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HOMODERIVATIONS ON LIE RINGS

Suffi Nuralesa[1*], Galang Riki Ramadhan[2], Carolus Pasha Lazuardi Jituprasojo Hatmakelana[3], Nurul Fadhilah[4], Benediktus Panji Pradipta[5], Nikken Prima Puspita[6]

[1,2,4,5,6]*Department of Mathematics, Faculty of Science and Mathematics, Universitas Diponegoro*
Jln. Prof. Jacub Rais, Semarang, 50275, Indonesia

[3] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada*
Sekip Utara BLS 21, Yogyakarta, 55281, Indonesia

*Corresponding author's e-mail: * suffinuralesa@students.undip.ac.id*

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ABSTRACT

Let L be a set equipped with two binary operations, addition $(+)$ and multiplication $[,]: L \times L \to L$. The triple $(L, +, [,])$ is a Lie ring if its multiplication $([,])$ is the Lie bracket commutator defined by $[x, y] = xy - yx$ for every $x, y \in L$. A homoderivation on a Lie ring is a mapping $H: L \to L$ that satisying the equation $H([x,y]) = [H(x), H(y)] + [H(x), y] + [x, H(y)]$, for all $x, y \in L$. The set of all homoderivations on L is denoted by HDL. This paper investigates the structure of HDL with the binary operations of addition and composition. It is shown that HDL does not form a ring. Therefore, we introduce a subset $HL \subseteq HDL$ satisfying the RH property. Under this condition, it is shown that $(HL, \oplus, \circ)$ forms a ring. Furthermore, using the bracket operator, the structure $(HL, \oplus, [,])$ is also a Lie ring. In addition, this paper formulates an iteration of homoderivations in terms of the sigma sum of Lie ring elements.

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1. INTRODUCTION

A ring is an algebraic structure that consists of a non-empty set equipped with two binary operations, namely the addition $(+)$ and the multiplication $(\cdot)$, that satisfying specific axioms. A ring (denoted by R) is usually written as a triple $(R,+,\cdot)$, where $(R,+)$ is an abelian group, $(R,\cdot)$ is a semigroup, and R with addition and multiplication operations satisfies the left and right distributive laws [1]. More developments in ring theory were introduced by Sophus Lie (1842-1899) through an operation known as the Lie bracket, commonly denoted by $[,]$. This operation defines a non-associative structure that is anticommutative and satisfies the Jacobi identity, leading to the concept of a Lie ring [2]. In specific, a Lie ring L satisfies the identity $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all $X,Y,Z \in L$ [3].

The notion of derivation, originating in analysis, has been extended to algebraic contexts such as rings and semirings. The concept of derivation was first introduced by two great figures, namely Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). Since that time, the concept of derivation has evolved not only in analysis but also in algebra, leading to the definition of derivation on rings [4] and semirings [5]. In general, an additive mapping $d: R \to R$ is called a derivation on the ring $(R,+,\cdot)$ if it satisfies Leibniz's rule $d(x+y)=d(x)+d(y)$ and $d(xy)=d(x)y+x\,d(y)$ for all $x,y \in R$ [6].

In previous research, Artemovych discussed the derivation rings of Lie rings [7]. Next, the research was explored by [8]. When applied to Lie rings, the definition of derivation is adapted to the bracket operation. A derivation on a Lie ring L is an additive mapping $d: L \to L$ that satisfies $d(x+y)=d(x)+d(y)$ and $d([x,y])=[d(x),y]+[x,d(y)]$ for all $x,y \in L$.

Moreover, the concept of homomorphic and derivational properties within ring structures. An additive mapping $f: R \to R$ is called a homomorphism if $f(x+y)=f(x)+f(y)$ and $f(xy)=f(x)f(y)$, for all $x,y \in R$ [9]. By combining the concepts of homomorphism and derivation, El-Sofy (2000) introduced the concept of homoderivation. A homoderivation combines homomorphic and derivational properties within ring structures. A mapping $h: R \to R$ is called a homoderivation if it satisfies the equation $h(xy)=h(x)h(y)+h(x)y+xh(y)$ for all $x,y \in R$ [10]. Based on this definition, there have been several previous studies related to homoderivations. Among them, Sarikaya investigated Results on Lie ideals of prime rings with homoderivations [11]. Furthermore, Said Belkadi, Shakir Ali, and Lahcen Taoufiq in 2023 explained nilpotent homoderivations in prime rings [12].

From [7], Artemovych also discusses the condition that the set of all derivations $DerA$, forms a Lie ring. This structure is equipped with point-wise addition and the Lie multiplication defined by $[d,\delta](r)=d(\delta(r))-\delta(d(r))$. However, unlike derivations, the structure of the set of all homoderivations has not been thoroughly explored. In particular, it is natural to ask whether the set of all homodoreviations on a Lie ring forms a similar structure to the set of all derivations.

Motivated by this problem, we investigate the algebraic structure of the set of all homoderivations on the Lie ring L, denoted by HDL, with the multiplication operation defined by $[H_1,H_2](a)=(H_1 \circ H_2)(a)-(H_2 \circ H_1)(a)$, where $H_1,H_2 \in HDL$ and $a \in L$. This research explores the construction and characterisation of Lie rings from sets of homoderivations, which generalize derivation on Lie rings. In addition, we present some recent examples of homoderivations on Lie rings, identify conditions that ensure the existence of HDL as Lie rings, and investigate the n-th iteration of a homoderivation on a Lie ring.

2. RESEARCH METHODS

This research investigates the algebraic structure of the set of homoderivations on a Lie ring L. We begin with a literature review that includes derivations, Lie rings, homoderivations, properties of Lie algebras, and several related theorems in [13], [14], and [15]. Following this, we introduce the construction process for the set of all homoderivations on L, denoted as HDL, and define the operations as $(H_1 \oplus H_2)(a)=H_1(a)+H_2(a)$ and $[H_1,H_2](a)=(H_1 \circ H_2)(a)-(H_2 \circ H_1)(a)$, where $H_1,H_2 \in HDL$ and $a \in L$. Next, we formulate conjectures regarding the properties of this multiplication operation and the necessary conditions for a subset of HDL to qualify as a Lie ring. We also provide several examples of Lie rings and their associated homoderivations to illustrate the construction process. We prove fundamental properties like antisymmetry and the Jacobi identity, to establish that a subset of HDL is a Lie ring under the defined operation.

In constructing *HDL*, it is important to understand the following definitions.

Definition 1. [2] Let $(R,+,\cdot)$ be a non-associative ring. The binary operation bracket $[\,,]: R\times R \to R$ is called the Lie bracket if for all $x,y \in R$ satisfy:

1. $[x,x] = 0_R$ (Anticommutativity),
2. Jacobi identity,
   a. $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ (Right-normed Jacobi identity),
   b. $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ (Left-normed Jacobi identity).

The triple $(R,+,[\,,])$ is called Lie ring if the bracket operation satisfies (1) and (2).

We begin with some examples of Lie rings illustrating Definition 1.

Example 1. Let $\mathcal{F}_2$ be the set of all functions $f:\mathbb{R}^2 \to \mathbb{R}$ such that f_x (the partial derivative with respect to x) and f_y (the partial derivative with respect to y) are continuous and differentiable. We can prove, based on the ring's axioms, that $(\mathcal{F}_2,+,\cdot)$ is ring. Define the Lie bracket operation "$[,]$" by $[f,g](x,y) = (f_x\, g_y)(x,y) - (f_y\, g_x)(x,y)$, for all $f,g \in \mathcal{F}_2$. If $f,g \in \mathcal{F}_2$, then f_x, f_y, g_x, g_y are continuous functions. Since sums and products of continuous functions are continuous, $(f_x\, g_y)(x,y) - (f_y g_x)(x,y)$ is continuous. Hence, $[f,g](x,y) \in \mathcal{F}_2$. To show the anti-commutative, consider $[f,g](x,y) = (f_x\, g_y)(x,y) - (f_y\, g_x)(x,y) = -(g_x f_y(x,y) - g_y f_x(x,y)) = -[g,f](x,y)$, since multiplication in $\mathbb{R}$ is commutative. Let $f,g,h \in \mathcal{F}_2$ with $[f,g](x,y) = (f_x\, g_y)(x,y) - (f_y\, g_x)(x,y)$. Then $[f,[g,h]](x,y) = (f_x(g_x h_y - g_y h_x)_y)(x,y) - (f_y(g_x h_y - g_y h_x)_x)(x,y)$. By the product rule, we obtain

$$(g_x h_y - g_y h_x)_y = g_{xy} h_y + g_x h_{yy} - g_{yy} h_x - g_y h_{xy},$$
$$(g_x h_y - g_y h_x)_x = g_{xx} h_y + g_x h_{yx} - g_{yx} h_x - g_y h_{xx}.$$

Therefore,

$[f,[g,h]](x,y) = (f_x g_{xy} h_y)(x,y) + (f_x g_x h_{yy})(x,y) - (f_x g_{yy} h_x)(x,y) - (f_x g_y h_{xy})(x,y) - (f_y g_{xx} h_y)(x,y) - (f_y g_x h_{yx})(x,y) + (f_y g_{yx} h_x)(x,y) + (f_y g_y h_{xx})(x,y).$

Similarly,

$[g,[h,f]](x,y) = (g_x h_{xy} f_y)(x,y) + (g_x h_x f_{yy})(x,y) - (g_x h_{yy} f_x)(x,y) - (g_x h_y f_{xy})(x,y) - (g_y h_{xx} f_y)(x,y) - (g_y h_x f_{yx})(x,y) + (g_y h_{yx} f_x)(x,y) + (g_y h_y f_{xx})(x,y),$

$[h,[f,g]](x,y) = (h_x f_{xy} g_y)(x,y) + (h_x f_x g_{yy})(x,y) - (h_x f_{yy} g_x)(x,y) - (h_x f_y g_{xy})(x,y) - (h_y f_{xx} g_y)(x,y) - (h_y f_x g_{yx})(x,y) + (h_y f_{yx} g_x)(x,y) + (h_y f_y g_{xx})(x,y).$

Therefore, $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$. Hence, $(\mathcal{F}_2,+,[\,,])$ is a Lie ring.

Example 2. Consider the ring of 2×2 matrices over the real numbers, denoted by $(M_{2\times 2}(\mathbb{R}),+,\cdot)$. We show that $(M_{2\times 2}(\mathbb{R}),+,\cdot)$ is a Lie ring with respect to the bracket operation "$[,]$" defined by $[A,B] = AB - BA$, for all $A,B \in M_{2\times 2}(\mathbb{R})$. For any $A,B \in M_{2\times 2}(\mathbb{R})$, $[A,B] = AB - BA \in M_{2\times 2}(\mathbb{R})$. If $A = C$ and $B = D$, then $[A,B] = AB - BA = CD - DC = [C,D]$, so the operation is well-defined. Next, we show the bracket operation is anti-commutative. For any $A,B \in M_{2\times 2}(\mathbb{R})$, $[A,B] = AB - BA = -(BA - AB) = -[B,A]$. In particular, $[A,A] = AA - AA = 0$. We now prove the Jacobi identity. Let $A,B,C \in M_{2\times 2}(\mathbb{R})$. Then, by using distributivity and associativity of matrix multiplication, $[[A,B],C] = ABC - BAC - CAB + CBA$. Similarly, $[[B,C],A] = BCA - CBA - ABC + ACB$, and $[[C,A],B] = CAB - ACB - BCA + BAC$. Adding the three expressions, we obtain $[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$. Hence, $(M_{2\times 2}(\mathbb{R}),+,[\,,])$ is a Lie ring.

Example 3. Consider the ring $(\mathbb{H},+,\cdot)$, where $\mathbb{H} = \{q = a + bi + cj + dk \mid a,b,c,d \in \mathbb{R}$ and i,j,k are imaginary units$\}$, with the Lie bracket operation defined by $[q_1,q_2] = q_1\, q_2 - q_2 q_1$ for all $q_1,q_2 \in \mathbb{H}$. Since $(\mathbb{H},+,\cdot)$ is ring, $[q_1,q_2] \in \mathbb{H}$. For all $q_1,q_2 \in \mathbb{H}$, $[q_1,q_2] = q_1\, q_2 - q_2 q_1 = -(q_2 q_1 - q_1 q_2) = -[q_2,q_1]$.

In particular, $[q,q] = 0$ for every $q \in \mathbb{H}$. Next, we show the Jacobi identity. For $q_1,q_2,q_3 \in \mathbb{H}$,

$[q_1,[q_2,q_3]] = q_1(q_2q_3 - q_3q_2) - (q_2q_3 - q_3q_2)q_1 = q_1q_2q_3 - q_1q_3q_2 - q_2q_3q_1 + q_3q_2q_1,$

and similarly, $[q_2,[q_3,q_1]] = q_2q_3q_1 - q_2q_1q_3 - q_3q_1q_2 + q_1q_3q_2$, and $[q_3,[q_1,q_2]] = q_3q_1q_2 - q_3q_2q_1 - q_1q_2q_3 + q_2q_1q_3$.

Adding these expressions and using the associativity of the quaternion multiplication, the terms will cancel out each other and give $[q_1,[q_2,q_3]] + [q_2,[q_3,q_1]] + [q_3,[q_1,q_2]] = 0$.

After defining the structure of the Lie ring, we now introduce a mapping that is closely related to the ring multiplication. We define a homoderivation on R.

Definition 2. [10] A homoderivation on R is an additive mapping $H: R \to R$ that satisfies $H(xy) = H(x)H(y) + H(x)y + xH(y)$ for all $x, y \in R$.

The following remark explains how a homoderivation interacts with the Lie bracket defined by the commutator. This observation will be useful in the subsequent results.

Remark. [11] *A homoderivation on a Lie ring with the bracket operation defined by* $[x,y] = xy - yx$ *satisfies*

$$\begin{aligned} H([x,y]) &= H(xy - yx) \\ &= H(xy) - H(yx) \\ &= H(x)H(y) + H(x)y + xH(y) - H(y)H(x) - H(y)x - yH(x) \\ &= [H(x),H(y)] + [H(x),y] + [x,H(y)]. \end{aligned}$$

3. RESULTS AND DISCUSSION

In this section, we discuss homoderivations on Lie rings. After that, we construct the set of all homoderivations on a Lie ring and explore the conditions under which it forms a Lie ring.

3.1 Homoderivations on Lie rings

Recall from Definition 2 that a homoderivation on a Lie ring L is a mapping $H: L \to L$ satisfying $H([x,y]) = [H(x),H(y)] + [H(x),y] + [x,H(y)]$, for all $x, y \in L$. We now present several examples of homoderivations on Lie rings.

Example 4. Consider the linear mapping $H : \mathcal{F}_2 \to \mathcal{F}_2$, where $H(f)(x,y) = -f(x,y)$. We will show that H is a homoderivation on the Lie ring $\mathcal{F}_2$. By Definition 2, a homoderivation on a ring $\mathcal{F}_2$ is a mapping $H: \mathcal{F}_2 \to \mathcal{F}_2$ satisfying $H(fg) = H(f)H(g) + H(f)g + fH(g)$, for all $f, g \in \mathcal{F}_2$. The Lie bracket on $\mathcal{F}_2$ is defined by $[f,g](x,y) = (f_x\, g_y)(x,y) - (f_y g_x)(x,y)$. For the left-hand side, we obtain

$$H([f,g](x,y)) = H((f_xg_y)(x,y) - (f_yg_x)(x,y)) = H(f_xg_y)(x,y) - H(f_yg_x)(x,y)$$

$$= [H(f_x)g_y + f_xH(g_y) + H(f_x)H(g_y)](x,y) - [H(f_y)g_x + f_yH(g_x) + H(f_y)H(g_x)](x,y)$$

$$= [(-f_x)g_y + f_x(-g_y) + (-f_x)(-g_y)](x,y) - [(-f_y)g_x + f_y(-g_x) + (-f_y)(-g_x)](x,y)$$

$$= [-f_xg_y - f_xg_y + f_xg_y](x,y) - [-f_yg_x - f_yg_x + f_yg_x](x,y)$$

$$= -(f_xg_y)(x,y) + (f_yg_x)(x,y)$$

$$= -[(f_xg_y)(x,y) - (f_yg_x)(x,y)]$$

$$= -[f,g](x,y).$$

For the right-hand side, we consider $[H(f_x)g_y + f_xH(g_y) + H(f_x)H(g_y)](x,y) - [H(f_y)g_x + f_yH(g_x) + H(f_y)H(g_x)](x,y) = [H(f)H(g)](x,y) + [H(f),g](x,y) + [f,H(g)](x,y)$, we obtain

$$[H(f)H(g)](x,y) = (-f)_x(-g)_y - (-f)_y(-g)_x = (f_xg_y - f_yg_x)(x,y) = [f,g](x,y),$$

$$[H(f),g](x,y) = (-f)_xg_y - (-f)_yg_x = -(f_xg_y - f_yg_x)(x,y) = -[f,g](x,y),$$

$$[f,H(g)](x,y) = f_x(-g)_y - f_y(-g)_x = -(f_xg_y - f_yg_x)(x,y) = -[f,g](x,y).$$

Comparing both sides, we obtain H is a homoderivation on the Lie ring $\mathcal{F}_2$.

Example 5. From **Example 2**, we know that $(M_{2\times 2}(\mathbb{R}), +, [\,,])$ is a Lie ring. Define a mapping $H: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ where $H(A) = trace(A) \cdot I$ for all $A \in M_{2\times 2}(\mathbb{R})$. We will show that the mapping H is a homoderivation. For the left-hand side, we have $H([A,B]) = trace([A,B]) \cdot I = trace(AB - BA) \cdot I = \big(trace(AB) - trace(BA)\big) \cdot I = \left[trace\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) - trace\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right)\right] \cdot I = [(a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}) - (b_{11}a_{11} + b_{12}a_{21} + b_{21}a_{12} + b_{22}a_{22})] \cdot I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and for the right-hand side, we obtain

$[H(A), H(B)] = (a_{11} + a_{22})I \cdot (b_{11} + b_{22})I - (b_{11} + b_{22})I \cdot (a_{11} + a_{22})I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

$[H(A), B] = H(A)B - BH(A) = (a_{11} + a_{22})I \cdot B - B \cdot (a_{11} + a_{22})I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

$[A, H(B)] = AH(B) - H(B)A = A(b_{11} + b_{22}) \cdot I - (b_{11} + b_{22})I \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Comparing both sides, we obtain $H([A,B]) = [H(A), H(B)] + [H(A), B] + [A, H(B)]$. Hence, H is a homoderivation on the Lie ring $M_{2\times 2}(\mathbb{R})$.

Example 6. Consider the set $GL_2(\mathbb{R}) = \{S \in M_{2\times 2}(\mathbb{R}) \mid det(S) \neq 0\}$. Define a mapping $H_S: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ where $H_S(X) = G(X) - X$ where $G(X) = SXS^{-1}$ with $S \in GL_2(\mathbb{R})$ and $X \in M_{2\times 2}(\mathbb{R})$. We want to show that $H([A,B]) = [H(A), H(B)] + [H(A), B] + [A, H(B)]$ for all $A, B \in M_{2\times 2}(\mathbb{R})$. Using the definition of $H_s(X)$, we obtain $H([A,B]) = G([A,B]) - [A,B]$. Now, for the right-hand side, we obtain,

$[H(A), H(B)] = [G(A) - A, G(B) - B] = [G(A), G(B)] + [G(A), -B] + [-A, G(B)] + [-A, -B] = [G(A), G(B)] - [G(A), B] - [A, G(B)] + [A, B]$,

$[H(A), B] = [G(A) - A, B] = [G(A), B] - [A, B]$,

$[A, H(B)] = [A, G(B) - B] = [A, G(B)] - [A, B]$.

Therefore,

$[H(A), H(B)] + [H(A), B] + [A, H(B)] = ([G(A), G(B)] - [G(A), B] - [A, G(B)] + [A, B]) + ([G(A), B] - [A, B]) + ([A, G(B)] - [A, B]) = [G(A), G(B)] - [A, B]$.

We can check that $[G(A), G(B)] = [SAS^{-1}, SBS^{-1}] = (SAS^{-1})(SBS^{-1}) - (SBS^{-1})(SAS^{-1}) = S(AB)S^{-1} - S(BA)S^{-1} = S(AB - BA)S^{-1} = S([A,B])S^{-1} = G([A,B])$.

Hence,

$[H(A), H(B)] + [H(A), B] + [A, H(B)] = G([A,B]) - [A,B]$,

and we obtain $H([A,B]) = [H(A), H(B)] + [H(A), B] + [A, H(B)]$. Therefore, H_s is a Lie homoderivation with $G(X) = SXS^{-1}$, for $S \in GL_2(\mathbb{R})$ and $X \in M_{2\times 2}(\mathbb{R})$.

3.2 Lie Ring Which is Constructed from Homoderivations on Lie Rings

Now, we focus on the construction of Lie rings from the set of all homoderivations on a Lie ring. The motivation for this construction is to investigate the algebraic structure formed by the operations of addition and composition.

Let $(L, +, [,])$ be a Lie ring. We define the following set of all homoderivations on Lie rings:

$$HDL = \{H : L \to L \mid H \text{ is a homoderivation on the Lie ring } L\}.$$

We define two operations on HDL as follows:

$\oplus$: $HDL \times HDL \to HDL$,

$(H_1, H_2) \mapsto (H_1 \oplus H_2)(a) = H_1(a) + H_2(a)$,

and

$\circ$: $HDL \times HDL \to HDL$,

$$(H_1, H_2) \mapsto (H_1 \circ H_2)(a) = H_1(H_2(a)),$$

for all $H_1, H_2 \in HDL$ and $a \in L$.

To show that the operations " $\oplus$ " and " $\circ$ " are binary operations on HDL, we prove that the result of applying these operations to two homoderivations is again a homoderivation on L.

Proposition 1. *For every homoderivation H_1 and H_2 on the ring lie L, and every $a, b \in L$ where $[H_1(a), H_2(b)] + [H_2(a), H_1(b)] = 0$, the addition $H_1 \oplus H_2$ is also a homoderivation.*

Proof. Let $H_1, H_2 \in HDL$ and $a, b \in L$, where $[H_1(a), H_2(b)] + [H_2(a), H_1(b)] = 0$. We obtain

$(H_1 \oplus H_2)([a, b]) = H_1([a, b]) + H_2([a, b])$

$= [H_1(a), H_1(b)] + [H_1(a), b] + [a, H_1(b)] + [H_2(a), H_2(b)] + [H_2(a), b] + [a, H_2(b)]$

$= [H_1(a) + H_2(a), H_1(b) + H_2(b)] + [H_1(a) + H_2(a), b] + [a, H_1(b) + H_2(b)] - ([H_1(a), H_2(b)]$
$+[H_2(a), H_1(b)])$

(since $[H_1(a) + H_2(a), H_1(b) + H_2(b)] = [H_1(a), H_1(b)] + [H_1(a), H_2(b)] + [H_2(a), H_1(b)] + [H_2(a), H_2(b)]$)

$= [(H_1 \oplus H_2)(a), (H_1 \oplus H_2)(b)] + [(H_1 \oplus H_2)(a), b] + [a, (H_1 \oplus H_2)(b)]$.

(by definition "$\oplus$" and $[H_1(a), H_2(b)] + [H_2(a), H_1(b)] = 0$)

Hence, $H_1 \oplus H_2 \in HDL$. We next verify that the operation is well defined. Let $H_1, H_2 \in HDL$. For each $a \in L$, $H_1(a), H_2(a)$. Since $(L, +)$ is an abelian group, the sum $H_1(a) + H_2(a)$ is uniquely determined in L. Therefore, the operation " $\oplus$ " is well defined and HDL is closed under " $\oplus$ ".■

Proposition 1 shows that, if $[H_1(a), H_2(b)] + [H_2(a), H_1(b)] = 0$ for all $a, b \in L$, the operation " $\oplus$ " is a binary operation on HDL. To demonstrate this fact, we examine the Lie ring $M_{2\times2}(\mathbb{R})$.

Example 7. We define $H_1, H_2: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$, where $H_1(X) = trace(X)I$ and $H_2(X) = -X$, for every $X \in M_{2\times2}(\mathbb{R})$. Then $(H_1 \oplus H_2)(X) = trace(X)I - X$. For arbitrary matrices $A, B \in M_{2\times2}(\mathbb{R})$, since $trace([A, B]) = 0$, we obtain $(H_1 \oplus H_2)([A, B]) = -[A, B]$. Because $trace(A)I$ and $trace(B)I$ are scalar matrices, they commute with every matrix in $M_{2\times2}(\mathbb{R})$, and therefore $[(H_1 \oplus H_2)(A), (H_1 \oplus H_2)(B)] = [-A, -B] = [A, B]$,

$[(H_1 \oplus H_2)(A), B] = [-A, B] = -[A, B]$, and $[A, (H_1 \oplus H_2)(B)] = [A, -B] = -[A, B]$. Hence,

$[(H_1 \oplus H_2)(A), (H_1 \oplus H_2)(B)] + [(H_1 \oplus H_2)(A), B] + [A, (H_1 \oplus H_2)(B)] = (H_1 \oplus H_2)([A, B])$.

Thus, $H_1 \oplus H_2$ is a homoderivation on the Lie ring $M_{2\times2}(\mathbb{R})$.

The previous proposition proves the closed property of the addition of homoderivations. Now, we try to investigate the composition property of homoderivations on Lie rings.

Proposition 2. *For every homoderivation H_1 and H_2 on the Lie ring L, for all $a, b \in L$, where $[H_1(a), H_2(b)] + [H_2(a), H_1(b)] = 0$ and $[(H_1 \circ H_2)(a), (H_1 \oplus H_2)(b)] + [(H_1 \oplus H_2)(a), (H_1 \circ H_2)(b)] = 0$, the composition $H_1 \circ H_2$ is also a homoderivation.*

Proof. Let H_1, H_2 be arbitrary homoderivations on a ring lie L. For $a, b \in L$, where $[H_1(a), H_2(b)] + [H_2(a), H_1(b)] = 0$ and $[(H_1 \circ H_2)(a), (H_1 \oplus H_2)(b)] + [(H_1 \oplus H_2)(a), (H_1 \circ H_2)(b)] = 0$, we have

$(H_1 \circ H_2)([a, b]) = H_1(H_2([a, b])) = H_1([H_2(a), H_2(b)] + [H_2(a), b] + [a, H_2(b)])$ (by the definition of homoderivation)

$= H_1([H_2(a), H_2(b)]) + H_1([H_2(a), b]) + H_1([a, H_2(b)])$ (since H_1 is additive mapping by the definition of homoderivation)

$= [H_1(H_2(a)), H_1(H_2(b))] + [H_1(H_2(a)), H_2(b)] + [H_2(a), H_1(H_2(b))] + [H_1(H_2(a)), H_1(b)] + [H_1(H_2(a)), b] + [H_2(a), H_1(b)] + [H_1(a), H_1(H_2(b))] + [H_1(a), H_2(b)] + [a, H_1(H_2(b))]$

(by the definition of homoderivation)

$= ([H_1(H_2(a)), H_1(H_2(b))] + [H_1(H_2(a)), b] + [a, H_1(H_2(b))]) + ([H_1(a), H_2(b)] + [H_2(a), H_1(b)]) + ([H_1(H_2(a)), H_1(b)] + [H_1(H_2(a)), H_2(b)]) + ([H_1(a), H_1(H_2(b))] + [H_2(a), H_1(H_2(b))])$

$= [H_1(H_2(a)), H_1(H_2(b))] + [H_1(H_2(a)), b] + [a, H_1(H_2(b))] + [H_1(a), H_2(b)] + [H_2(a), H_1(b)] + [H_1(H_2(a)), H_1(b) + H_2(b)] + [H_1(a) + H_2(a), H_1(H_2(b))]$ (since bilinear property)

$= [(H_1 \circ H_2)(a), (H_1 \circ H_2)(b)] + [(H_1 \circ H_2)(a), b] + [a, (H_1 \circ H_2)(b)] + [H_1(a), H_2(b)] + [H_2(a), H_1(b)] + [(H_1 \circ H_2)(a), (H_1 \oplus H_2)(b)] + [(H_1 \oplus H_2)(a), (H_1 \circ H_2)(b)]$ (by the definition of operation " $\circ$ " and " $\oplus$ ")

$= [(H_1 \circ H_2)(a), (H_1 \circ H_2)(b)] + [(H_1 \circ H_2)(a), b] + [a, (H_1 \circ H_2)(b)] + 0$ (by assumption)

$= [(H_1 \circ H_2)(a), (H_1 \circ H_2)(b)] + [(H_1 \circ H_2)(a), b] + [a, (H_1 \circ H_2)(b)]$

Therefore, $H_1 \circ H_2 \in HDL$. We next verify that the operation is well defined. Let $H_1, H_2 \in HDL$. For each $a \in L$, we have that $H_1(H_2(a)) \in L$. Hence, $H_1 \circ H_2$ is a well-defined mapping from L to L. Thus, the composition $\mathrm{H}_1 \circ \mathrm{H}_2$ is also a homoderivation. ■

From Proposition 2, the composition of two homoderivations is again a homoderivation on L under the assumptions that $[H_1(a), H_2(b)] + [H_2(a), H_1(b)] = 0$ and $[(H_1 \circ H_2)(a), (H_1 \oplus H_2)(b)] + [(H_1 \oplus H_2)(a), (H_1 \circ H_2)(b)] = 0$ for all $H_1, H_2 \in HDL$ and $a, b \in L$. Thus, under these assumptions, the operation " $\circ$ " defines a binary operation on HDL. To illustrate this result, we provide the following example.

Example 8. Let $L = M_{2\times 2}(\mathbb{R})$ with the Lie bracket defined by $[X, Y] = XY - YX$ for any $X, Y \in L$. Define two mappings $H_1, H_2: L \to L$, where $H_1(X) = trace(X)I$ and $H_2(X) = 0_{2\times 2}$, for any $X \in L$. From the definition of operations " $\oplus$ " and " $\circ$ ", it follows that $(H_1 \oplus H_2)(X) = trace(X)I$ and $(H_1 \circ H_2)(X) = 0_{2\times 2}$. Hence, for every $A, B \in L$, we obtain $[H_1(a), H_2(b)] + [H_2(a), H_1(b)] = 0_{2\times 2}$, $[(H_1 \circ H_2)(\mathrm{A}), (H_1 \oplus H_2)(\mathrm{B})] = 0_{2\times 2}$, and $[(H_1 \oplus H_2)(A), (H_1 \circ H_2)(B)] = 0_{2\times 2}$. Therefore, the condition in Proposition 2 is satisfied. Next, we check that $H_1 \circ H_2 \in HDL$. Observe that $(H_1 \circ H_2)([A, B]) = 0_{2\times 2}$, and on the right-hand side, we also have $[(H_1 \circ H_2)(A), (H_1 \circ H_2)(B)] + [(H_1 \circ H_2)(A), B] + [A, (H_1 \circ H_2)(B)] = 0_{2\times 2}$. Thus, $H_1 \circ H_2$ is a homoderivation on L.

From Proposition 1 and Proposition 2, we can observe that the set of all homoderivations (HDL) on a Lie ring (L) does not directly form a ring structure. Therefore, to construct a ring, a subset of HDL with an additional condition that must be satisfied can be considered. This condition is different from the condition for forming a ring from the set of derivations on a Lie ring, which is explained in [7], [8]. This condition can be defined as the RH property that satisfies the subset of HDL is closed over the composition and addition operations of HDL, and allows the construction of a ring structure. These conditions arise because the addition operation " $\oplus$ " and composition operation " $\circ$ " on homoderivations, do not always keep the homoderivations structure. In other words, if H_1 and H_2 are homoderivations on L, then $H_1 \oplus H_2$ or $H_1 \circ H_2$ may not satisfy the definition of homoderivations. As a result, the set of all homoderivations is not guaranteed to be closed under these operations, and therefore does not directly form a ring. The RH property is introduced to ensure that a subset of HDL remains closed under addition and composition. With this property, algebraic operations are closed and well defined within the subset, allowing the formation of a ring structure from homoderivations

Definition 3. Let $HL \subseteq HDL$. We say that HL satisfies the RH property if for every $H_1, H_2 \in HL$ and for all $a, b \in L$, $[H_1(a), H_2(b)] + [H_2(a), H_1(b)] = 0$ and $[(H_1 \circ H_2)(a), (H_1 \oplus H_2)(b)] + [(H_1 \oplus H_2)(a), (H_1 \circ H_2)(b)] = 0$.

From Definition 3, if a subset of HDL satisfies the RH property, then that subset can be constructed into an algebraic structure. This is shown in Theorem 1 below.

Theorem 1. *Given a lie ring $(L, +, [,])$ and the set $HDL = \{H: L \to L | H \text{ homoderivation on } L\}$. If $HL \subseteq HDL$ be a nonempty subset that is closed under operations "$\oplus$" and "$\circ$", and also satisfies the RH property, then $(HL, \oplus, \circ)$ is a ring.*

Proof. Let $H_1, H_2, H_3 \in HL$ and $a \in L$. We obtain

$$\begin{aligned}(H_1 \oplus (H_2 \oplus H_3))(a) &= H_1(a) + (H_2 + H_3)(a)\\ &= H_1(a) + (H_2(a) + H_3(a))\\ &= (H_1(a) + H_2(a)) + H_3(a)\end{aligned}$$

$$= ((H_1 \oplus H_2) \oplus H_3)(a).$$

Consider $e: L \to L$ defined by $e(a) = 0$ for every $a \in L$. It is easy to see that $e \in HDL$. For all $H \in HL$ we have

$$(H \oplus e)(a) = H(a) + e(a) = H(a) \text{ and } (e \oplus H)(a) = e(a) + H(a) = H(a).$$

Since L is an additive group, every element has an additive inverse. For each $H \in HL$, we define $(-H)(a) = -(H(a))$, for all $a \in L$. To verify that $-H \in HL$, we must check that $-H$ is also a homoderivation. By applying the RH property, in particular, the case $H_1 = H_2 = H$, we obtain $2[H(x), H(y)] = 0$, which shows that $-H$ satisfies the required condition. Therefore, $-H \in HL$ and we have

$$\big(\mathrm{H} \oplus (-H)\big)(a) = 0 = e(a) \text{ and } \big((-H) \oplus H\big)(a) = 0 = e(a).$$

For the commutativity, we can show that for every $H_1, H_2 \in HL$,

$$(H_1 \oplus H_2)(a) = H_1(a) + H_2(a) = H_2(a) + H_1(a) = (H_2 \oplus H_1)(a).$$

Now, we will prove that $(HL,\circ)$ is a semigroup. Let $H_1, H_2, H_3 \in HL$ and $a \in L$. Then we obtain

$$\begin{aligned}\big(H_1 \circ (H_2 \circ H_3)\big)(a) &= H_1((H_2 \circ H_3)(a)) \\ &= H_1(H_2(H_3(a))) \\ &= (H_1 \circ H_2)(H_3(a)) \\ &= ((H_1 \circ H_2) \circ H_3)(a).\end{aligned}$$

And last, we will prove that the operations " $\oplus$ " and " $\circ$ " on HL satisfy left and right distributive laws. Let $H_1, H_2, H_3 \in HL$ and $a \in L$. We have

$$\begin{aligned}(H_1 \circ (H_2 \oplus H_3))(a) &= H_1((H_2 \oplus H_3)(a)) \\ &= H_1(H_2(a) + H_3(a)) \\ &= H_1\big(H_2(a)\big) \oplus H_1(H_3(a)) \\ &= (H_1 \circ H_2)(a) \oplus (H_1 \circ H_3)(a) \\ ((H_1 \oplus H_2) \circ H_3)(a) &= (H_1 \oplus H_2)(H_3(a)) \\ &= H_1\big(H_3(a)\big) \oplus H_2(H_3(a)) \\ &= (H_1 \circ H_3)(a) \oplus (H_2 \circ H_3)(a).\end{aligned}$$

Hence, $(HL,\oplus,\circ)$ is a ring. ■

Example 9. Let $L = M_{2\times 2}(R)$ with the Lie bracket defined by $[A, B] = AB - BA$ for all $A, B \in M_{2\times 2}(R)$. Let HDL be the set of all homoderivations on L. Define $HL = \left\{H_t: L \to L \mid H_t(X) = \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} X \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}^{-1}, t \in \mathbb{R}, t \neq 0, X \in L\right\}$. From **Example 6**, $H_t(X)$ is a homoderivation, because $\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} \in GL_2(\mathbb{R})$.

However, HL does not satisfy the RH property. Let $H_{t_1}, H_{t_2} \in HL$ with $t_1 \neq t_2$ and $A = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $B = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. We obtain:

$$H_t(A) = \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} A \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}^{-1} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{t}\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{t}\\ 0 & 0\end{pmatrix},$$

$$H_t(B) = \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} B \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}^{-1} = \begin{pmatrix}0 & 0\\ t & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{t}\end{pmatrix} = \begin{pmatrix}0 & 0\\ t & 0\end{pmatrix}.$$

Consequently,

$$\left[H_{t_1}(A), H_{t_2}(B)\right] = \begin{pmatrix}0 & \frac{1}{t_1}\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ t_2 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ t_2 & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{t_1}\\ 0 & 0\end{pmatrix} = \begin{pmatrix}\frac{t_2}{t_1} & 0\\ 0 & -\frac{t_2}{t_1}\end{pmatrix},$$

$$[H_{t_2}(A), H_{t_1}(B)] = \begin{pmatrix} 0 & \frac{1}{t_2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ t_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ t_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{t_2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{t_1}{t_2} & 0 \\ 0 & -\frac{t_1}{t_2} \end{pmatrix}.$$

Thus,

$$[H_{t_1}(A), H_{t_2}(B)] + [H_{t_2}(A), H_{t_1}(B)] = \begin{pmatrix} \frac{t_2}{t_1} + \frac{t_1}{t_2} & 0 \\ 0 & -\left(\frac{t_2}{t_1} + \frac{t_1}{t_2}\right) \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the RH property is not satisfied, and hence HL is not a ring.

In Theorem 1, it has been proven that HL forms a ring. Next, we will show that HL also forms a Lie ring using the bracket operator [,] defined in that structure.

Theorem 2. Let $(HL, \oplus, \circ)$ be a ring. The bracket operator $[,]$ on HL defined as $[H_1, H_2] = (H_1 \circ H_2) - (H_2 \circ H_1)$, for all $H_1, H_2 \in HL$. The ring HL with the bracket operator $[,]$ is a Lie ring.

Proof. Let L be a Lie ring and $HL \subseteq HDL$ where $(HL, \oplus, \circ)$ is a ring. We define a bracket operation on HL by $[H_1, H_2] = (H_1 \circ H_2) - (H_2 \circ H_1)$, for all $H_1, H_2 \in HL$. We will show that HL is a Lie ring with a bracket operator "$[,]$".

1. Let $H \in HL$ and $a \in L$. We have

   $$\begin{aligned}[H,H](a) &= H(H(a)) - H(H(a)) \\ &= 0_{HDL}\end{aligned}$$

   Hence, the bracket operation "$[,]$" satisfies the anti-commutative property on HL.

2. Let $H_1, H_2, H_3 \in HL$ and $a \in L$. We can show that it satisfies the right-normed Jacobi identity,

   $[H_1,[H_2,H_3]](a) + [H_2,[H_3,H_1]](a) + [H_3,[H_1,H_2]](a)$

   $= H_1[H_2,H_3] - [H_2,H_3]H_1 + H_2[H_3,H_1] - [H_3,H_1]H_2 + H_3[H_1,H_2] - [H_1,H_2]H_3$

   $= H_1(H_2H_3 - H_3H_2) - (H_2H_3 - H_3H_2)H_1 + H_2(H_3H_1 - H_1H_3) - (H_3H_1 - H_1H_3)H_2 + H_3(H_1H_2 - H_2H_1) - (H_1H_2 - H_2H_1)H_3$

   $= H_1H_2H_3 - H_1H_3H_2 - H_2H_3H_1 + H_3H_2H_1 + H_2H_3H_1 - H_2H_1H_3 - H_3H_1H_2 + H_1H_3H_2$

   $+ H_3H_1H_2 - H_3H_2H_1 - H_1H_2H_3 + H_2H_1H_3$

   $= (H_1H_2H_3 - H_1H_2H_3) + (-H_1H_3H_2 + H_1H_3H_2) + (-H_2H_3H_1 + H_2H_3H_1)$

   $+ (H_3H_2H_1 - H_3H_2H_1) + (-H_2H_1H_3 + H_2H_1H_3) + (-H_3H_1H_2 + H_3H_1H_2)$

   $= 0_{HL},$

   and it also satisfies the left-normed Jacobi identity,

   $[[H_1,H_2],H_3](a) + [[H_2,H_3],H_1](a) + [[H_3,H_1],H_2](a)$

   $= [H_1,H_2]H_3 - H_3[H_1,H_2] + [H_2,H_3]H_1 - H_1[H_2,H_3] + [H_3,H_1]H_2 - H_2[H_3,H_1]$

   $= (H_1H_2 - H_2H_1)H_3 - H_3(H_1H_2 - H_2H_1) + (H_2H_3 - H_3H_2)H_1 - H_1(H_2H_3 - H_3H_2) + (H_3H_1 - H_1H_3)H_2 - H_2(H_3H_1 - H_1H_3)$

   $= H_1H_2H_3 - H_2H_1H_3 - H_3H_1H_2 + H_3H_2H_1 + H_2H_3H_1 - H_3H_2H_1 - H_1H_2H_3 + H_1H_3H_2 + H_3H_1H_2 - H_1H_3H_2 - H_2H_3H_1 + H_2H_1H_3$

   $= (H_1H_2H_3 - H_1H_2H_3) + (-H_2H_1H_3 + H_2H_1H_3) + (-H_3H_1H_2 + H_3H_1H_2) + (H_3H_2H_1 - H_3H_2H_1) + (H_2H_3H_1 - H_2H_3H_1) + (H_1H_3H_2 - H_1H_3H_2)$

   $= 0_{HL}.$

Therefore, HL is a Lie ring with respect to the bracket operation "$[,]$". ■

Example 10. Let $(M_{2\times 2}(\mathbb{R}), +, [,])$ be the Lie ring with the bracket $[A,B] = AB - BA$, for all $A, B \in M_{2\times 2}(\mathbb{R})$. For $c \in \mathbb{R}$, define a mapping $H_c : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ where $H_c(A) = c\, trace(A)I$, where I is the 2×2 identity matrix. Analog with **Example 5**, we can show that H_c is a homoderivation. Consider the subset $HL = \{H_c \mid c \in \mathbb{R}\} \subseteq HDL$. We will show that $(HL, \oplus, \circ)$. Let $H_c, H_d \in HL$ and $A, B \in M_{2\times 2}(\mathbb{R})$. Since I commutes with every matrix, we obtain $[H_c(A), H_d(B)] = [c\, trace(A)I, d\, trace(B)I] = 0$. Similarly, $[H_d(A), H_c(B)] = 0$. Hence $[H_c(A), H_d(B)] + [H_d(A), H_c(B)] = 0$. Next, $(H_c \oplus H_d)(A) = H_c(A) + H_d(A) = (c+d)\, trace(A)I$, and $(H_c \circ H_d)(A) = H_c\big(H_d(A)\big) = c\, trace(d\, trace(A)I)I = 2cd\, trace(A)I$. Thus, $(H_c \oplus H_d)(A)$ and $(H_c \circ H_d)(A)$ are scalar multiples of I. Consequently, $[(H_c \circ H_d)(A), (H_c \oplus H_d)(B)] = 0$ and $[(H_c \oplus H_d)(A), (H_c \circ H_d)(B)] = 0$. Therefore, $[(H_c \circ H_d)(A), (H_c \oplus H_d)(B)] + [(H_c \oplus H_d)(A), (H_c \circ H_d)(B)] = 0$. Hence, HL satisfies the RH property. Moreover, HL is closed under " $\oplus$ " and " $\circ$ ". Since " $\oplus$ " is associative and commutative, the zero mapping H_0 acts as the additive identity, and for every $H_c \in HL$ there exists an additive inverse $H_{-c} \in HL$. Furthermore, the composition $\circ$ is associative and distributes over " $\oplus$ ". Therefore, all ring axioms are satisfied. Hence, $(HL, \oplus, \circ)$ forms a ring.

Now, we analyze how repeated composition acts on the Lie bracket on L. The following theorem provides a general formula for the n-th power action of homoderivations on the Lie bracket.

Theorem 3. *Let L be a Lie ring and let $H : L \to L$ be a homoderivation on the Lie ring L. If n is a non-negative integer, then for each n it satisfies that*

$$H^n([x,y]) = \sum_{r+s+t=n} \frac{n!}{r!\,s!\,t!}[H^{r+s}(x), H^{r+t}(y)],$$

for all $x, y \in L$.

Proof. We prove this using mathematical induction. Suppose $n = 0$. Since $H \in HDL$ and H^0 denotes the identity element with respect to the composition of functions, it follows that H^0 is the identity map. Hence,

$$H^0([x,y]) = [x,y] = [H^0(x), H^0(y)] = \sum_{\substack{r,s,t\geq 0 \\ r+s+t=0}} \frac{0!}{0!\,0!\,0!}[H^{0+0}(x), H^{0+0}(y)]$$

This shows that the statement holds for $n = 0$.

Now, assume that the statement is true for $n = k$ with $k \geq 1$, that is,

$$H^k([x,y]) = \sum_{r+s+t=k} \frac{k!}{r!\,s!\,t!}[H^{r+s}(x), H^{r+t}(y)].$$

We will prove the case $n = k+1$. Observe that $H^{k+1}([x,y]) = H\Big(H^k([x,y])\Big)$. Thus, using the assumption for $n = k$ with $k \geq 1$, we obtain:

$$\begin{aligned}
H^{k+1}([x,y]) &= H\left(\sum_{r+s+t=k} \frac{k!}{r!\,s!\,t!}[H^{r+s}(x), H^{r+t}(y)]\right) \\
&= \sum_{r+s+t=k} \frac{k!}{r!\,s!\,t!} H[H^{r+s}(x), H^{r+t}(y)] \\
&= \sum_{r+s+t=k} \frac{k!}{r!\,s!\,t!} \big([H^{r+s+1}(x), H^{r+t+1}(y)] + [H^{r+s+1}(x), H^{r+t}(y)] \\
&\qquad + [H^{r+s}(x), H^{r+t+1}(y)]\big) \\
&= \sum_{r+s+t=k} \frac{k!}{r!\,s!\,t!}[H^{r+s+1}(x), H^{r+t+1}(y)] + \sum_{r+s+t=k} \frac{k!}{r!\,s!\,t!}[H^{r+s+1}(x), H^{r+t}(y)] \\
&\qquad + \sum_{r+s+t=k} \frac{k!}{r!\,s!\,t!}[H^{r+s}(x), H^{r+t+1}(y)]
\end{aligned}$$

$$= \sum_{r+s+t=k} \frac{k!\,(r+1)}{(r+1)!\,s!\,t!}[H^{r+s+1}(x), H^{r+t+1}(y)] + \sum_{r+s+t=k} \frac{k!\,(s+1)}{r!\,(s+1)!\,t!}[H^{r+s+1}(x), H^{r+t}(y)] + \sum_{r+s+t=k} \frac{k!\,(t+1)}{r!\,s!\,(t+1)!}[H^{r+s}(x), H^{r+t+1}(y)]$$

$$= \sum_{u+s+t=k+1} \frac{k!\,u}{u!\,s!\,t!}[H^{u+s}(x), H^{u+t}(y)] + \sum_{r+v+t=k+1} \frac{k!\,v}{r!\,v!\,t!}[H^{r+v}(x), H^{r+t}(y)] + \sum_{r+s+w=k+1} \frac{k!\,w}{r!\,s!\,w!}[H^{r+s}(x), H^{r+w}(y)]$$

$$= \sum_{a+b+c=k+1} \frac{k!\,a}{a!\,b!\,c!}[H^{a+b}(x), H^{a+c}(y)] + \sum_{a+b+c=k+1} \frac{k!\,b}{a!\,b!\,c!}[H^{a+b}(x), H^{a+c}(y)] + \sum_{a+b+c=k+1} \frac{k!\,c}{a!\,b!\,c!}[H^{a+b}(x), H^{a+c}(y)]$$

$$= \sum_{a+b+c=k+1} \left(\frac{k!\,a}{a!\,b!\,c!}[H^{a+b}(x), H^{a+c}(y)] + \frac{k!\,b}{a!\,b!\,c!}[H^{a+b}(x), H^{a+c}(y)] + \frac{k!\,c}{a!\,b!\,c!}[H^{a+b}(x), H^{a+c}(y)]\right)$$

$$= \sum_{a+b+c=k+1} \frac{k!}{a!\,b!\,c!}\left([H^{a+b}(x), H^{a+c}(y)]\right)(a+b+c)$$

$$= \sum_{a+b+c=k+1} \frac{k!}{a!\,b!\,c!}\left([H^{a+b}(x), H^{a+c}(y)]\right)(k+1)$$

$$= \sum_{a+b+c=k+1} \frac{(k+1)!}{a!\,b!\,c!}\left([H^{a+b}(x), H^{a+c}(y)]\right)$$

$$= \sum_{r+s+t=k+1} \frac{(k+1)!}{r!\,s!\,t!}\left([H^{r+s}(x), H^{r+t}(y)]\right)$$

Thus, by mathematical induction, the proposition is proved.■

In Theorem 3, H^n denotes the composition of n times of H. The formula in Theorem 3 provides an expression for $H^n([x,y])$ in terms of brackets involving high-order iterations of H applied to x and y. In particular, the appearance of multinomial coefficients reflects the combinatorial pattern of interactions between the composition operation " $\circ$ " in HL and the Lie bracket on L. The following is an example to clarify Theorem 3.

Example 11. From the definition of homoderivation on a Lie ring, for $n = 2$, we obtain

$H^2([x,y]) = H(\,[H(x),H(y)] + [H(x),y] + [x,H(y)]) = H([H(x),H(y)]) + H([H(x),y]) + H([x,H(y)])$. By applying the definition of H to each term, we obtain:

$$H^2([x,y]) = [H^2(x),H(y)] + 2[H(x),H^2(y)] + 2[H(x),H(y)] + [H^2(x),y] + 2[H(x),H(y)] + [x,H^2(y)].$$

From Theorem 3, for $n = 2$, all possible values of r, s, and t are given with $r + s + t = 2$.

For $r = 2, s = 0, t = 0$, then $[H^2(x),H^2(y)]$;

for $r = 1, s = 1, t = 0$, then $2[H^2(x),H(y)]$;

for $r = 1, s = 0, t = 1$, then $2[H(x),H^2(y)]$;

for $r = 0, s = 2, t = 0$, then $[H^2(x),y]$;

for $r=0, s=1, t=1$, then $2[H(x), H(y)]$;

for $r=0, s=0, t=2$, then $[x, H^2(y)]$.

Thus, $H^2([x,y]) = [H^2(x), H(y)] + 2[H(x), H^2(y)] + 2[H(x), H(y)] + [H^2(x), y] + 2[H(x), H(y)] + [x, H^2(y)]$.

In this research, Proposition 1 and Proposition 2 define the conditions under which the operation " $\oplus$ " and " $\circ$ " are binary operations. Furthermore, Theorem 1 and Theorem 2 show that the set of homoderivations on a Lie ring forms a ring structure, and hence a Lie ring. Additionally, the iteration of homoderivations on a Lie ring up to order n is observed in Theorem 3 for structural properties.

4. CONCLUSION

This research begins with the problem of how to construct a set of homoderivations on a Lie ring L in order to construct a ring structure, and then a Lie ring. The main research objective is to investigate the conditions that must be satisfied to construct the set of all homoderivatives on a Lie ring, denoted by $HDL = \{H : L \to L \mid H \text{ is a homoderivation on the Lie ring } L\}$, into an algebraic structure. The results show that with the addition operation " $\oplus$ " and the composition operation " $\circ$ ", the set HDL is not always closed. Therefore, the set $HL \subseteq HDL$ is introduced, which satisfies the RH property, which is $[H_1(a), H_2(b)] + [H_2(a), H_1(b)] = 0$ and $[(H_1 \circ H_2)(a), (H_1 \oplus H_2)(b)] + [(H_1 \oplus H_2)(a), (H_1 \circ H_2)(b)] = 0$, for every $H_1, H_2 \in HL$ and $a, b \in L$. With this condition, it is proven that $(HL, \oplus, \circ)$ is a ring. Furthermore, with the bracket operator $[H_1, H_2] = (H_1 \circ H_2) - (H_2 \circ H_1)$, the triple $(HL, \oplus, [,])$ is a Lie ring. In addition, this research also formulates the results of homoderivation iteration, that is $H^n([x,y]) = \sum_{r+s+t=n} \frac{n!}{r!s!t!} [H^{r+s}(x), H^{r+t}(y)]$, for all $x, y \in L$. The formula provides knowledge related to homoderivations on Lie rings, particularly in iteration. In this paper, we have developed a concept of homoderivations as an algebraic structure that constructs a ring and a Lie ring.

Author Contributions

Suffi Nuralesa: Conceptualization, Formal Analysis, Funding Acquisition, Investigation, Writing – Original Draft. Galang Riki Ramadhan: Formal Analysis, Methodology, Writing – Original Draft. Carolus Pasha Lazuardi Jituprasojo Hatmakelana: Formal Analysis, Validation, Writing – Review and Editing. Nurul Fadhilah: Methodology, Project Administration, Resources. Benediktus Panji Pradipta: Supervision, Validation, Writing – Review and Editing. Nikken Prima Puspita: Supervision, Validation, Writing – Review and Editing. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare no competing interests.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools (e.g., ChatGPT) were used solely for language refinement, including grammar, spelling, and clarity. The scientific content, analysis, interpretation, and conclusions were developed entirely by the authors. All final text was reviewed and approved by the authors.

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