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HOMODERIVATIONS ON RINGS: STRUCTURAL PROPERTIES AND CHARACTERIZATIONS

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ABSTRACT

Research in ring theory has produced major advances, particularly in the study of derivations. A related notion is that of a homoderivation, which combines aspects of derivations and ring homomorphisms. An additive map $h: R \rightarrow R$ *is called a homoderivation if* $h(ab) = h(a)h(b) + h(a)b + ah(b)$, *for every* $a, b \in R$. *This study provides a comprehensive analysis of the fundamental structural properties of homoderivation mappings. The main results show that the class of homoderivations is closed under direct sums on component rings. In addition, the sum of two homoderations is again a homoderivation, provided the maps are orthogonal. However, scalar multiplication of a homoderivation generally does not yield another homoderivation, highlighting a key structural distinction from classical derivations.*

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1. INTRODUCTION

The concept of derivation plays a fundamental role in ring theory, and research in this area continues to develop rapidly. A derivation on a ring R, as stated by Felzenszwalb and Lanski [1], is an additive mapping $d: R \to R$ that satisfies the Leibniz rule. Further studies on additive mappings in rings with identity, such as the work of Dhara and Sharma [2], have contributed to a deeper understanding of ring structures and their fundamental properties.

Numerous researchers have investigated derivations from various perspectives. For instance, Chung and Kobayashi [3] examined nil derivations and their influence on ideals in prime rings, while Ernanto [4] studied derivations in factor rings. Comprehensive treatments of derivations can be found in Ali et al. [5] and Ashraf et al. [6]. Other works include Ber et al. [7], as well as Chung [8] and Al-Khalaf et al. [9], who focused on derivations in prime rings. Additional contributions by Thomas et al. [10] further highlight the diversity of derivation theory, and Huang et al. [11], who analyzed derivations on Murray–von Neumann algebras.

Research has also extended to specialized types of derivations. Al-Khalaf et al. [12] studied Lie and Jordan derivations, while Helmi et al. [13] investigated differential polynomial rings. Ansari et al. [14] explored n-derivations with automorphisms, and Sayin and Feride [15] examined Jordan derivations in matrix subrings. Watase [16] contributed to derivations on commutative rings. More recent studies include Fitriani et al. ([17], [18]) on f-derivations, Sitompul et al. [19] on Jordan derivations in polynomial rings, and Mursyidah et al. [20] on nil derivations. Further developments by Faisol and Fitriani [21] and Waluyo et al. [22] extend derivations to skew modules and group rings, respectively.

A natural extension of derivations is the notion of a homoderivation, which combines the Leibniz rule with the multiplicative property of ring homomorphisms. Specifically, an additive mapping $h: R \to R$ is called a homoderivation if $h(ab) = h(a)h(b) + h(a)b + a\,h(b)$, for every $a, b \in R$. Here, the term $h(a)h(b)$ reflects the homomorphic property, while the remaining terms correspond to the Leibniz rule. Homoderivations have been studied in various algebraic settings. Engin and Aydin [23] investigated homoderivations on prime rings, while Belkadi and Taoufiq [24] and Taoufiq et al. [25] examined higher-order and nilpotent homoderivations in semiprime rings. Alharfie and Muthana ([26], [27]) explored their connections with commutativity, and Turkmen [28] analyzed their relationship with semiderivations. Additional contributions include Melaibari et al. [29] on associative rings, Hummdi et al. [30] on Lie ideals, El-Sayiad and Almulhem [31] on generalized homoderivations, and Ali et al. [32] on reverse homoderivations.

Given the central role of derivations and their generalizations in revealing ring structures, the study of homoderivations is both natural and necessary. As an extension of classical derivations, homoderivations provide a framework for structure-preserving transformations in algebra. Their relevance extends beyond pure mathematics to applications such as coding theory, cryptography, and data analysis, as illustrated by Manju and Sharma [33]. Therefore, a systematic investigation of the structural properties of homoderivations is essential for advancing both theoretical understanding and practical applications.

2. RESEARCH METHODS

This study adopts a literature-based and theoretical analysis approach. Relevant references, including peer-reviewed journals, scientific articles, monographs, and other authoritative sources, are systematically reviewed to establish a rigorous theoretical foundation. The methodological steps undertaken in this research are as follows:

1. Examine the algebraic properties of homoderivations on the ring R, focusing on their structural characteristics and operational behavior.
2. Develop and present explicit examples of homoderivations on the ring R to illustrate and validate the theoretical framework.
3. Explore the relationship between homoderivations and classical derivations on the ring R, highlighting similarities, distinctions, and potential generalizations.

This methodological design ensures that the study not only establishes fundamental properties but also contextualizes homoderivations within the broader theory of derivations in ring structures.

3. RESULTS AND DISCUSSION

In this section, we give some properties of homoderivation on a ring. Given two homoderivation maps, μ_1 and μ_2. We investigate whether the sum $\mu_1 + \mu_2$ is a homoderivation on the ring R or not, as defined below:

$$(\mu_1 + \mu_2)(\alpha) = \mu_1(\alpha) + \mu_2(\alpha).$$

Therefore, for the sum of two homoderivations μ_1 and μ_2 to be a homoderivation, it is necessary that they be orthogonal (mutually null), that is, $\mu_1(\alpha)\mu_2(\theta) + \mu_2(\alpha)\mu_1(\theta) = 0$, for every $\alpha, \theta \in R$.This is demonstrated in the following theorem.

Theorem 1. *Let a ring R, a map $\mu: R \to R$ is a homoderivation on ring R. If μ_1 and μ_2 are homoderivations mapping on ring R, where $\mu_1(\alpha)\mu_2(\theta) + \mu_2(\alpha)\mu_1(\theta) = 0$, for every $\alpha, \theta \in R$, then $\mu_1 + \mu_2$ is a homoderivation on ring R.*

Proof. We will show that, $\mu_1 + \mu_2$ is a homoderivation mapping, where $(\mu_1 + \mu_2)(\alpha) = \mu_1(\alpha) + \mu_2(\alpha)$, for every $\alpha \in R$.

1. Given any $\alpha, \theta \in R$.

$$\begin{aligned}(\mu_1 + \mu_2)(\alpha + \theta) &= \mu_1(\alpha + \theta) + \mu_2(\alpha + \theta)\\ &= \mu_1(\alpha) + \mu_1(\theta) + \mu_2(\alpha) + \mu_2(\theta)\\ &= \mu_1(\alpha) + \mu_2(\alpha) + \mu_1(\theta) + \mu_2(\theta)\\ &= (\mu_1 + \mu_2)(\alpha) + (\mu_1 + \mu_2)(\theta).\end{aligned}$$

2. $(\mu_1 + \mu_2)(\alpha\theta) = \mu_1(\alpha\theta) + \mu_2(\alpha\theta)$

$$\begin{aligned}&= \mu_1(\alpha)\mu_1(\theta) + \mu_1(\alpha)\theta + \alpha\mu_1(\theta) + \mu_2(\alpha)\mu_2(\theta) + \mu_2(\alpha)\theta + \alpha\mu_2(\theta)\\ &= \mu_1(\alpha)\mu_1(\theta) + \mu_2(\alpha)\mu_2(\theta) + (\mu_1 + \mu_2)(\alpha)\theta + \alpha(\mu_1 + \mu_2)(\theta)\\ &= (\mu_1 + \mu_2)(\alpha)(\mu_1 + \mu_2)(\theta) - \big(\mu_1(\alpha)\mu_2(\theta) + \mu_2(\alpha)\mu_1(\theta)\big) + (\mu_1 + \mu_2)(\alpha)\theta +\\ &\quad \alpha(\mu_1 + \mu_2)(\theta).\end{aligned}$$

So, $\mu_1 + \mu_2$ is a homoderivation under the condition $\mu_1(\alpha)\mu_2(\theta) + \mu_2(\alpha)\mu_1(\theta) = 0$ which are referred to as mutually annihilating or orthogonal homomorphisms. ■

Example 1. Given a ring $M = \left\{\begin{bmatrix}\alpha & 0\\ 0 & \theta\end{bmatrix} \mid \alpha, \theta \in \mathbb{Z}\right\}$. We define $\mu_1,\ \mu_2: M \to M$, where: $\mu_1\left(\begin{bmatrix}\alpha & 0\\ 0 & \theta\end{bmatrix}\right) = \begin{bmatrix}-\alpha & 0\\ 0 & 0\end{bmatrix}$ and $\mu_2\left(\begin{bmatrix}\alpha & 0\\ 0 & \theta\end{bmatrix}\right) = \begin{bmatrix}0 & 0\\ 0 & -\theta\end{bmatrix}$, for every $\begin{bmatrix}\alpha & 0\\ 0 & \theta\end{bmatrix} \in M$. We can show that μ_1 and μ_2 are homoderivations on M. Now, we can show that $\mu_1 + \mu_2$ is a homoderivation on M.

Let $U = \begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}$, $V = \begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix} \in M$. We have:

$$\begin{aligned}(\mu_1 + \mu_2)(U + V) &= (\mu_1 + \mu_2)\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix} + \begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)\\ &= (\mu_1 + \mu_2)\left(\begin{bmatrix}\alpha_1 + \alpha_2 & 0\\ 0 & \theta_1 + \theta_2\end{bmatrix}\right)\\ &= \mu_1\left(\begin{bmatrix}\alpha_1 + \alpha_2 & 0\\ 0 & \theta_1 + \theta_2\end{bmatrix}\right) + \mu_2\left(\begin{bmatrix}\alpha_1 + \alpha_2 & 0\\ 0 & \theta_1 + \theta_2\end{bmatrix}\right)\\ &= \begin{bmatrix}-(\alpha_1 + \alpha_2) & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & -(\theta_1 + \theta_2)\end{bmatrix}\\ &= \begin{bmatrix}-\alpha_1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}-\alpha_2 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & -\theta_1\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & -\theta_2\end{bmatrix}\\ &= \mu_1\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right) + \mu_1\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right) + \mu_2\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right) + \mu_2\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)\end{aligned}$$

$$= \mu_1\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right)+\mu_2\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right)+\mu_1\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)+\mu_2\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)$$

$$= (\mu_1+\mu_2)\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right)+(\mu_1+\mu_2)\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)$$

$$= (\mu_1+\mu_2)(U)+(\mu_1+\mu_2)(V).$$

Furthermore,

$$\begin{aligned}(\mu_1+\mu_2)(UV) &= (\mu_1+\mu_2)\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)\\ &= (\mu_1+\mu_2)\left(\begin{bmatrix}\alpha_1\alpha_2 & 0\\ 0 & \theta_1\theta_2\end{bmatrix}\right)\\ &= \mu_1\left(\begin{bmatrix}\alpha_1\alpha_2 & 0\\ 0 & \theta_1\theta_2\end{bmatrix}\right)+\mu_2\left(\begin{bmatrix}\alpha_1\alpha_2 & 0\\ 0 & \theta_1\theta_2\end{bmatrix}\right)\\ &= \begin{bmatrix}-\alpha_1\alpha_2 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & -\theta_1\theta_2\end{bmatrix}\\ &= \begin{bmatrix}-\alpha_1\alpha_2 & 0\\ 0 & -\theta_1\theta_2\end{bmatrix}.\end{aligned}$$

On the other hand,

$$(\mu_1+\mu_2)(U)(\mu_1+\mu_2)(V)+(\mu_1+\mu_2)(U)V+U(h_1+h_2)(V)$$

$$\begin{aligned}&= (\mu_1+\mu_2)\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right)(\mu_1+\mu_2)\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)+(\mu_1+\mu_2)\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right)\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\\ &\quad+\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}(\mu_1+\mu_2)\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)\end{aligned}$$

$$\begin{aligned}&= \left(\mu_1\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right)+\mu_2\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right)\right)\left(\mu_1\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)+\mu_2\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)\right)\\ &\quad+\left(\mu_1\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right)+\mu_2\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right)\right)\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\\ &\quad+\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\left(\mu_1\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)+\mu_2\left(\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\right)\right)\end{aligned}$$

$$\begin{aligned}&= \left(\begin{bmatrix}-\alpha_1 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & -\theta_1\end{bmatrix}\right)\left(\begin{bmatrix}-\alpha_2 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & -\theta_2\end{bmatrix}\right)+\left(\begin{bmatrix}-\alpha_1 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & -\theta_1\end{bmatrix}\right)\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}\\ &\quad+\begin{bmatrix}\alpha_1 & 0\\ 0 & b_1\end{bmatrix}\left(\begin{bmatrix}-\alpha_2 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & -b_2\end{bmatrix}\right)\end{aligned}$$

$$= \begin{bmatrix}-\alpha_1 & 0\\ 0 & -\theta_1\end{bmatrix}\begin{bmatrix}-\alpha_2 & 0\\ 0 & -\theta_2\end{bmatrix}+\begin{bmatrix}-\alpha_1 & 0\\ 0 & -\theta_1\end{bmatrix}\begin{bmatrix}\alpha_2 & 0\\ 0 & \theta_2\end{bmatrix}+\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\begin{bmatrix}-\alpha_2 & 0\\ 0 & -\theta_2\end{bmatrix}$$

$$= \begin{bmatrix}\alpha_1\alpha_2 & 0\\ 0 & \theta_1\theta_2\end{bmatrix}+\begin{bmatrix}-\alpha_1\alpha_2 & 0\\ 0 & -\theta_1\theta_2\end{bmatrix}+\begin{bmatrix}-\alpha_1\alpha_2 & 0\\ 0 & -\theta_1\theta_2\end{bmatrix}$$

$$= \begin{bmatrix}-\alpha_1\alpha_2 & 0\\ 0 & -\theta_1\theta_2\end{bmatrix}.$$

Hence, $(\mu_1+\mu_2)(UV)=(\mu_1+\mu_2)(U)(\mu_1+\mu_2)(V)+(\mu_1+\mu_2)(U)V+U(\mu_1+\mu_2)(V)$, for every $U,V \in M$. Thus, $\mu_1+\mu_2$ is a homoderivation on M.

Moreover, scalar multiplication of a homoderivation μ on a ring does not, in general, yield another homoderivation, as illustrated by the following counterexample. For a homoderivation $\mu : R \to R$, we define $k\mu$ and $-\mu$ by $(k\mu)(r)=k\mu(r)$ and $-\mu$: $R \to R$ by $(-\mu)(r)=-\mu(r)$, for every $r \in R$.

Example 2. Let a ring $M_2(\mathbb{Z})$ with a mapping $\mu : M_2(\mathbb{Z}) \to M_2(\mathbb{Z})$, defined as follows: $\mu\left(\begin{bmatrix}\alpha & \theta\\ \tau & \sigma\end{bmatrix}\right)=\begin{bmatrix}0 & 0\\ 0 & -\sigma\end{bmatrix}$, for every $\begin{bmatrix}\alpha & \theta\\ \tau & \sigma\end{bmatrix} \in M_2(\mathbb{Z})$. Choose $K=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$, $U=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}, V=\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix} \in M_2(\mathbb{Z})$, to be used in examining the homoderivation property of the mapping $K\mu$.

It will be examined whether the equality $K\mu(UV)=K\mu(U)K\mu(V)+K\mu(U)V+UK\mu(V)$ holds.

$$\begin{aligned} Kh(UV) &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mu\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mu\left(\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\right) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(-\mu)(U)(-\mu)(V) + (-\mu)(U)V + U(-\mu)(V) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mu\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mu\left(\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right) + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mu\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \\ &\qquad + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mu\left(\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 12 \end{bmatrix}. \end{aligned}$$

Thus, it is obtained that the left-hand side and the right-hand side are not equal. Therefore, it is proven that $K\mu$ is not a homoderivation mapping on the ring $M_2(\mathbb{Z})$.

Next, if μ_1 and μ_2 are given as homoderivation maps, their direct sum is defined by:

$$(\mu_1 \oplus \mu_2)(\alpha_1, \alpha_2) = (\mu_1(\alpha_1), \mu_2(\alpha_2)).$$

It has been investigated that the direct sum of homoderivation maps $\mu_1, \mu_2, \dots, \mu_n$ is also a homoderivation map on the rings $R_1 \oplus R_2 \oplus \dots \oplus R_n$, as demonstrated in the following theorem.

Theorem 2. *Let $\mu_1, \mu_2, \dots, \mu_n$ be homoderivations on rings $R_1, R_2, \dots, R_n$, respectively. Then $\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n$ is a homoderivation on the direct sum $R_1 \oplus R_2 \oplus \dots \oplus R_n$.*

Proof. We will show that, $\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n$ on $R_1 \oplus R_2 \oplus \dots \oplus R_n$.

1. We will show that $\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n$ is additive on ring $R_1 \oplus R_2 \oplus \dots \oplus R_n$. Given any $(\alpha_1, \alpha_2, \dots, \alpha_n), (\theta_1, \theta_2, \dots, \theta_n) \in R_1 \oplus R_2 \oplus \dots \oplus R_n$.

$$\begin{aligned} &\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n((\alpha_1, \alpha_2, \dots, \alpha_n) + (\theta_1, \theta_2, \dots, \theta_n)) \\ &= (\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\alpha_1 + \theta_1,\ \alpha_2 + \theta_2, \dots, \alpha_n + \theta_n) \\ &= \left(\mu_1(\alpha_1 + \theta_1), \mu_2(\alpha_2 + \theta_2), \dots, \mu_n(\alpha_n + \theta_n)\right) \\ &= \left(\mu_1(\alpha_1) + \mu_1(\theta_1), \mu_2(\alpha_2) + \mu_2(\theta_2), \dots, \mu_n(\alpha_n) + h_n(\theta_n)\right) \\ &= \left(\mu_1(\alpha_1), \mu_2(\alpha_2), \dots, \mu_n(\alpha_n)\right) + \left(\mu_1(\theta_1), \mu_2(\theta_2), \dots, \mu_n(\theta_n)\right) \\ &= (\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\alpha_1, \alpha_2, \dots, \alpha_n) + (\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\theta_1, \theta_2, \dots, \theta_n). \end{aligned}$$

Thus, $\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n$ is additive.

2. $(\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_n)(\alpha_1\theta_1,\ \alpha_2\theta_2, \dots, \alpha_n\theta_n)$

$$\begin{aligned} &= \left(\mu_1(\alpha_1\theta_1), \mu_2(\alpha_2\theta_2), \dots, \mu_n(\alpha_n\theta_n)\right) \\ &= \left(\mu_1(\alpha_1)\mu_1(\theta_1) + \mu_1(\alpha_1)\theta_1 + \alpha_1\mu_1(\theta_1), \mu_2(\alpha_2)\mu_2(\theta_2) + \mu_2(\alpha_2)\theta_2 + \alpha_2\mu_2(\theta_2), \dots, \mu_n(\alpha_n)\mu_n(\theta_n)\right. \\ &\qquad \left. + \mu_n(\alpha_n)\theta_n + \alpha_n\mu_n(\theta_n)\right) \\ &= \left(\mu_1(\alpha_1)\mu_1(\theta_1), \mu_2(\alpha_2)\mu_2(\theta_2), \dots, \mu_n(\alpha_n)\mu_n(\theta_n)\right) + \left(\mu_1(\alpha_1)\theta_1, \mu_2(\alpha_2)\theta_2, \dots, \mu_n(\alpha_n)\theta_n\right) \\ &\qquad + \left(\alpha_1\mu_1(\theta_1), \alpha_2\mu_2(\theta_2), \dots, \alpha_n\mu_n(\theta_n)\right) \end{aligned}$$

$$= \big(\mu_1(\alpha_1), \mu_2(\alpha_2), \ldots, \mu_n(\alpha_n)\big)\big(\mu_1(\theta_1), \mu_2(\theta_2), \ldots, \mu_n(\theta_n)\big) + \big(\mu_1(\alpha_1), \mu_2(\alpha_2), \ldots, \mu_n(\alpha_n)\big)(\theta_1, \theta_2, \ldots, \theta_n) + (\alpha_1, \alpha_2, \ldots, \alpha_n)\big(\mu_1(\theta_1), \mu_2(\theta_2), \ldots, \mu_n(\theta_n)\big)$$

$$= (\mu_1 \oplus \mu_2 \oplus \ldots \oplus \mu_n)(\alpha_1, \alpha_2, \ldots, \alpha_n)(\mu_1 \oplus \mu_2 \oplus \ldots \oplus \mu_n)(\theta_1, \theta_2, \ldots, \theta_n) + (\mu_1 \oplus \mu_2 \oplus \ldots \oplus \mu_n)(\alpha_1, \alpha_2, \ldots, \alpha_n)(\theta_1, \theta_2, \ldots, \theta_n) + (\alpha_1, \alpha_2, \ldots, \alpha_n)(\mu_1 \oplus \mu_2 \oplus \ldots \oplus \mu_n)(\theta_1, \theta_2, \ldots, \theta_n).$$

Hence, $\mu_1 \oplus \mu_2 \oplus \ldots \oplus \mu_n$ is a homoderivation on ring $R_1 \oplus R_2 \oplus \ldots \oplus R_n$. ∎

We now provide an example demonstrating Theorem 2.

Example 3. Let $M_1 = \left\{\begin{bmatrix}\alpha & 0\\ 0 & \theta\end{bmatrix} \mid \alpha, \theta \in \mathbb{Z}\right\}$ and $M_2 = \left\{\begin{bmatrix}\alpha & \theta\\ 0 & 0\end{bmatrix} \mid \alpha, \theta \in \mathbb{Z}\right\}$. We define $\mu_1: M_1 \to M_1$ and $\mu_2: M_2 \to M_2$ by $\mu_1\left(\begin{bmatrix}\alpha & 0\\ 0 & \theta\end{bmatrix}\right) = \begin{bmatrix}0 & 0\\ 0 & -\theta\end{bmatrix}$, $\mu_2\left(\begin{bmatrix}\alpha & \theta\\ 0 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & -\alpha\\ 0 & 0\end{bmatrix}$.

We will show that $\mu_1 \oplus \mu_2$ is a homoderivation on $M_1 \oplus M_2$.

Given any $U_1 = \begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}, V_1 = \begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix} \in M_1$ and $U_2 = \begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}, V_2 = \begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix} \in M_2$.

We will show that $\mu_1 \oplus \mu_2$ is an additive mapping on ring $M_1 \oplus M_2$.

$$(\mu_1 \oplus \mu_2)(U_1 + V_1, U_2 + V_2)$$

$$= (\mu_1 \oplus \mu_2)\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix} + \begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix}, \begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix} + \begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)$$

$$= (\mu_1 \oplus \mu_2)\left(\begin{bmatrix}\alpha_1 + \sigma_1 & 0\\ 0 & \theta_1 + \tau_1\end{bmatrix}, \begin{bmatrix}\alpha_2 + \sigma_2 & \theta_2 + \tau_2\\ 0 & 0\end{bmatrix}\right)$$

$$= \left(\mu_1\left(\begin{bmatrix}\alpha_1 + \sigma_1 & 0\\ 0 & \theta_1 + \tau_1\end{bmatrix}\right), \mu_2\left(\begin{bmatrix}\alpha_2 + \sigma_2 & \theta_2 + \tau_2\\ 0 & 0\end{bmatrix}\right)\right)$$

$$= \left(\begin{bmatrix}0 & 0\\ 0 & -(\theta_1 + \tau_1)\end{bmatrix}, \begin{bmatrix}0 & -(\alpha_2 + \sigma_2)\\ 0 & 0\end{bmatrix}\right)$$

$$= \left(\begin{bmatrix}0 & 0\\ 0 & -\theta_1\end{bmatrix}, \begin{bmatrix}0 & -\alpha_2\\ 0 & 0\end{bmatrix}\right) + \left(\begin{bmatrix}0 & 0\\ 0 & -\tau_1\end{bmatrix}, \begin{bmatrix}0 & -\sigma_2\\ 0 & 0\end{bmatrix}\right)$$

$$= \left(\mu_1\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right), \mu_2\left(\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}\right)\right) + \left(\mu_1\left(\begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix}\right), \mu_2\left(\begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)\right)$$

$$= (\mu_1 \oplus \mu_2)\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}, \begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}\right) + (\mu_1 \oplus \mu_2)\left(\begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix}, \begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)$$

$$= (\mu_1 \oplus \mu_2)(U_1, U_2) + (\mu_1 \oplus \mu_2)(V_1, V_2).$$

Now,

$$(\mu_1 \oplus \mu_2)(U_1V_1, U_2V_2) = (\mu_1 \oplus \mu_2)\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix}, \begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}\begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)$$

$$= (\mu_1 \oplus \mu_2)\left(\begin{bmatrix}\alpha_1\sigma_1 & 0\\ 0 & \theta_1\tau_1\end{bmatrix}, \begin{bmatrix}\alpha_2\sigma_2 & \alpha_2\tau_2\\ 0 & 0\end{bmatrix}\right)$$

$$= \left(\mu_1\left(\begin{bmatrix}\alpha_1\sigma_1 & 0\\ 0 & \theta_1\tau_1\end{bmatrix}\right), \mu_2\left(\begin{bmatrix}\alpha_2\sigma_2 & \alpha_2\tau_2\\ 0 & 0\end{bmatrix}\right)\right)$$

$$= \left(\begin{bmatrix}0 & 0\\ 0 & -\theta_1\tau_1\end{bmatrix}, \begin{bmatrix}0 & -\alpha_2\sigma_2\\ 0 & 0\end{bmatrix}\right).$$

On the other hand,

$$(\mu_1 \oplus \mu_2)(U_1,U_2)(\mu_1 \oplus \mu_2)(V_1,V_2) + (\mu_1 \oplus \mu_2)(U_1,U_2)(V_1,V_2) + (U_1,U_2)(\mu_1 \oplus \mu_2)(V_1,V_2)$$

$$\begin{aligned}
&= (\mu_1 \oplus \mu_2)\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix},\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}\right)(\mu_1 \oplus \mu_2)\left(\begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix},\begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)\\
&\quad + (\mu_1 \oplus \mu_2)\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix},\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}\right)\left(\begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix},\begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)\\
&\quad + \left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix},\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}\right)(\mu_1 \oplus \mu_2)\left(\begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix},\begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)\\
&= \left(\mu_1\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right),\mu_2\left(\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}\right)\right)\left(\mu_1\left(\begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix}\right),h_2\left(\begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)\right)\\
&\quad + \left(\mu_1\left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix}\right),\mu_2\left(\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}\right)\right)\left(\begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix},\begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)\\
&\quad + \left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix},\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}\right)\left(\mu_1\left(\begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix}\right),\mu_2\left(\begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)\right)\\
&= \left(\begin{bmatrix}0 & 0\\ 0 & -\theta_1\end{bmatrix},\begin{bmatrix}0 & -\alpha_2\\ 0 & 0\end{bmatrix}\right)\left(\begin{bmatrix}0 & 0\\ 0 & -\tau_1\end{bmatrix},\begin{bmatrix}0 & -\sigma_2\\ 0 & 0\end{bmatrix}\right) + \left(\begin{bmatrix}0 & 0\\ 0 & -\theta_1\end{bmatrix},\begin{bmatrix}0 & -\alpha_2\\ 0 & 0\end{bmatrix}\right)\left(\begin{bmatrix}\sigma_1 & 0\\ 0 & \tau_1\end{bmatrix},\begin{bmatrix}\sigma_2 & \tau_2\\ 0 & 0\end{bmatrix}\right)\\
&\quad + \left(\begin{bmatrix}\alpha_1 & 0\\ 0 & \theta_1\end{bmatrix},\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & 0\end{bmatrix}\right)\left(\begin{bmatrix}0 & 0\\ 0 & -\tau_1\end{bmatrix},\begin{bmatrix}0 & -\sigma_2\\ 0 & 0\end{bmatrix}\right)\\
&= \left(\begin{bmatrix}0 & 0\\ 0 & \theta_1\tau_1\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right) + \left(\begin{bmatrix}0 & 0\\ 0 & -\theta_1\tau_1\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right) + \left(\begin{bmatrix}0 & 0\\ 0 & -\theta_1\tau_1\end{bmatrix},\begin{bmatrix}0 & -\alpha_2\sigma_2\\ 0 & 0\end{bmatrix}\right)\\
&= \left(\begin{bmatrix}0 & 0\\ 0 & -\theta_1\tau_1\end{bmatrix},\begin{bmatrix}0 & -\alpha_2\sigma_2\\ 0 & 0\end{bmatrix}\right).
\end{aligned}$$

Hence, $\mu_1 \oplus \mu_2$ is a homoderivation on $M_1 \oplus M_2$.

A homoderivation on a ring M reduces to a derivation whenever $\mu(\alpha)\mu(\theta) = 0$, for every $\alpha, \theta \in M$, that is, the multiplicative term in the definition of a homoderivation vanishes. This condition holds, for example, when μ is the zero mapping, when one of the images vanishes, or when both images are nonzero. However, their product is zero, as may occur in the presence of zero divisors or orthogonal homomorphisms. We now present an illustrative example.

Example 4. Let $M = \left\{\begin{bmatrix}\alpha & \theta\\ 0 & \delta\end{bmatrix} \mid \alpha, \theta, \delta \in \mathbb{Z}\right\}$. We define a mapping $\mu: M \to M$ by

$$\mu\left(\begin{bmatrix}\alpha & \theta\\ 0 & \delta\end{bmatrix}\right) = \begin{bmatrix}0 & -\theta\\ 0 & 0\end{bmatrix}, \text{ for very } \begin{bmatrix}\alpha & \theta\\ 0 & \delta\end{bmatrix} \in M.$$

We will show that μ is a homoderivation on M.

Let $U = \begin{bmatrix}\alpha_1 & \theta_1\\ 0 & \delta_1\end{bmatrix}, V = \begin{bmatrix}\alpha_2 & \theta_2\\ 0 & \delta_2\end{bmatrix} \in M$.

We have:

$$\begin{aligned}
\mu(UV) &= \mu\left(\begin{bmatrix}\alpha_1 & \theta_1\\ 0 & \delta_1\end{bmatrix}\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & \delta_2\end{bmatrix}\right)\\
&= \mu\left(\begin{bmatrix}\alpha_1\alpha_2 & \alpha_1\theta_2 + \theta_1\delta_2\\ 0 & \delta_1\delta_2\end{bmatrix}\right)\\
&= \begin{bmatrix}0 & -(\alpha_1\theta_2 + \theta_1\delta_2)\\ 0 & 0\end{bmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mu(U)\mu(V) + \mu(U)V + U\mu(V)\\
&= \mu\left(\begin{bmatrix}\alpha_1 & \theta_1\\ 0 & \delta_1\end{bmatrix}\right)\mu\left(\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & \delta_2\end{bmatrix}\right) + \mu\left(\begin{bmatrix}\alpha_1 & \theta_1\\ 0 & \delta_1\end{bmatrix}\right)\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & \delta_2\end{bmatrix} + \begin{bmatrix}\alpha_1 & \theta_1\\ 0 & \delta_1\end{bmatrix}\mu\left(\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & \delta_2\end{bmatrix}\right)\\
&= \begin{bmatrix}0 & -\theta_1\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & -\theta_2\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & -\theta_1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\alpha_2 & \theta_2\\ 0 & \delta_2\end{bmatrix} + \begin{bmatrix}\alpha_1 & \theta_1\\ 0 & \delta_1\end{bmatrix}\begin{bmatrix}0 & -\theta_2\\ 0 & 0\end{bmatrix} \text{ [33]}
\end{aligned}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\theta_1\delta_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_1\theta_2 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -(\alpha_1\theta_2 + \theta_1\delta_2) \\ 0 & 0 \end{bmatrix}.$$

This shows that $\mu(UV) = \mu(U)\mu(V) + \mu(U)V + U\mu(V)$, for every $U, V \in M$. It is proven that μ is a homoderivation mapping on ring M.

In **Example 5**, we obtain $\mu(U)\mu(V) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, by the previous result, the homoderivation μ is also a derivation on the ring M whenever $\mu(U)\mu(V) = 0$.

In the next result, we establish the relationship between homoderivations and inner derivations.

Theorem 3. *Let a ring R and $\alpha \in Z(R) = \{x \in R \mid x\gamma = \gamma x, \forall \gamma \in R\}$. Inner derivation $\mu_\alpha : R \to R$, where:*

$$\mu_\alpha : \gamma \mapsto [\gamma, \alpha] = \gamma\alpha - \alpha\gamma$$

is a homoderivation on ring R.

Proof. We will show that μ_α is a homoderivation on ring R. First, we will show that μ_α is an additive mapping on ring R. Given any $\gamma, \pi \in R$. We have:

$$\begin{aligned} \mu_\alpha(\gamma + \pi) &= [\gamma + \pi, \alpha] \\ &= (\gamma + \pi)\alpha - \alpha(\gamma + \pi) \\ &= \gamma\alpha + \pi\alpha - \alpha\gamma - \alpha\pi \\ &= \gamma\alpha - \alpha\gamma + \pi\alpha - \pi\alpha \\ &= [\gamma, \alpha] + [\pi, \alpha] \\ &= \mu_\alpha(\gamma) + \mu_\alpha(\pi) \end{aligned}$$

Now, $\mu_\alpha(\gamma\pi) = [\gamma\pi, \alpha] = \gamma\pi\alpha - \alpha\gamma\pi$.

On the other hand,

$$\begin{aligned} \mu_\alpha(\gamma)\mu_\alpha(\pi) + \mu_\alpha(\gamma)\pi + \gamma\mu_\alpha(\pi) &= [\gamma, \alpha][\pi, \alpha] + [\gamma, \alpha]\pi + \gamma[\pi, \alpha] \\ &= (\gamma\alpha - \alpha\gamma)(\pi\alpha - \alpha\pi) + (\gamma\alpha - \alpha\gamma)\pi + \gamma(\pi\alpha - \alpha\pi) \\ &= (\gamma\alpha - \gamma\alpha)(\pi\alpha - \alpha\pi) + (\gamma\alpha - \gamma\alpha)\pi + \gamma(\pi\alpha - \alpha\pi) \\ &= \gamma\pi\alpha - \gamma\alpha\pi \\ &= \gamma\pi\alpha - \alpha\gamma\pi \\ &= \mu_\alpha(\gamma\pi). \end{aligned}$$

Hence, μ_α is a homoderivation on ring R. ∎

To illustrate Theorem 3, we present the following example.

Example 5. Given ring $M_2(\mathbb{Z}) = \left\{ \begin{bmatrix} \varepsilon & \tau \\ \delta & \pi \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$ and choose $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \in M$. We define $\mu_A : M_2(\mathbb{Z}) \to M_2(\mathbb{Z})$, by:

$$\mu_A : \mapsto [X, A] = XA - AX, \text{ for every } X \in M_2(\mathbb{Z}).$$

We will show that μ_A is a homoderivation on ring $M_2(\mathbb{Z})$.

Given any $X = \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix}, Y = \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} \in M_2(\mathbb{Z})$.

We have:

$$\begin{aligned} \mu_A(XY) &= \mu_A\left(\begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix} \begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} \right) \\ &= \mu_A\left(\begin{bmatrix} \varepsilon_1\varepsilon_2 + \tau_1\delta_2 & \varepsilon_1\tau_2 + \tau_1\pi_2 \\ \delta_1\varepsilon_2 + \pi_1\delta_2 & \delta_1\tau_2 + \pi_1\pi_2 \end{bmatrix} \right) \end{aligned}$$

$$= \left[\begin{bmatrix} \varepsilon_1\varepsilon_2 + \tau_1\delta_2 & \varepsilon_1\tau_2 + \tau_1\pi_2 \\ \delta_1\varepsilon_2 + \pi_1\delta_2 & \delta_1\tau_2 + \pi_1\pi_2 \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]$$
$$= \begin{bmatrix} \varepsilon_1\varepsilon_2 + \tau_1\delta_2 & \varepsilon_1\tau_2 + \tau_1\pi_2 \\ \delta_1\varepsilon_2 + \pi_1\delta_2 & \delta_1\tau_2 + \pi_1\pi_2 \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \varepsilon_1\varepsilon_2 + \tau_1\delta_2 & \varepsilon_1\tau_2 + \tau_1\pi_2 \\ \delta_1\varepsilon_2 + \pi_1\delta_2 & \delta_1\tau_2 + \pi_1\pi_2 \end{bmatrix}$$
$$= \begin{bmatrix} \varepsilon_1\varepsilon_2\lambda + \tau_1\delta_2\lambda & \varepsilon_1\tau_2\lambda + \tau_1\pi_2\lambda \\ \delta_1\varepsilon_2\lambda + \pi_1\delta_2\lambda & \delta_1\tau_2 + \pi_1\pi_2\lambda \end{bmatrix} - \begin{bmatrix} \lambda\varepsilon_1\varepsilon_2 + \lambda\tau_1\delta_2 & \lambda\varepsilon_1\tau_2 + \lambda\tau_1\pi_2 \\ \lambda\delta_1\varepsilon_2 + \lambda\pi_1\delta_2 & \lambda\delta_1\tau_2 + \lambda\pi_1\pi_2 \end{bmatrix}.$$

On the other hand,

$\mu_A(X)\mu_A(Y) + \mu_A(X)Y + X\mu_A(Y)$

$$= \left[\begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]\left[\begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right] + \left[\begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]\begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix}\left[\begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix}\begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix}$$
$$= \begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix}\begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \varepsilon_1 & \tau_1 \\ \delta_1 & \pi_1 \end{bmatrix}\begin{bmatrix} \varepsilon_2 & \tau_2 \\ \delta_2 & \pi_2 \end{bmatrix}$$
$$= \begin{bmatrix} \varepsilon_1\varepsilon_2 + \tau_1\delta_2 & \varepsilon_1\tau_2 + \tau_1\pi_2 \\ \delta_1\varepsilon_2 + \pi_1\delta_2 & \delta_1\tau_2 + \pi_1\pi_2 \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \varepsilon_1\varepsilon_2 + \tau_1\delta_2 & \varepsilon_1\tau_2 + \tau_1\pi_2 \\ \delta_1\varepsilon_2 + \pi_1\delta_2 & \delta_1\tau_2 + \pi_1\pi_2 \end{bmatrix}$$
$$= \begin{bmatrix} \varepsilon_1\varepsilon_2\lambda + \tau_1\delta_2\lambda & \varepsilon_1\tau_2\lambda + \tau_1\pi_2\lambda \\ \delta_1\varepsilon_2\lambda + \pi_1\delta_2\lambda & \delta_1\tau_2 + \pi_1\pi_2\lambda \end{bmatrix} - \begin{bmatrix} \lambda\varepsilon_1\varepsilon_2 + \lambda\tau_1\delta_2 & \lambda\varepsilon_1\tau_2 + \lambda\tau_1\pi_2 \\ \lambda\delta_1\varepsilon_2 + \lambda\pi_1\delta_2 & \lambda\delta_1\tau_2 + \lambda\pi_1\pi_2 \end{bmatrix}.$$

We have $\mu_A(XY) = \mu_A(X)\mu_A(Y) + \mu_A(X)Y + X\mu_A(Y)$ and hence, μ_A is a homoderivation on the ring $M_2(\mathbb{Z})$.

In the following theorem, we describe the structure of the set of constants of a homoderivation on a ring R.

Theorem 4. *Let R be a ring and μ be a homoderivation on R. If $C_\mu = \{\alpha \in \mathbb{Z} \,|\mu(\alpha) = 0\}$ set of all constant elements of μ, then C_μ is a $\mathbb{Z}$-module.*

Proof. Since every Abelian group is a $\mathbb{Z}$-module, it suffices to show that C_d is an Abelian group in order to conclude that C_d is a $\mathbb{Z}$-module.

1. Given $\alpha, \theta \in C_\mu$. We have:
$$\begin{aligned} \mu(\alpha + \theta) &= \mu(\alpha) + \mu(\theta) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$
2. Given $\alpha, \theta, \gamma \in C_\mu$. We have:
$$\begin{aligned} \mu\big((\alpha + \theta) + \gamma\big) &= \mu(\alpha + \theta) + \mu(\gamma) \\ &= \mu(\alpha) + \mu(\theta) + \mu(\gamma) \\ &= \mu(\alpha) + \mu(\theta + \gamma) \\ &= \mu(\alpha + (\theta + \gamma)). \end{aligned}$$
3. There exists an identity element, that is $0 \in C_\mu$, such that $\mu(0) + \mu(\alpha) = \mu(0 + \alpha) = \mu(\alpha)$.
4. For every $\alpha \in C_\mu$, there exists $-\alpha \in C_\mu$, such that $\mu(\alpha) + \mu(-\alpha) = \mu\big(\alpha + (-\alpha)\big) = \mu(0) = 0$.
5. Given any $\alpha, \theta \in C_\mu$. We have $\mu(\alpha + \theta) = \mu(\alpha) + \mu(\theta) = 0 = \mu(\theta) + \mu(\alpha) = \mu(\theta + \alpha)$.

Thus, it is proven that $\langle C_\mu, +\rangle$ is an Abelian group. Consequently, $\langle C_\mu, +\rangle$ is a $\mathbb{Z}$-module. ■

4. CONCLUSION

Based on the preceding discussion, a homoderivation on a ring combines the Leibniz rule with the multiplicative property of a ring homomorphism. We show that the sum of two homoderivations is again a homoderivation, provided the maps h_1 and h_2 are orthogonal (mutually annihilating). Moreover, the direct sum of homoderivations on their respective rings is also a homoderivation. In contrast, scalar multiplication of a homoderivation does not, in general, yield another homoderivation. Additionally, the set of constants of a homoderivation forms a $\mathbb{Z}$-module.

Author Contributions

Desi Widiarti: Formal Analysis, Writing-Original Draft. Fitriani: Conceptualization, Methodology, Validation, Writing-Review and Editing. Ahmad Faisol: Formal Analysis, Validation. All authors considered the findings and provided input for the final manuscript

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Declarations

The authors declare no competing interests.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools (e.g., ChatGPT) were used solely for language refinement, including grammar, spelling, and clarity. The scientific content, analysis, interpretation, and conclusions were developed entirely by the authors. All final text was reviewed and approved by the authors.

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