

HOMODERIVATIONS ON RINGS: STRUCTURAL PROPERTIES AND CHARACTERIZATIONS

Desi Widiarti ¹, Fitriani ^{2*}, Ahmad Faisol ³

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lampung
Jln. Prof. Dr. Ir. Sumantri Brojonegoro No. 1, Bandar Lampung, 35141, Indonesia

Corresponding author's e-mail: *fitriani.1984@fmipa.unila.ac.id

Article Info

Article History:

Received: 27th September 2025

Revised: 4th May 2026

Accepted: 7th June 2026

Available online: 1st July 2026

Keywords:

Derivation;
Homoderivation;
Homomorphism;
Ring.

ABSTRACT

Research in ring theory has produced major advances, particularly in the study of derivations. A related notion is that of a homoderivation, which combines aspects of derivations and ring homomorphisms. An additive map $h: R \rightarrow R$ is called a homoderivation if $h(ab) = h(a)h(b) + h(a)b + ah(b)$, for every $a, b \in R$. This study provides a comprehensive analysis of the fundamental structural properties of homoderivation mappings. The main results show that the class of homoderivations is closed under direct sums on component rings. In addition, the sum of two homoderivations is again a homoderivation, provided the maps are orthogonal. However, scalar multiplication of a homoderivation generally does not yield another homoderivation, highlighting a key structural distinction from classical derivations.



This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution-ShareAlike 4.0 International License](https://creativecommons.org/licenses/by-sa/4.0/).

How to cite this article:

D. Widiarti, Fitriani, and A. Faisol, "HOMODERIVATIONS ON RINGS: STRUCTURAL PROPERTIES AND CHARACTERIZATIONS", *BAREKENG: J. Math. & App.*, vol. 20, no. 4, pp. 3083-3094, Dec. 2026.

Copyright © 2026 Author(s)

Journal homepage: <https://ojs3.unpatti.ac.id/index.php/barekeng/>

Journal e-mail: barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id

Research Article • **Open Access**

1. INTRODUCTION

The concept of derivation plays a fundamental role in ring theory, and research in this area continues to develop rapidly. A derivation on a ring R , as stated by Felzenszwalb and Lanski [1], is an additive mapping $d: R \rightarrow R$ that satisfies the Leibniz rule. Further studies on additive mappings in rings with identity, such as the work of Dhara and Sharma [2], have contributed to a deeper understanding of ring structures and their fundamental properties.

Numerous researchers have investigated derivations from various perspectives. For instance, Chung and Kobayashi [3] examined nil derivations and their influence on ideals in prime rings, while Ernanto [4] studied derivations in factor rings. Comprehensive treatments of derivations can be found in Ali et al. [5] and Ashraf et al. [6]. Other works include Ber et al. [7], as well as Chung [8] and Al-Khalaf et al. [9], who focused on derivations in prime rings. Additional contributions by Thomas et al. [10] further highlight the diversity of derivation theory, and Huang et al. [11], who analyzed derivations on Murray–von Neumann algebras.

Research has also extended to specialized types of derivations. Al-Khalaf et al. [12] studied Lie and Jordan derivations, while Helmi et al. [13] investigated differential polynomial rings. Ansari et al. [14] explored n -derivations with automorphisms, and Sayin and Feride [15] examined Jordan derivations in matrix subrings. Watase [16] contributed to derivations on commutative rings. More recent studies include Fitriani et al. ([17], [18]) on f -derivations, Sitompul et al. [19] on Jordan derivations in polynomial rings, and Mursyidah et al. [20] on nil derivations. Further developments by Faisol and Fitriani [21] and Waluyo et al. [22] extend derivations to skew modules and group rings, respectively.

A natural extension of derivations is the notion of a homoderivation, which combines the Leibniz rule with the multiplicative property of ring homomorphisms. Specifically, an additive mapping $h: R \rightarrow R$ is called a homoderivation if $h(ab) = h(a)h(b) + h(a)b + ah(b)$, for every $a, b \in R$. Here, the term $h(a)h(b)$ reflects the homomorphic property, while the remaining terms correspond to the Leibniz rule. Homoderivations have been studied in various algebraic settings. Engin and Aydin [23] investigated homoderivations on prime rings, while Belkadi and Taoufiq [24] and Taoufiq et al. [25] examined higher-order and nilpotent homoderivations in semiprime rings. Alharfie and Muthana ([26], [27]) explored their connections with commutativity, and Turkmen [28] analyzed their relationship with semiderivations. Additional contributions include Melaibari et al. [29] on associative rings, Hummdi et al. [30] on Lie ideals, El-Sayiad and Almulhem [31] on generalized homoderivations, and Ali et al. [32] on reverse homoderivations.

Given the central role of derivations and their generalizations in revealing ring structures, the study of homoderivations is both natural and necessary. As an extension of classical derivations, homoderivations provide a framework for structure-preserving transformations in algebra. Their relevance extends beyond pure mathematics to applications such as coding theory, cryptography, and data analysis, as illustrated by Manju and Sharma [33]. Therefore, a systematic investigation of the structural properties of homoderivations is essential for advancing both theoretical understanding and practical applications.

2. RESEARCH METHODS

This study adopts a literature-based and theoretical analysis approach. Relevant references, including peer-reviewed journals, scientific articles, monographs, and other authoritative sources, are systematically reviewed to establish a rigorous theoretical foundation. The methodological steps undertaken in this research are as follows:

1. Examine the algebraic properties of homoderivations on the ring R , focusing on their structural characteristics and operational behavior.
2. Develop and present explicit examples of homoderivations on the ring R to illustrate and validate the theoretical framework.
3. Explore the relationship between homoderivations and classical derivations on the ring R , highlighting similarities, distinctions, and potential generalizations.

This methodological design ensures that the study not only establishes fundamental properties but also contextualizes homoderivations within the broader theory of derivations in ring structures.

3. RESULTS AND DISCUSSION

In this section, we give some properties of homoderivation on a ring. Given two homoderivation maps, μ_1 and μ_2 . We investigate whether the sum $\mu_1 + \mu_2$ is a homoderivation on the ring R or not, as defined below:

$$(\mu_1 + \mu_2)(\alpha) = \mu_1(\alpha) + \mu_2(\alpha).$$

Therefore, for the sum of two homoderivations μ_1 and μ_2 to be a homoderivation, it is necessary that they be orthogonal (mutually null), that is, $\mu_1(\alpha)\mu_2(\theta) + \mu_2(\alpha)\mu_1(\theta) = 0$, for every $\alpha, \theta \in R$. This is demonstrated in the following theorem.

Theorem 1. *Let a ring R , a map $\mu: R \rightarrow R$ is a homoderivation on ring R . If μ_1 and μ_2 are homoderivations mapping on ring R , where $\mu_1(\alpha)\mu_2(\theta) + \mu_2(\alpha)\mu_1(\theta) = 0$, for every $\alpha, \theta \in R$, then $\mu_1 + \mu_2$ is a homoderivation on ring R .*

Proof. We will show that, $\mu_1 + \mu_2$ is a homoderivation mapping, where $(\mu_1 + \mu_2)(\alpha) = \mu_1(\alpha) + \mu_2(\alpha)$, for every $\alpha \in R$.

1. Given any $\alpha, \theta \in R$.

$$\begin{aligned} (\mu_1 + \mu_2)(\alpha + \theta) &= \mu_1(\alpha + \theta) + \mu_2(\alpha + \theta) \\ &= \mu_1(\alpha) + \mu_1(\theta) + \mu_2(\alpha) + \mu_2(\theta) \\ &= \mu_1(\alpha) + \mu_2(\alpha) + \mu_1(\theta) + \mu_2(\theta) \\ &= (\mu_1 + \mu_2)(\alpha) + (\mu_1 + \mu_2)(\theta). \end{aligned}$$

2. $(\mu_1 + \mu_2)(\alpha\theta) = \mu_1(\alpha\theta) + \mu_2(\alpha\theta)$

$$\begin{aligned} &= \mu_1(\alpha)\mu_1(\theta) + \mu_1(\alpha)\theta + \alpha\mu_1(\theta) + \mu_2(\alpha)\mu_2(\theta) + \mu_2(\alpha)\theta + \alpha\mu_2(\theta) \\ &= \mu_1(\alpha)\mu_1(\theta) + \mu_2(\alpha)\mu_2(\theta) + (\mu_1 + \mu_2)(\alpha)\theta + \alpha(\mu_1 + \mu_2)(\theta) \\ &= (\mu_1 + \mu_2)(\alpha)(\mu_1 + \mu_2)(\theta) - (\mu_1(\alpha)\mu_2(\theta) + \mu_2(\alpha)\mu_1(\theta)) + (\mu_1 + \mu_2)(\alpha)\theta + \alpha(\mu_1 + \mu_2)(\theta). \end{aligned}$$

So, $\mu_1 + \mu_2$ is a homoderivation under the condition $\mu_1(\alpha)\mu_2(\theta) + \mu_2(\alpha)\mu_1(\theta) = 0$ which are referred to as mutually annihilating or orthogonal homomorphisms. ■

Example 1. Given a ring $M = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \theta \end{bmatrix} \mid \alpha, \theta \in \mathbb{Z} \right\}$. We define $\mu_1, \mu_2: M \rightarrow M$, where: $\mu_1\left(\begin{bmatrix} \alpha & 0 \\ 0 & \theta \end{bmatrix}\right) = \begin{bmatrix} -\alpha & 0 \\ 0 & 0 \end{bmatrix}$ and $\mu_2\left(\begin{bmatrix} \alpha & 0 \\ 0 & \theta \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & -\theta \end{bmatrix}$, for every $\begin{bmatrix} \alpha & 0 \\ 0 & \theta \end{bmatrix} \in M$. We can show that μ_1 and μ_2 are homoderivations on M . Now, we can show that $\mu_1 + \mu_2$ is a homoderivation on M .

Let $U = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}$, $V = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \in M$. We have:

$$\begin{aligned} (\mu_1 + \mu_2)(U + V) &= (\mu_1 + \mu_2)\left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix}\right) \\ &= (\mu_1 + \mu_2)\left(\begin{bmatrix} \alpha_1 + \alpha_2 & 0 \\ 0 & \theta_1 + \theta_2 \end{bmatrix}\right) \\ &= \mu_1\left(\begin{bmatrix} \alpha_1 + \alpha_2 & 0 \\ 0 & \theta_1 + \theta_2 \end{bmatrix}\right) + \mu_2\left(\begin{bmatrix} \alpha_1 + \alpha_2 & 0 \\ 0 & \theta_1 + \theta_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} -(\alpha_1 + \alpha_2) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -(\theta_1 + \theta_2) \end{bmatrix} \\ &= \begin{bmatrix} -\alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\alpha_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\theta_2 \end{bmatrix} \\ &= \mu_1\left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}\right) + \mu_1\left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix}\right) + \mu_2\left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}\right) + \mu_2\left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix}\right) \end{aligned}$$

$$\begin{aligned}
&= \mu_1 \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right) + \mu_2 \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right) + \mu_1 \left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \right) + \mu_2 \left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \right) \\
&= (\mu_1 + \mu_2) \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right) + (\mu_1 + \mu_2) \left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \right) \\
&= (\mu_1 + \mu_2)(U) + (\mu_1 + \mu_2)(V).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(\mu_1 + \mu_2)(UV) &= (\mu_1 + \mu_2) \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \right) \\
&= (\mu_1 + \mu_2) \left(\begin{bmatrix} \alpha_1 \alpha_2 & 0 \\ 0 & \theta_1 \theta_2 \end{bmatrix} \right) \\
&= \mu_1 \left(\begin{bmatrix} \alpha_1 \alpha_2 & 0 \\ 0 & \theta_1 \theta_2 \end{bmatrix} \right) + \mu_2 \left(\begin{bmatrix} \alpha_1 \alpha_2 & 0 \\ 0 & \theta_1 \theta_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} -\alpha_1 \alpha_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \theta_2 \end{bmatrix} \\
&= \begin{bmatrix} -\alpha_1 \alpha_2 & 0 \\ 0 & -\theta_1 \theta_2 \end{bmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\mu_1 + \mu_2)(U)(\mu_1 + \mu_2)(V) + (\mu_1 + \mu_2)(U)V + U(\mu_1 + \mu_2)(V) \\
&= (\mu_1 + \mu_2) \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right) (\mu_1 + \mu_2) \left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \right) + (\mu_1 + \mu_2) \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right) \begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} (\mu_1 + \mu_2) \left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \right) \\
&= \left(\mu_1 \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right) + \mu_2 \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right) \right) \left(\mu_1 \left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \right) + \mu_2 \left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \right) \right) \\
&\quad + \left(\mu_1 \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right) + \mu_2 \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right) \right) \begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \left(\mu_1 \left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \right) + \mu_2 \left(\begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \right) \right) \\
&= \left(\begin{bmatrix} -\alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \end{bmatrix} \right) \left(\begin{bmatrix} -\alpha_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\theta_2 \end{bmatrix} \right) + \left(\begin{bmatrix} -\alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \end{bmatrix} \right) \begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \left(\begin{bmatrix} -\alpha_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\theta_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\theta_1 \end{bmatrix} \begin{bmatrix} -\alpha_2 & 0 \\ 0 & -\theta_2 \end{bmatrix} + \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\theta_1 \end{bmatrix} \begin{bmatrix} \alpha_2 & 0 \\ 0 & \theta_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \begin{bmatrix} -\alpha_2 & 0 \\ 0 & -\theta_2 \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 \alpha_2 & 0 \\ 0 & \theta_1 \theta_2 \end{bmatrix} + \begin{bmatrix} -\alpha_1 \alpha_2 & 0 \\ 0 & -\theta_1 \theta_2 \end{bmatrix} + \begin{bmatrix} -\alpha_1 \alpha_2 & 0 \\ 0 & -\theta_1 \theta_2 \end{bmatrix} \\
&= \begin{bmatrix} -\alpha_1 \alpha_2 & 0 \\ 0 & -\theta_1 \theta_2 \end{bmatrix}.
\end{aligned}$$

Hence, $(\mu_1 + \mu_2)(UV) = (\mu_1 + \mu_2)(U)(\mu_1 + \mu_2)(V) + (\mu_1 + \mu_2)(U)V + U(\mu_1 + \mu_2)(V)$, for every $U, V \in M$. Thus, $\mu_1 + \mu_2$ is a homoderivation on M .

Moreover, scalar multiplication of a homoderivation μ on a ring does not, in general, yield another homoderivation, as illustrated by the following counterexample. For a homoderivation $\mu : R \rightarrow R$, we define $k\mu$ and $-\mu$ by $(k\mu)(r) = k\mu(r)$ and $-\mu : R \rightarrow R$ by $(-\mu)(r) = -\mu(r)$, for every $r \in R$.

Example 2. Let a ring $M_2(\mathbb{Z})$ with a mapping $\mu : M_2(\mathbb{Z}) \rightarrow M_2(\mathbb{Z})$, defined as follows: $\mu \left(\begin{bmatrix} \alpha & \theta \\ \tau & \sigma \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & -\sigma \end{bmatrix}$, for every $\begin{bmatrix} \alpha & \theta \\ \tau & \sigma \end{bmatrix} \in M_2(\mathbb{Z})$. Choose $K = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \in M_2(\mathbb{Z})$, to be used in examining the homoderivation property of the mapping $K\mu$.

It will be examined whether the equality $K\mu(UV) = K\mu(U)K\mu(V) + K\mu(U)V + UK\mu(V)$ holds.

$$\begin{aligned}
Kh(UV) &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mu \left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right) \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mu \left(\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \right) \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(-\mu)(U)(-\mu)(V) + (-\mu)(U)V + U(-\mu)(V) \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mu \left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mu \left(\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right) + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mu \left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mu \left(\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right) \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 12 \end{bmatrix}.
\end{aligned}$$

Thus, it is obtained that the left-hand side and the right-hand side are not equal. Therefore, it is proven that $K\mu$ is not a homoderivation mapping on the ring $M_2(\mathbb{Z})$.

Next, if μ_1 and μ_2 are given as homoderivation maps, their direct sum is defined by:

$$(\mu_1 \oplus \mu_2)(\alpha_1, \alpha_2) = (\mu_1(\alpha_1), \mu_2(\alpha_2)).$$

It has been investigated that the direct sum of homoderivation maps $\mu_1, \mu_2, \dots, \mu_n$ is also a homoderivation map on the rings $R_1 \oplus R_2 \oplus \dots \oplus R_n$, as demonstrated in the following theorem.

Theorem 2. Let $\mu_1, \mu_2, \dots, \mu_n$ be homoderivations on rings $R_1, R_2, \dots, R_n$, respectively. Then $\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n$ is a homoderivation on the direct sum $R_1 \oplus R_2 \oplus \dots \oplus R_n$.

Proof. We will show that, $\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n$ on $R_1 \oplus R_2 \oplus \dots \oplus R_n$.

1. We will show that $\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n$ is additive on ring $R_1 \oplus R_2 \oplus \dots \oplus R_n$. Given any $(\alpha_1, \alpha_2, \dots, \alpha_n), (\theta_1, \theta_2, \dots, \theta_n) \in R_1 \oplus R_2 \oplus \dots \oplus R_n$.

$$\begin{aligned}
&\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n((\alpha_1, \alpha_2, \dots, \alpha_n) + (\theta_1, \theta_2, \dots, \theta_n)) \\
&= (\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \dots, \alpha_n + \theta_n) \\
&= (\mu_1(\alpha_1 + \theta_1), \mu_2(\alpha_2 + \theta_2), \dots, \mu_n(\alpha_n + \theta_n)) \\
&= (\mu_1(\alpha_1) + \mu_1(\theta_1), \mu_2(\alpha_2) + \mu_2(\theta_2), \dots, \mu_n(\alpha_n) + \mu_n(\theta_n)) \\
&= (\mu_1(\alpha_1), \mu_2(\alpha_2), \dots, \mu_n(\alpha_n)) + (\mu_1(\theta_1), \mu_2(\theta_2), \dots, \mu_n(\theta_n)) \\
&= (\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\alpha_1, \alpha_2, \dots, \alpha_n) + (\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\theta_1, \theta_2, \dots, \theta_n).
\end{aligned}$$

Thus, $\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n$ is additive.

$$\begin{aligned}
&2. (\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\alpha_1\theta_1, \alpha_2\theta_2, \dots, \alpha_n\theta_n) \\
&= (\mu_1(\alpha_1\theta_1), \mu_2(\alpha_2\theta_2), \dots, \mu_n(\alpha_n\theta_n)) \\
&= (\mu_1(\alpha_1)\mu_1(\theta_1) + \mu_1(\alpha_1)\theta_1 + \alpha_1\mu_1(\theta_1), \mu_2(\alpha_2)\mu_2(\theta_2) + \mu_2(\alpha_2)\theta_2 + \alpha_2\mu_2(\theta_2), \dots, \mu_n(\alpha_n)\mu_n(\theta_n) \\
&\quad + \mu_n(\alpha_n)\theta_n + \alpha_n\mu_n(\theta_n)) \\
&= (\mu_1(\alpha_1)\mu_1(\theta_1), \mu_2(\alpha_2)\mu_2(\theta_2), \dots, \mu_n(\alpha_n)\mu_n(\theta_n)) + (\mu_1(\alpha_1)\theta_1, \mu_2(\alpha_2)\theta_2, \dots, \mu_n(\alpha_n)\theta_n) \\
&\quad + (\alpha_1\mu_1(\theta_1), \alpha_2\mu_2(\theta_2), \dots, \alpha_n\mu_n(\theta_n))
\end{aligned}$$

$$\begin{aligned}
&= (\mu_1(\alpha_1), \mu_2(\alpha_2), \dots, \mu_n(\alpha_n))(\mu_1(\theta_1), \mu_2(\theta_2), \dots, \mu_n(\theta_n)) + (\mu_1(\alpha_1), \mu_2(\alpha_2), \dots, \mu_n(\alpha_n))(\theta_1, \theta_2, \dots, \theta_n) \\
&\quad + (\alpha_1, \alpha_2, \dots, \alpha_n)(\mu_1(\theta_1), \mu_2(\theta_2), \dots, \mu_n(\theta_n)) \\
&= (\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\alpha_1, \alpha_2, \dots, \alpha_n)(\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\theta_1, \theta_2, \dots, \theta_n) \\
&\quad + (\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\alpha_1, \alpha_2, \dots, \alpha_n)(\theta_1, \theta_2, \dots, \theta_n) \\
&\quad + (\alpha_1, \alpha_2, \dots, \alpha_n)(\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\theta_1, \theta_2, \dots, \theta_n).
\end{aligned}$$

Hence, $\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n$ is a homoderivation on ring $R_1 \oplus R_2 \oplus \dots \oplus R_n$. ■

We now provide an example demonstrating [Theorem 2](#).

Example 3. Let $M_1 = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \theta \end{bmatrix} \mid \alpha, \theta \in \mathbb{Z} \right\}$ and $M_2 = \left\{ \begin{bmatrix} \alpha & \theta \\ 0 & 0 \end{bmatrix} \mid \alpha, \theta \in \mathbb{Z} \right\}$. We define $\mu_1: M_1 \rightarrow M_1$ and $\mu_2: M_2 \rightarrow M_2$ by $\mu_1\left(\begin{bmatrix} \alpha & 0 \\ 0 & \theta \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & -\theta \end{bmatrix}$, $\mu_2\left(\begin{bmatrix} \alpha & \theta \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -\alpha \\ 0 & 0 \end{bmatrix}$.

We will show that $\mu_1 \oplus \mu_2$ is a homoderivation on $M_1 \oplus M_2$.

Given any $U_1 = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}, V_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix} \in M_1$ and $U_2 = \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix}, V_2 = \begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix} \in M_2$.

We will show that $\mu_1 \oplus \mu_2$ is an additive mapping on ring $M_1 \oplus M_2$.

$$\begin{aligned}
&(\mu_1 \oplus \mu_2)(U_1 + V_1, U_2 + V_2) \\
&= (\mu_1 \oplus \mu_2)\left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix}\right) \\
&= (\mu_1 \oplus \mu_2)\left(\begin{bmatrix} \alpha_1 + \sigma_1 & 0 \\ 0 & \theta_1 + \tau_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 + \sigma_2 & \theta_2 + \tau_2 \\ 0 & 0 \end{bmatrix}\right) \\
&= \left(\mu_1\left(\begin{bmatrix} \alpha_1 + \sigma_1 & 0 \\ 0 & \theta_1 + \tau_1 \end{bmatrix}\right), \mu_2\left(\begin{bmatrix} \alpha_2 + \sigma_2 & \theta_2 + \tau_2 \\ 0 & 0 \end{bmatrix}\right)\right) \\
&= \left(\begin{bmatrix} 0 & 0 \\ 0 & -(\theta_1 + \tau_1) \end{bmatrix}, \begin{bmatrix} 0 & -(\alpha_2 + \sigma_2) \\ 0 & 0 \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \end{bmatrix}, \begin{bmatrix} 0 & -\alpha_2 \\ 0 & 0 \end{bmatrix}\right) + \left(\begin{bmatrix} 0 & 0 \\ 0 & -\tau_1 \end{bmatrix}, \begin{bmatrix} 0 & -\sigma_2 \\ 0 & 0 \end{bmatrix}\right) \\
&= \left(\mu_1\left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}\right), \mu_2\left(\begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix}\right)\right) + \left(\mu_1\left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix}\right), \mu_2\left(\begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix}\right)\right) \\
&= (\mu_1 \oplus \mu_2)\left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix}\right) + (\mu_1 \oplus \mu_2)\left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix}, \begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix}\right) \\
&= (\mu_1 \oplus \mu_2)(U_1, U_2) + (\mu_1 \oplus \mu_2)(V_1, V_2).
\end{aligned}$$

Now,

$$\begin{aligned}
(\mu_1 \oplus \mu_2)(U_1 V_1, U_2 V_2) &= (\mu_1 \oplus \mu_2)\left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix}\right) \\
&= (\mu_1 \oplus \mu_2)\left(\begin{bmatrix} \alpha_1 \sigma_1 & 0 \\ 0 & \theta_1 \tau_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 \sigma_2 & \alpha_2 \tau_2 \\ 0 & 0 \end{bmatrix}\right) \\
&= \left(\mu_1\left(\begin{bmatrix} \alpha_1 \sigma_1 & 0 \\ 0 & \theta_1 \tau_1 \end{bmatrix}\right), \mu_2\left(\begin{bmatrix} \alpha_2 \sigma_2 & \alpha_2 \tau_2 \\ 0 & 0 \end{bmatrix}\right)\right) \\
&= \left(\begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \tau_1 \end{bmatrix}, \begin{bmatrix} 0 & -\alpha_2 \sigma_2 \\ 0 & 0 \end{bmatrix}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
 & (\mu_1 \oplus \mu_2)(U_1, U_2)(\mu_1 \oplus \mu_2)(V_1, V_2) + (\mu_1 \oplus \mu_2)(U_1, U_2)(V_1, V_2) + (U_1, U_2)(\mu_1 \oplus \mu_2)(V_1, V_2) \\
 &= (\mu_1 \oplus \mu_2) \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix} \right) (\mu_1 \oplus \mu_2) \left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix}, \begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix} \right) \\
 &\quad + (\mu_1 \oplus \mu_2) \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix}, \begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix} \right) \\
 &\quad + \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix} \right) (\mu_1 \oplus \mu_2) \left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix}, \begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix} \right) \\
 &= \left(\mu_1 \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right), \mu_2 \left(\begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix} \right) \right) \left(\mu_1 \left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix} \right), \mu_2 \left(\begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix} \right) \right) \\
 &\quad + \left(\mu_1 \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix} \right), \mu_2 \left(\begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix} \right) \right) \left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix}, \begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix} \right) \\
 &\quad + \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix} \right) \left(\mu_1 \left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix} \right), \mu_2 \left(\begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix} \right) \right) \\
 &= \left(\begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \end{bmatrix}, \begin{bmatrix} 0 & -\alpha_2 \\ 0 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} 0 & 0 \\ 0 & -\tau_1 \end{bmatrix}, \begin{bmatrix} 0 & -\sigma_2 \\ 0 & 0 \end{bmatrix} \right) + \left(\begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \end{bmatrix}, \begin{bmatrix} 0 & -\alpha_2 \\ 0 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \tau_1 \end{bmatrix}, \begin{bmatrix} \sigma_2 & \tau_2 \\ 0 & 0 \end{bmatrix} \right) \\
 &\quad + \left(\begin{bmatrix} \alpha_1 & 0 \\ 0 & \theta_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} 0 & 0 \\ 0 & -\tau_1 \end{bmatrix}, \begin{bmatrix} 0 & -\sigma_2 \\ 0 & 0 \end{bmatrix} \right) \\
 &= \left(\begin{bmatrix} 0 & 0 \\ 0 & \theta_1 \tau_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) + \left(\begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \tau_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) + \left(\begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \tau_1 \end{bmatrix}, \begin{bmatrix} 0 & -\alpha_2 \sigma_2 \\ 0 & 0 \end{bmatrix} \right) \\
 &= \left(\begin{bmatrix} 0 & 0 \\ 0 & -\theta_1 \tau_1 \end{bmatrix}, \begin{bmatrix} 0 & -\alpha_2 \sigma_2 \\ 0 & 0 \end{bmatrix} \right).
 \end{aligned}$$

Hence, $\mu_1 \oplus \mu_2$ is a homoderivation on $M_1 \oplus M_2$.

A homoderivation on a ring M reduces to a derivation whenever $\mu(\alpha)\mu(\theta) = 0$, for every $\alpha, \theta \in M$, that is, the multiplicative term in the definition of a homoderivation vanishes. This condition holds, for example, when μ is the zero mapping, when one of the images vanishes, or when both images are nonzero. However, their product is zero, as may occur in the presence of zero divisors or orthogonal homomorphisms. We now present an illustrative example.

Example 4. Let $M = \left\{ \begin{bmatrix} \alpha & \theta \\ 0 & \delta \end{bmatrix} \mid \alpha, \theta, \delta \in \mathbb{Z} \right\}$. We define a mapping $\mu: M \rightarrow M$ by

$$\mu \left(\begin{bmatrix} \alpha & \theta \\ 0 & \delta \end{bmatrix} \right) = \begin{bmatrix} 0 & -\theta \\ 0 & 0 \end{bmatrix}, \text{ for every } \begin{bmatrix} \alpha & \theta \\ 0 & \delta \end{bmatrix} \in M.$$

We will show that μ is a homoderivation on M .

$$\text{Let } U = \begin{bmatrix} \alpha_1 & \theta_1 \\ 0 & \delta_1 \end{bmatrix}, V = \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & \delta_2 \end{bmatrix} \in M.$$

We have:

$$\begin{aligned}
 \mu(UV) &= \mu \left(\begin{bmatrix} \alpha_1 & \theta_1 \\ 0 & \delta_1 \end{bmatrix} \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & \delta_2 \end{bmatrix} \right) \\
 &= \mu \left(\begin{bmatrix} \alpha_1 \alpha_2 & \alpha_1 \theta_2 + \theta_1 \delta_2 \\ 0 & \delta_1 \delta_2 \end{bmatrix} \right) \\
 &= \begin{bmatrix} 0 & -(\alpha_1 \theta_2 + \theta_1 \delta_2) \\ 0 & 0 \end{bmatrix}.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \mu(U)\mu(V) + \mu(U)V + U\mu(V) \\
 &= \mu \left(\begin{bmatrix} \alpha_1 & \theta_1 \\ 0 & \delta_1 \end{bmatrix} \right) \mu \left(\begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & \delta_2 \end{bmatrix} \right) + \mu \left(\begin{bmatrix} \alpha_1 & \theta_1 \\ 0 & \delta_1 \end{bmatrix} \right) \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & \delta_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \theta_1 \\ 0 & \delta_1 \end{bmatrix} \mu \left(\begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & \delta_2 \end{bmatrix} \right) \\
 &= \begin{bmatrix} 0 & -\theta_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\theta_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\theta_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_2 & \theta_2 \\ 0 & \delta_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \theta_1 \\ 0 & \delta_1 \end{bmatrix} \begin{bmatrix} 0 & -\theta_2 \\ 0 & 0 \end{bmatrix} \quad [33]
 \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\theta_1 \delta_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_1 \theta_2 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -(\alpha_1 \theta_2 + \theta_1 \delta_2) \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

This shows that $\mu(UV) = \mu(U)\mu(V) + \mu(U)V + U\mu(V)$, for every $U, V \in M$. It is proven that μ is a homoderivation mapping on ring M .

In **Example 5**, we obtain $\mu(U)\mu(V) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, by the previous result, the homoderivation μ is also a derivation on the ring M whenever $\mu(U)\mu(V) = 0$.

In the next result, we establish the relationship between homoderivations and inner derivations.

Theorem 3. Let a ring R and $\alpha \in Z(R) = \{x \in R \mid x\gamma = \gamma x, \forall \gamma \in R\}$. Inner derivation $\mu_\alpha: R \rightarrow R$, where:

$$\mu_\alpha: \gamma \mapsto [\gamma, \alpha] = \gamma\alpha - \alpha\gamma$$

is a homoderivation on ring R .

Proof. We will show that μ_α is a homoderivation on ring R . First, we will show that μ_α is an additive mapping on ring R . Given any $\gamma, \pi \in R$. We have:

$$\begin{aligned}
\mu_\alpha(\gamma + \pi) &= [\gamma + \pi, \alpha] \\
&= (\gamma + \pi)\alpha - \alpha(\gamma + \pi) \\
&= \gamma\alpha + \pi\alpha - \alpha\gamma - \alpha\pi \\
&= \gamma\alpha - \alpha\gamma + \pi\alpha - \pi\alpha \\
&= [\gamma, \alpha] + [\pi, \alpha] \\
&= \mu_\alpha(\gamma) + \mu_\alpha(\pi)
\end{aligned}$$

Now, $\mu_\alpha(\gamma\pi) = [\gamma\pi, \alpha] = \gamma\pi\alpha - \alpha\gamma\pi$.

On the other hand,

$$\begin{aligned}
\mu_\alpha(\gamma)\mu_\alpha(\pi) + \mu_\alpha(\gamma)\pi + \gamma\mu_\alpha(\pi) &= [\gamma, \alpha][\pi, \alpha] + [\gamma, \alpha]\pi + \gamma[\pi, \alpha] \\
&= (\gamma\alpha - \alpha\gamma)(\pi\alpha - \alpha\pi) + (\gamma\alpha - \alpha\gamma)\pi + \gamma(\pi\alpha - \alpha\pi) \\
&= (\gamma\alpha - \gamma\alpha)(\pi\alpha - \alpha\pi) + (\gamma\alpha - \gamma\alpha)\pi + \gamma(\pi\alpha - \alpha\pi) \\
&= \gamma\pi\alpha - \gamma\alpha\pi \\
&= \gamma\pi\alpha - \alpha\gamma\pi \\
&= \mu_\alpha(\gamma\pi).
\end{aligned}$$

Hence, μ_α is a homoderivation on ring R . ■

To illustrate **Theorem 3**, we present the following example.

Example 5. Given ring $M_2(\mathbb{Z}) = \left\{ \begin{bmatrix} \varepsilon & \tau \\ \delta & \pi \end{bmatrix} \mid \varepsilon, \tau, \delta, \pi \in \mathbb{Z} \right\}$ and choose $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \in M$. We define $\mu_A: M_2(\mathbb{Z}) \rightarrow M_2(\mathbb{Z})$, by:

$$\mu_A: X \mapsto [X, A] = XA - AX, \text{ for every } X \in M_2(\mathbb{Z}).$$

We will show that μ_A is a homoderivation on ring $M_2(\mathbb{Z})$.

Given any $X = \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix}, Y = \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} \in M_2(\mathbb{Z})$.

We have:

$$\begin{aligned}
\mu_A(XY) &= \mu_A \left(\begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix} \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} \right) \\
&= \mu_A \left(\begin{bmatrix} \varepsilon_1 \varepsilon_2 + \tau_1 \delta_2 & \varepsilon_1 \tau_2 + \tau_1 \pi_2 \\ \delta_1 \varepsilon_2 + \pi_1 \delta_2 & \delta_1 \tau_2 + \pi_1 \pi_2 \end{bmatrix} \right)
\end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} \varepsilon_1 \varepsilon_2 + \tau_1 \delta_2 & \varepsilon_1 \tau_2 + \tau_1 \pi_2 \\ \delta_1 \varepsilon_2 + \pi_1 \delta_2 & \delta_1 \tau_2 + \pi_1 \pi_2 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\
&= \begin{bmatrix} \varepsilon_1 \varepsilon_2 + \tau_1 \delta_2 & \varepsilon_1 \tau_2 + \tau_1 \pi_2 \\ \delta_1 \varepsilon_2 + \pi_1 \delta_2 & \delta_1 \tau_2 + \pi_1 \pi_2 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \varepsilon_1 \varepsilon_2 + \tau_1 \delta_2 & \varepsilon_1 \tau_2 + \tau_1 \pi_2 \\ \delta_1 \varepsilon_2 + \pi_1 \delta_2 & \delta_1 \tau_2 + \pi_1 \pi_2 \end{bmatrix} \\
&= \begin{bmatrix} \varepsilon_1 \varepsilon_2 \lambda + \tau_1 \delta_2 \lambda & \varepsilon_1 \tau_2 \lambda + \tau_1 \pi_2 \lambda \\ \delta_1 \varepsilon_2 \lambda + \pi_1 \delta_2 \lambda & \delta_1 \tau_2 + \pi_1 \pi_2 \lambda \end{bmatrix} - \begin{bmatrix} \lambda \varepsilon_1 \varepsilon_2 + \lambda \tau_1 \delta_2 & \lambda \varepsilon_1 \tau_2 + \lambda \tau_1 \pi_2 \\ \lambda \delta_1 \varepsilon_2 + \lambda \pi_1 \delta_2 & \lambda \delta_1 \tau_2 + \lambda \pi_1 \pi_2 \end{bmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mu_A(X)\mu_A(Y) + \mu_A(X)Y + X\mu_A(Y) \\
&= \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix} \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix} \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} \\
&= \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix} \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix} \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} \\
&= \begin{bmatrix} \varepsilon_1 \varepsilon_2 + \tau_1 \delta_2 & \varepsilon_1 \tau_2 + \tau_1 \pi_2 \\ \delta_1 \varepsilon_2 + \pi_1 \delta_2 & \delta_1 \tau_2 + \pi_1 \pi_2 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \varepsilon_1 \varepsilon_2 + \tau_1 \delta_2 & \varepsilon_1 \tau_2 + \tau_1 \pi_2 \\ \delta_1 \varepsilon_2 + \pi_1 \delta_2 & \delta_1 \tau_2 + \pi_1 \pi_2 \end{bmatrix} \\
&= \begin{bmatrix} \varepsilon_1 \varepsilon_2 \lambda + \tau_1 \delta_2 \lambda & \varepsilon_1 \tau_2 \lambda + \tau_1 \pi_2 \lambda \\ \delta_1 \varepsilon_2 \lambda + \pi_1 \delta_2 \lambda & \delta_1 \tau_2 + \pi_1 \pi_2 \lambda \end{bmatrix} - \begin{bmatrix} \lambda \varepsilon_1 \varepsilon_2 + \lambda \tau_1 \delta_2 & \lambda \varepsilon_1 \tau_2 + \lambda \tau_1 \pi_2 \\ \lambda \delta_1 \varepsilon_2 + \lambda \pi_1 \delta_2 & \lambda \delta_1 \tau_2 + \lambda \pi_1 \pi_2 \end{bmatrix}.
\end{aligned}$$

We have $\mu_A(XY) = \mu_A(X)\mu_A(Y) + \mu_A(X)Y + X\mu_A(Y)$ and hence, μ_A is a homoderivation on the ring $M_2(\mathbb{Z})$.

In the following theorem, we describe the structure of the set of constants of a homoderivation on a ring R .

Theorem 4. Let R be a ring and μ be a homoderivation on R . If $C_\mu = \{\alpha \in \mathbb{Z} \mid \mu(\alpha) = 0\}$ set of all constant elements of μ , then C_μ is a $\mathbb{Z}$ -module.

Proof. Since every Abelian group is a $\mathbb{Z}$ -module, it suffices to show that C_d is an Abelian group in order to conclude that C_d is a $\mathbb{Z}$ -module.

1. Given $\alpha, \theta \in C_\mu$. We have:

$$\begin{aligned}
\mu(\alpha + \theta) &= \mu(\alpha) + \mu(\theta) \\
&= 0 + 0 \\
&= 0.
\end{aligned}$$

2. Given $\alpha, \theta, \gamma \in C_\mu$. We have:

$$\begin{aligned}
\mu((\alpha + \theta) + \gamma) &= \mu(\alpha + \theta) + \mu(\gamma) \\
&= \mu(\alpha) + \mu(\theta) + \mu(\gamma) \\
&= \mu(\alpha) + \mu(\theta + \gamma) \\
&= \mu(\alpha + (\theta + \gamma)).
\end{aligned}$$

3. There exists an identity element, that is $0 \in C_\mu$, such that $\mu(0) + \mu(\alpha) = \mu(0 + \alpha) = \mu(\alpha)$.

4. For every $\alpha \in C_\mu$, there exists $-\alpha \in C_\mu$, such that $\mu(\alpha) + \mu(-\alpha) = \mu(\alpha + (-\alpha)) = \mu(0) = 0$.

5. Given any $\alpha, \theta \in C_\mu$. We have $\mu(\alpha + \theta) = \mu(\alpha) + \mu(\theta) = 0 = \mu(\theta) + \mu(\alpha) = \mu(\theta + \alpha)$.

Thus, it is proven that $\langle C_\mu, + \rangle$ is an Abelian group. Consequently, $\langle C_\mu, + \rangle$ is a $\mathbb{Z}$ -module. ■

4. CONCLUSION

Based on the preceding discussion, a homoderivation on a ring combines the Leibniz rule with the multiplicative property of a ring homomorphism. We show that the sum of two homoderivations is again a homoderivation, provided the maps h_1 and h_2 are orthogonal (mutually annihilating). Moreover, the direct

sum of homoderivations on their respective rings is also a homoderivation. In contrast, scalar multiplication of a homoderivation does not, in general, yield another homoderivation. Additionally, the set of constants of a homoderivation forms a $\mathbb{Z}$ -module.

Author Contributions

Desi Widiarti: Formal Analysis, Writing-Original Draft. Fitriani: Conceptualization, Methodology, Validation, Writing-Review and Editing. Ahmad Faisol: Formal Analysis, Validation. All authors considered the findings and provided input for the final manuscript

Funding Statement

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Acknowledgment

This research was funded by the Directorate of Research, Technology, and Community Service, under the Directorate General of Higher Education, Research, and Technology, Ministry of Education, Culture, Research, and Technology, in accordance with Research Contract No. 076/C3/DT.05.00/PL/2025.

Declarations

The authors declare no competing interests.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools (e.g., ChatGPT) were used solely for language refinement, including grammar, spelling, and clarity. The scientific content, analysis, interpretation, and conclusions were developed entirely by the authors. All final text was reviewed and approved by the authors.

REFERENCES

- [1] B. Felzenszwalb and C. Lanski, "ON THE CENTRALIZERS OF IDEALS AND NIL DERIVATIONS," *J. Algebr.*, vol. 83, pp. 520–530, 1983. doi: [https://doi.org/10.1016/0021-8693\(83\)90237-5](https://doi.org/10.1016/0021-8693(83)90237-5)
- [2] B. Dhara and R. K. Sharma, "ON ADDITIVE MAPPINGS IN RINGS WITH IDENTITY ELEMENT," *Int. Math. Forum*, vol. 4, no. 15, pp. 727–732, 2009.
- [3] L. O. Chung and Y. Kobayasi, "NIL DERIVATIONS AND CHAIN CONDITIONS IN PRIME RINGS," in *Proceedings of the American Mathematical Society*, 1985, pp. 201–205. doi: <https://doi.org/10.1090/S0002-9939-1985-0784162-4>
- [4] I. Ernanto, "SIFAT-SIFAT RING FAKTOR YANG DILENGKAPI DERIVASI," *J. Fundam. Math. Appl.*, vol. 1, no. 1, p. 12, Jun. 2018, doi: <https://doi.org/10.14710/jfma.v1i1.3>.
- [5] S. Ali, N. N. Rafiquee, and V. Varshney, "CERTAIN TYPES OF DERIVATIONS IN RINGS: A SURVEY," *J. Indones. Math. Soc.*, vol. 30, no. 02, pp. 256–306, 2024. doi: <https://doi.org/10.22342/jims.30.2.1623.256-306>
- [6] M. Ashraf, S. Ali, and C. Haetinger, "ON DERIVATIONS IN RINGS AND THEIR APPLICATIONS," *The Aligarh Bull. of Math.*, vol. 25, no. 2, pp. 79–107, 2006.
- [7] A. Ber, K. Kudaybergenov, and F. Sukochev, "NOTES ON DERIVATIONS OF MURRAY-VON NEUMANN ALGEBRAS," *J. Funct. Anal.*, vol. 279, no. 1, pp. 1–26, 2020. doi: <https://doi.org/10.1016/j.jfa.2020.108589>
- [8] L. O. Chung, "NIL DERIVATIONS," *J. Algebr.*, vol. 95, pp. 20–30, 1985. doi: [https://doi.org/10.1016/0021-8693\(85\)90089-4](https://doi.org/10.1016/0021-8693(85)90089-4)
- [9] A. Al-Khalaf, O. D. Artemavych, and I. Taha, "RINGS WITH SIMPLE LIE RINGS OF LIE AND JORDAN DERIVATIONS," *J. Algebr. its Appl.*, vol. 17, no. 4, 2018. doi: <https://doi.org/10.1142/S0219498818500780>
- [10] A. Thomas, N. P. Puspita, and Fitriani, "DERIVATION ON SEVERAL RINGS," *Barekeng J. Math. App.*, vol. 18, no. 3, pp. 1729–1738, 2024. doi: <https://doi.org/10.30598/barekengvol18iss3pp1729-1738>
- [11] J. Huang, K. Kudaybergenov, and F. Sukochev, "RING DERIVATIONS OF MURRAY-VON NEUMANN ALGEBRAS," *Linear Algebra Appl.*, vol. 672, no. 1, pp. 28–52, 2023. doi: <https://doi.org/10.1016/j.laa.2023.04.011>
- [12] A. Al-Khalaf, O. D. Artemavych, and I. Taha, "DERIVATIONS IN DIFFERENTIALLY PRIME RINGS," *J. Algebr. its Appl.*, vol. 17, no. 7, p. 1850129, 2018. doi: <https://doi.org/10.1142/S0219498818501293>
- [13] M. R. Helmi, H. Marubayashi, and A. Ueda, "DIFFERENTIAL POLYNOMIAL RINGS WHICH ARE GENERALIZED ASANO PRIME RINGS," *Indian J. Pure Appl. Math.*, vol. 44, no. 5, pp. 673–681, 2013. doi: <https://doi.org/10.1007/s13226-013-0035-6>
- [14] Ansari A. Z., F. Shujat, and A. Fallatah, "STRUCTURE OF (Σ, P) -N-DERIVATIONS ON RINGS," *Contemp. Math.*, vol. 5, no. 4, pp. 5666–5678, 2025.

- [15] U. Sayın and F. Kuzucuoğlu, "JORDAN DERIVATIONS OF SPECIAL SUBRINGS OF MATRIX RINGS," *Algebr. Colloq.*, vol. 26, no. 1, pp. 83–92, 2019, doi: <https://doi.org/10.1142/S1005386719000087>.
- [16] Y. Watase, "DERIVATION OF COMMUTATIVE RINGS AND THE LEIBNIZ FORMULA FOR POWER OF DERIVATION," *Formuized Math.*, vol. 29, no. 1, pp. 1–8, 2021. doi: <https://doi.org/10.2478/forma-2021-0001>
- [17] Fitriani, I. E. Wijayanti, A. Faisol, and S. Ali, "ON F-DERIVATIONS ON POLYNOMIAL MODULES," *J. Algebr. its Appl.*, vol. 24, no. 6, pp. 1–14, 2024, doi: <https://doi.org/10.1142/S0219498825501555>.
- [18] Fitriani, I. E. Wijayanti, A. Faisol, and S. Ali, "COMMUTING AND CENTRALIZING MAPS ON MODULES," vol. 10, no. 3, pp. 691–697, 2025. doi: <https://doi.org/10.26554/sti.2025.10.3.690-697>
- [19] D. E. Sitompul, F. Fitriani, S. L. Chasanah, and A. Faisol, "JORDAN DERIVATION ON THE POLYNOMIAL RING $R[X]$," *Integr. J. Integr. Math. Comput. Sci.*, vol. 2, no. 2, pp. 41–47, 2024. doi: <https://doi.org/10.26554/integracijms.20252229>
- [20] D. L. Mursyidah, F. Fitriani, B. H. S. Utami, and A. Faisol, "NIL DERIVATION AND Δ -IDEAL ON POLYNOMIAL RING," *Barekeng J. Math. App.*, vol. accepted, 2025. doi: <https://doi.org/10.30598/barekengvol20iss1pp0325-0334>
- [21] A. Faisol and F. Fitriani, "A STUDY OF DERIVATIONS AND LINEAR MAPPINGS ON SKEW GENERALIZED POWER SERIES MODULES," *Barekeng J. Math. App.*, vol. 19, no. 4, pp. 3047–3058, 2025. doi: <https://doi.org/10.30598/barekengvol19iss4pp3047-3058>
- [22] R. Waluyo, A. Faisol, and F. Fitriani, " (σ, τ) -DERIVASI PADA RING GRUP," *Euler J. Ilm. Mat. Sains dan Teknol.*, vol. 13, no. 2, pp. 142–146, 2025. doi: <https://doi.org/10.37905/euler.v13i2.31564>
- [23] A. Engin and N. Aydin, "HOMODERIVATIONS IN PRIME RINGS," *J. New Theory*, vol. 43, pp. 23–34, 2023. doi: <https://doi.org/10.53570/jnt.1258402>
- [24] S. Belkadi and I. Taoufiq, "ON HIGHER ORDER HOMODERIVATIONS IN SEMI-PRIME RINGS," *Jordan J. Math. Stat.*, vol. 17, no. 1, pp. 99–112, 2024. doi: <https://doi.org/10.1515/gmj-2023-2065>
- [25] S. Belkadi, S. Ali, and L. Taoufiq, "ON NILPOTENT HOMODERIVATIONS IN PRIME RINGS," *Georg. Math. J.*, vol. 1, pp. 17–24, 2024. doi: <https://doi.org/10.1515/gmj-2023-2065>
- [26] E. F. Alharfie and N. M. Muthana, "THE COMMUNITATIVITY OF PRIME RINGS WITH HOMODERIVATIONS," *Int. J. Adv. Appl. Sci.*, vol. 5, no. 5, pp. 79–81, 2018. doi: <https://doi.org/10.21833/ijaas.2018.05.010>
- [27] E. F. Alharfie and N. M. Muthana, "ON HOMODERIVATIONS AND COMMUNITATIVITY OF RINGS," *Bull. Int. Math. Virtual Inst.*, vol. 5, pp. 301–304, 2019.
- [28] S. Türkmen, "RELATIONSHIP BETWEEN A HOMODERIVATION AND A SEMI-DERIVATION," *J. New Theory*, vol. 47, pp. 28–38, 2024. doi: <https://doi.org/10.53570/jnt.1467690>
- [29] A. Melaibari, N. Muthana, and A. Al-Kenani, "HOMODERIVATIONS ON RINGS," *Gen. Math. Notes*, vol. 35, no. 1, pp. 1–8, 2016. doi: <https://doi.org/10.56947/gjom.v4i2.67>
- [30] A. Y. Hummdi, Z. Bedir, E. K. Söğütçü, Ö. Gölbaşı, and N. U. Rehman, "LIE IDEALS AND HOMODERIVATIONS IN SEMIPRIME RINGS," *Mathematics*, vol. 13, no. 4, p. 548, 2025, doi: <https://doi.org/10.3390/math13040548>.
- [31] M. S. T. El-sayiad and M. Almulhem, "ON CENTRALLY-EXTENDED N-HOMODERIVATIONS ON RINGS," vol. 10, no. December 2024, pp. 7191–7205, 2025. doi: <https://doi.org/10.3934/math.2025328>
- [32] S. Ali, N. N. Rafiquee, and V. Varshney, "Reverse Homoderivations on (Semi)-prime Rings," *J. Indones. Math. Soc.*, vol. 31, no. 3, p. 1867, 2025, doi: <https://doi.org/10.22342/jims.v31i2.1867>.
- [33] P. Manju and K. Sharma, " (σ, τ) -Derivations of Number Rings with Coding Theory Applications," *Palestine Journal of Mathematics*, vol. 13, no. 2, pp. 191–202, 2024.

