v) + \cdots + vzl_n\vartheta_n(v)] \\
&= [l_1\vartheta_1(v) + l_2\vartheta_2(v) + \cdots + l_n\vartheta_n(v)]zv + v[l_1\vartheta_1(z) + l_2\vartheta_2(z) + \cdots + l_n\vartheta_n(z)]v + vz[l_1\vartheta_1(v) \\
&\quad + l_2\vartheta_2(v) + \cdots + l_n\vartheta_n(v)] \\
&= [(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(v)]zv + v[(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(z)]\,v + vz[(l_1\vartheta_1 + l_2\vartheta_2 + \cdots \\
&\qquad + l_n\vartheta_n)(v)] \\
&= (\textstyle\sum_{i=1}^{n} l_i\vartheta_i)(v)zv + v(\sum_{i=1}^{n} l_i\vartheta_i)(z)v + vz(\sum_{i=1}^{n} l_i\vartheta_i)(v).
\end{aligned}$$

Hence,
$(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(vzv) = [(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(v)]zv + v[(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(z)]\,v + vz[(l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n)(v)]$. Therefore, it is proven that $l_1\vartheta_1 + l_2\vartheta_2 + \cdots + l_n\vartheta_n$ is a Jordan triple derivation. ∎

Example 3. Given $R = \mathbb{Z}[x]$. We define a map $\vartheta_1, \vartheta_2 : R \to R$, with $\vartheta_1(h(x)) = h'(x)$ and $\vartheta_2(p(x)) = p'(x)$. Both ϑ_1 and ϑ_2 are Jordan triple derivations on the ring R. Take $l_1 = v$ and $l_2 = z$, then the linear combination $\vartheta = l_1\vartheta_1 + l_2\vartheta_2 = v\vartheta_1 + z\vartheta_2$ is given by $\vartheta(h(x)) = vh'(x) + zp'(x)$. Hence, $v\vartheta_1 + z\vartheta_2$ is a Jordan triple derivation on R.

After constructing a linear combination of Jordan triple derivations, an alternative approach to forming another structure of Jordan triple derivations is through the $\oplus$ (direct sum) operation. This approach will form the basis for the theorem that will be discussed next.

Theorem 1. *Given a ring R. If $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ are Jordan triple derivations on the ring $R_1, R_2, \dots, R_n$, then $\vartheta_1 \oplus \vartheta_2 \oplus \dots \oplus \vartheta_n$ is Jordan triple derivation on $R_1 \oplus R_2 \oplus \dots \oplus R_n$.*

Proof. Given $(v_1, v_2, \dots, v_n) \in R_1 \oplus R_2 \oplus \dots \oplus R_n$. We will investigate that $\vartheta_1 \oplus \vartheta_2 \oplus \dots \oplus \vartheta_n$ is a Jordan triple derivation.

We define:

$$\vartheta_1 \oplus \vartheta_2 \oplus \dots \oplus \vartheta_n : R_1 \oplus R_2 \oplus \dots \oplus R_n \to R_1 \oplus R_2 \oplus \dots \oplus R_n$$

by $\vartheta_1 \oplus \vartheta_2 \oplus \dots \oplus \vartheta_n\ (v_1, v_2, \dots, v_n) = (\vartheta_1(v_1), \vartheta_2(v_2), \dots, \vartheta_n(v_n))$, for every $(v_1, v_2, \dots, v_n) \in R_1 \oplus R_2 \oplus \dots \oplus R_n$.

Given any $(v_1, v_2, \dots, v_n), (z_1, z_2, \dots, z_n) \in R_1 \oplus R_2 \oplus \dots \oplus R_n$. We have:

$$\begin{aligned}
(\vartheta_1 \oplus \dots \oplus \vartheta_n)((v_1, \dots, v_n) + (z_1, \dots, z_n)) &= (\vartheta_1 \oplus \dots \oplus \vartheta_n)(v_1 + z_1, \dots, v_n + z_n) \\
&= (\vartheta_1(v_1 + z_1), \dots, \vartheta_n(v_n + z_n)) \\
&= (\vartheta_1(v_1) + \vartheta_1(z_1), \dots, \vartheta_n(v_n) + \vartheta_n(z_n)) \\
&= (\vartheta_1(v_1), \dots, \vartheta_n(v_n)) + (\vartheta_1(z_1), \dots, \vartheta_n(z_n)) \\
&= (\vartheta_1 \oplus \dots \oplus \vartheta_n)((v_1, \dots, v_n)) + (\vartheta_1 \oplus \dots \oplus \vartheta_n) \\
&\quad ((z_1, \dots, z_n)).
\end{aligned}$$

It is shown that $\vartheta_1 \oplus \dots \oplus \vartheta_n$ is additive.

Furthermore, we can show that:

$$(\oplus_{i=1}^{n} \vartheta_n)((v_1, \dots, v_n)(z_1, \dots, z_n)(v_1, \dots, v_n)) = (\oplus_{i=1}^{n} \vartheta_n(v_1, \dots, v_n))(z_1, \dots, z_n)(v_1, \dots, v_n) + (v_1, \dots, v_n)(\oplus_{i=1}^{n} \vartheta_n(z_1, \dots, z_n))(v_1, \dots, v_n) + (v_1, \dots, v_n)(z_1, \dots, z_n)(\oplus_{i=1}^{n} \vartheta_n(v_1, \dots, v_n),$$

for every $(v_1, \dots, v_n)(z_1, \dots, z_n) \in R_1 \oplus R_2 \oplus \dots \oplus R_n$. Hence, $\vartheta_1 \oplus \dots \oplus \vartheta_n$ is a Jordan triple derivation on the ring $R_1 \oplus R_2 \oplus \dots \oplus R_n$. ■

In the previous discussion on Jordan derivatives, it was shown that every inner derivative is a Jordan derivative. Based on these results, it is interesting to further examine whether Inner derivatives also satisfy the properties of triple Jordan derivatives. The following is the discussion.

Theorem 2. *A mapping $\vartheta : R \to R$, with $\vartheta(c) = [c, v]$, for any $c \in R$ and $v \in R$. If ϑ is an inner derivation, then ϑ is also a Jordan triple derivation on R.*

Proof. We define the mapping $\vartheta : R \to R$ by $\vartheta(c) = [c, v] = vc - cv$ and $\vartheta(d) = [d, v] = vd - dv$, for all $c, d \in R$ and $v \in R$. We will show that $\vartheta(cdc) = \vartheta(c)dc + c\vartheta(d)c + cd\vartheta(c)$.

$$\begin{aligned}
\vartheta(c)dc + c\vartheta(d)c + cd\vartheta(c) &= [c, v]dc + c[d, v]c + cd[c, v] \\
&= (vc - cv)dc + c(vd - dv)c + cd(vc - cv) \\
&= vcdc - cvdc + cvdc - cdvc + cdvc - cdcv \\
&= vcdc - cdcv \\
&= [cdc, v] \\
&= \vartheta(cdc).
\end{aligned}$$

Hence, $\vartheta(cdc) = \vartheta(c)dc + c\vartheta(d)c + cd\vartheta(c)$. So, if ϑ is an inner derivation, then ϑ is a Jordan triple derivation on R.

■

Example 4. *Let $R = M_2(\mathbb{R})$. Take $A = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \in R$. Define the mapping $\vartheta: R \to R$ by $\vartheta(c) = [c, A] = Ac - cA$, for all $c \in R$. If, ϑ is an inner derivation on R. Then, by Theorem 3, ϑ is also a Jordan triple derivation on R.*

After constructing Jordan triple derivations on various algebraic structures, we will now discuss how a Jordan triple derivation can be defined on the polynomial ring $R[x]$. The aim is to show that every Jordan triple derivation defined on the ring R can be carried over to the polynomial ring $R[x]$ without losing its fundamental properties. Through this construction, a foundation is obtained for further analysis of the properties of Jordan triple derivation on the polynomial ring $R[x]$, which in this case opens up space for generalization to more complex algebraic structures. The following is the discussion.

Definition 2. Given a polynomial ring $R[x]$. If an additive mapping $\vartheta: R[x] \to R[x]$ is a triple Jordan derivation, then the following holds $\vartheta\big(h(x)k(x)h(x)\big) = \vartheta\big(h(x)\big)k(x)h(x) + h(x)\vartheta\big(k(x)\big)h(x) + h(x)k(x)\vartheta\big(h(x)\big)$, for all $h(x), k(x) \in R[x]$.

We now present an illustrative example of a Jordan triple derivation defined on the polynomial ring $R[x]$.

Example 5. Let $R = \mathbb{Z}$ and consider the polynomial ring $R[x] = \mathbb{Z}[x]$. We define a map $\vartheta: R[x] \to R[x]$, with $\vartheta\big(f(x)\big) = f'(x)$. As an illustration, we take $f(x) = a_2x^2 + a_1x + a_0$ and $g(x) = b_1x + b_0$. Then, $f'(x) = 2a_2x + a_1$ *and* $g'(x) = b_1$. Then,

$$\vartheta\big(f(x)g(x)f(x)\big)$$

$$= \vartheta\big((a_2x^2 + a_1x + a_0)(b_1x + b_0)(a_2x^2 + a_1x + a_0)\big)$$

$$= \vartheta\Big(\big((a_2x^2 + a_1x + a_0)(b_1x + b_0)\big)(a_2x^2 + a_1x + a_0)\Big)$$

$$= \vartheta\big((a_2x^2 + a_1x + a_0)(b_1x + b_0)\big)(a_2x^2 + a_1x + a_0) + (a_2x^2 + a_1x + a_0)(b_1x + b_0)\vartheta\big((a_2x^2 + a_1x + a_0)\big)$$

$$= \big[\vartheta(a_2x^2 + a_1x + a_0)(b_1x + b_0) + (a_2x^2 + a_1x + a_0)\vartheta\big((b_1x + b_0)\big)\big](a_2x^2 + a_1x + a_0) + (a_2x^2 + a_1x + a_0)(b_1x + b_0)\vartheta(a_2x^2 + a_1x + a_0)$$

$$= \vartheta(a_2x^2 + a_1x + a_0)(b_1x + b_0)(a_2x^2 + a_1x + a_0) + (a_2x^2 + a_1x + a_0)d\big((b_1x + b_0)\big)(a_2x^2 + a_1x + a_0) + (a_2x^2 + a_1x + a_0)(b_1x + b_0)\vartheta(a_2x^2 + a_1x + a_0)$$

$$= (2a_2x + a_1)(b_1x + b_0)(a_2x^2 + a_1x + a_0) + (a_2x^2 + a_1x + a_0)(b_1)(a_2x^2 + a_1x + a_0) + (a_2x^2 + a_1x + a_0)(b_1x + b_0)(2a_2x + a_1)$$

$$= f'(x)g(g)f(x) + f(x)g'(x)f(x) + f(x)g(x)f'(x)$$

$$= \vartheta(f(x))g(x)f(x) + f(x)\vartheta(g(x))f(x) + f(x)g(x)\vartheta(f(x)).$$

It is shown that ϑ is a Jordan triple derivation on $R[x]$.

The next theorem demonstrates how a Jordan triple derivation on the polynomial ring $R[x]$ can be constructed from a Jordan triple derivation on the ring R.

Theorem 3. *Let R be a ring. If $d : R \to R$ is a Jordan triple derivation, then there exists a Jordan triple derivation $\hat{d} : R[x] \to R[x]$ on $R[x]$.*

Proof. Let R be a ring. *If* $d : R \to R$ is a Jordan triple derivation, then it satisfies $d(aba) = d(a)ba + ad(b)a + abd(a)$, for all $a, b \in R$. We define $\hat{d} : R[x] \to R[x]$ by $\hat{d}(\sum_{i=0}^{n} a_ix^i) = \sum_{i=0}^{n} d(a_i)x^i$, for every $\sum_{i=0}^{n} a_ix^i \in R[x]$ We will show that $\hat{d}$ is a Jordan triple derivation on $R[x]$.

Let $p(x), q(x) \in R[x]$, where $p(x) = \sum_{i=0}^{n} a_ix^i$ and $q(x) = \sum_{i=0}^{n} b_ix^i$. We have:

$$\begin{aligned} \hat{d}(p(x) + q(x)) &= \hat{d}(\Sigma\, a_ix^i + \Sigma\, b_ix^i) \\ &= \hat{d}(\Sigma\, (a_i + b_i)x^i) \end{aligned}$$

$$= \sum_{i=0}^{n} d(a_i + b_i)x^i$$
$$= \sum_{i=0}^{n} (d(a_i) + d(b_i))x^i$$
$$= \sum_{i=0}^{n} d(a_i)x^i + \sum_{i=0}^{n} d(b_i)x^i$$
$$= \hat{d}(p(x)) + \hat{d}(q(x)).$$

Thus, d is additive.

Next, we will show that

$$\hat{d}\big(p(x)q(x)p(x)\big) = \hat{d}\big(p(x)\big)q(x)p(x) + p(x)\hat{d}\big(q(x)\big)p(x) + p(x)q(x)\hat{d}\big(p(x)\big),$$
$$p(x)q(x)p(x) = \left(\sum_{i=0}^{n} a_i\, x^i\right)\left(\sum_{j=0}^{m} b_j\, x^j\right)\left(\sum_{i=0}^{n} a_i\, x^i\right) = \sum_{k+0}^{n+m+n}\left(\sum_{i+j+i=k} a_i\, b_j a_i\right)x^k.$$

Applying $\hat{d}$, we get

$$\hat{d}(p(x)q(x)p(x)) = \hat{d}\left(\sum_{k+0}^{n+m+n}\left(\sum_{i+j+i=k} a_i\, b_j a_i\right)x^k\right) = \sum_{k=0}^{n+m+n} d\left(\sum_{i+j+i=k} a_i\, b_j a_i\right)x^k.$$

Since d is a Jordan triple derivation on R, we have

$$d(a_i b_j a_i) = d(a_i)b_j a_i + a_i d(b_j)a_i + a_i b_j d(a_i),$$

for every $a_i, b_j \in R$.

Consequently,

$$d\left(\sum_{i+j+i=k} a_i\, b_j a_i\right) = \sum_{i+j+i=k} d(\,a_i b_j a_i) = \sum_{i+j+i=k} (d(\,a_i)b_j a_i + a_i d(b_j)a_i + a_i b_j d(a_i)).$$

Therefore,

$$\hat{d}\big(p(x)q(x)p(x)\big) = \sum_{k+0}^{n+m+n}\left(\sum_{i+j+i=k} (\,d(a_i)b_j a_i + a_i d(b_j)a_i + a_i b_j d(a_i))\right)x^k.$$

Based on the definition of $\hat{d}$, it can be shown that

$$\hat{d}(p(x)) = \hat{d}\left(\sum_{i=0}^{n} a_i\, x^i\right) = \sum_{i=0}^{n} d\,(a_i)x^i$$

and

$$\hat{d}(q(x)) = \hat{d}\left(\sum_{j=0}^{m} b_j\, x^j\right) = \sum_{j=0}^{m} d\,(b_j)x^j.$$

Then we obtain,

$$\hat{d}(p(x))q(x)p(x) + p(x)\hat{d}(q(x))p(x) + p(x)q(x)\hat{d}(p(x))$$
$$= \left(\sum_{i=0}^{n} d(a_i)x^i\right)\left(\sum_{j=0}^{m} b_j\, x^j\right)\left(\sum_{i=0}^{n} a_i\, x^i\right) + \left(\sum_{i=0}^{n} a_i\, x^i\right)\left(\sum_{j=0}^{m} d(b_j)x^j\right)\left(\sum_{i=0}^{n} a_i\, x^i\right)$$

$$+\left(\sum_{i=0}^{n} a_i\, x^i\right)\left(\sum_{j=0}^{m} b_j\, x^j\right)\left(\sum_{i=0}^{n} d(a_i)x^i\right)$$

$$= \sum_{k+0}^{n+m+n}\left(\sum_{i+j+i=k} d(a_i)b_j a_i\right)x^k + \sum_{k+0}^{n+m+n}\left(\sum_{i+j+i=k} a_i\, d(b_j)a_i\right)x^k$$

$$+ \sum_{k+0}^{n+m+n}\left(\sum_{i+j+i=k} a_i\, b_j d(a_i)\right)x^k$$

$$= \sum_{k+0}^{n+m+n}\left(\sum_{i+j+i=k} (\, d(a_i)b_j a_i + a_i d(b_j)a_i + a_i b_j d(a_i))\right)x^k$$

$$= \sum_{k+0}^{n+m+n}\left(\sum_{i+j+i=k} d(a_i b_j a_i)\right)x^k$$

$$= \hat{d}\big(p(x)q(x)p(x)\big).$$

Hence, $\hat{d}\big(p(x)q(x)p(x)\big) = \hat{d}\big(p(x)\big)q(x)p(x) + p(x)\hat{d}\big(q(x)\big)p(x) + p(x)q(x)\hat{d}\big(p(x)\big)$.

Thus, $\hat{d}$ is a Jordan triple derivation on $R[x]$. ■

4. CONCLUSION

Based on the results and discussion, it has been established that the set of Jordan triple derivations is closed under finite sums, linear combinations, and finite direct sums. Furthermore, it is shown that every inner derivation also satisfies the conditions of a Jordan triple derivation. Moreover, any Jordan triple derivation on the ring R induces a Jordan triple derivation on the polynomial ring $R[x]$. These findings contribute to a deeper understanding of the structure of Jordan triple derivations and can be effectively applied in the study of such derivations on polynomial rings $R[x]$.

These results not only reinforce the algebraic robustness of Jordan triple derivations but also provide a foundational framework for further exploration in polynomial ring structures. Future research may extend these findings to more general classes of rings or investigate potential applications in related areas of algebra.

Author Contributions

Rachma Allya Sieffa: Writing-Original Draft, Formal Analysis. Fitriani: Conceptualization, Methodology, Validation, Writing-Review and Editing. Ahmad Faisol: Formal Analysis, Validation. Wamiliana: Validation, Writing-Review. Thomas Juliansyah: Validation, Writing-Review and Editing. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare no competing interests.

Declaration of Generative AI and AI-assisted technologies

Generative AI tools (e.g., ChatGPT) were used solely for language refinement, including grammar, spelling, and clarity. The scientific content, analysis, interpretation, and conclusions were developed entirely by the authors. All final text was reviewed and approved by the authors.

REFERENCES

[1] M. Bresar, "JORDAN DERIVATIONS REVISITED," *Dep. Math. Univ. Maribor, PEF, Koroska 160, Maribor, Slov.*, pp. 1–16, 2005.

[2] M. Hongan, N. U. Rehman, and R. M. Al-Omary, "LIE IDEALS AND JORDAN TRIPLE DERIVATIONS IN RINGS," *Rend. Sem. Mat. Univ. Padova*, vol. 125, pp. 147–156, 2011. doi: https://doi.org/10.4171/rsmup/125-9

[3] M. R. Helmi, H. Marubayashi, and A. Ueda, "DIFFERENTIAL POLYNOMIAL RINGS WHICH ARE GENERALIZED ASANO PRIME RING," *Indian J. Pure Appl. Math.*, vol. 44(5), pp. 673–681, 2013. doi: https://doi.org/10.1007/s13226-013-0035-6

[4] N. U. Rehman, "JORDAN TRIPLE (\ALPHA,\ \BETA)\AST-DERIVATIONS ON SEMIPRIME RINGS WITH INVOLUTION," *Hacettepe J. Math. Stat.*, vol. 42, no. 6, pp. 641–651, 2013.

[5] M. Atteya, "COMMUTATIVITY WITH DERIVATIONS OF SEMIPRIME RINGS," *Discuss. Math. Gen. Algebr. Appl.*, vol. 40, pp. 165–175, 2020. doi: https://doi.org/10.7151/dmgaa.1333

[6] X. Zhao and X. Qi, "CHARACTERIZATION OF JORDAN \AST-DERIVATIONS BY LOCAL ACTION ON RINGS WITH INVOLUTION," *J. Hyperstructures*, vol. 6, no. 2, pp. 120–127, 2017.

[7] R. N. Ferreira and B. L. M. Ferreira, "JORDAN TRIPLE DERIVATION ON ALTERNATIVE RINGS," *Proyecciones J. Math.*, vol. 37, no. 1, pp. 171–180, 2018. doi: https://doi.org/10.4067/S0716-09172018000100171

[8] U. Sayin and F. Kuzucuoglu, "JORDAN DERIVATIONS OF SPECIAL SUBRINGS OF MATRIX RINGS," *Algebr. Colloq.*, vol. 26, no. 1, pp. 83–92, 2019. doi: https://doi.org/10.1142/S1005386719000087

[9] A. Bagheri and H. Emami, "THE APPLICATIONS OF ALGEBRAIC POLYNOMIAL RINGS IN SATELLITE CODING AND CRYPTOGRAPHY," *Math. Interdiscip. Res.*, vol. 7, pp. 301–329, 2022.

[10] V. Darvish, M. Nouri, M. Razeghi, and A. Taghavi, "NONLINEAR \AST-JORDAN TRIPLE DERIVATION ON PRIME\ \AST-ALGEBRAS VOL. 50, NO. 2, PP. 543-549, 2020.," *J. Math.*, vol. 50, no. 2, pp. 543–549, 2020. doi: https://doi.org/10.1216/rmj.2020.50.543

[11] A. Ma, L. Chen, and Z. Qin, "JORDAN SEMI-TRIPLE DERIVATIONS AND JORDAN CENTRALIZERS ON GENERALIZED QUATERNION ALGEBRAS," *AIMS Math.*, vol. 8, no. 3, pp. 6026–6035, 2022. doi: https://doi.org/10.3934/math.2023304

[12] J. Huang, K. Kudaybergenov, and F. Sukochev, "RING DERIVATIONS OF MURRAY–VON NEUMANN ALGEBRAS," *Linear Algebra Appl.*, vol. 672, no. 1, pp. 28–52, 2023. doi: https://doi.org/10.1016/j.laa.2023.04.011

[13] A. R. Rahnaward, S. Kaheshzad, and S. Rahimi, "NON-LINEAR \LAMBDA-JORDAN TRIPLE DERIVATION ON PRIME ALGEBRAS," *J. Res. Appl. Sci. Biotecnol.*, vol. 3, no. 5, pp. 303–306, 2024. doi: https://doi.org/10.55544/jrasb.3.5.31

[14] Fitriani, I. E. Wijayanti, A. Faisol, and S. Ali, "ON F-DERIVATIONS ON POLYNOMIAL MODULES," *J. Algebr. its Appl.*, vol. 24, no. 6, pp. 1–14, 2024, doi: https://doi.org/10.1142/S0219498825501555.

[15] Fitriani, I. E. Wijayanti, A. Faisol, and S. Ali, "COMMUTING AND CENTRALIZING MAPS ON MODULES," vol. 10, no. 3, pp. 691–697, 2025. doi: https://doi.org/10.26554/sti.2025.10.3.690-697

[16] D. E. Sitompul, F. Fitriani, S. L. Chasanah, and A. Faisol, "JORDAN DERIVATION ON THE POLYNOMIAL RING R [X]," *Integr. J. Integr. Math. Comput. Sci.*, vol. 2, no. 2, pp. 41–47, 2024. doi: https://doi.org/10.26554/integrajimcs.20252229

[17] D. L. Mursyidah, F. Fitriani, B. H. S. Utami, and A. Faisol, "NIL DERIVATION AND \DELTA-IDEAL ON POLYNOMIAL RING," *Barekeng J. Math. App.*, vol. accepted, 2025. doi: https://doi.org/10.30598/barekengvol20iss1pp0325-0334

[18] A. Faisol and F. Fitriani, "A STUDY OF DERIVATIONS AND LINEAR MAPPINGS ON SKEW GENERALIZED POWER SERIES MODULES," *Barekeng J. Math. App.*, vol. 19, no. 4, pp. 3047–3058, 2025. doi: https://doi.org/10.30598/barekengvol19iss4pp3047-3058

[19] R. Waluyo, A. Faisol, and F. Fitriani, "(σ, τ)-DERIVASI PADA RING GRUP," *Euler J. Ilm. Mat. Sains dan Teknol.*, vol. 13, no. 2, pp. 142–146, 2025. doi: https://doi.org/10.37905/euler.v13i2.31564