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# THE EIGENVECTORS OF REDUCIBLE MATRICES OVER MIN-PLUS ALGEBRA

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*ABSTRACT*

*The eigenvalues and the eigenvectors of a matrix are fundamental concepts in max-plus algebra, especially for square matrices. A communication graph is classified as strongly connected or not, corresponding to irreducible and reducible matrices, respectively. This study aims to develop methods for determining the eigenvectors of reducible matrices over min-plus algebra. Before determining the eigenvectors, we need to determine the eigenvalues. The methodology used is based on the Frobenius Normal Form and graph condensation to decompose the reducible matrix into spectral classes. The eigenvalues can be found using the graphical method. Eigenvectors can be determined for each corresponding eigenvalue. Once the eigenvectors are known, we can find the set of eigenvectors as a map of a matrix. The main results include a characterization of eigenvector bases for each spectral class, criteria for the existence of finite eigenvectors, and an analysis of computational complexity. An isomorphic correspondence exists between min-plus algebra and max-plus algebra. Hence, the eigenvectors of reducible matrices over min-plus algebra are able to be evaluated according to a concept from eigenvectors of reducible matrices over max-plus algebra. This study is limited to min-plus algebra and does not extend to interval min-plus algebra.*



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## 1. INTRODUCTION

Max-plus algebra is an algebraic structure defined on the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$, where $\mathbb{R}$ denotes the set of all real numbers and the epsilon ($\varepsilon$) represents negative infinity ($-\infty$) [1]. This structure is equipped with two binary operations: maximum (denoted by $\oplus$) and addition (denoted by $\otimes$). The algebraic system ($\mathbb{R}_\varepsilon$,$\oplus$,$\otimes$) is commonly denoted by $\mathbb{R}_\varepsilon$. It forms an idempotent and commutative semiring [2]. Furthermore, a matrix whose entries are elements of $\mathbb{R}_\varepsilon$ is referred to as a matrix over max-plus algebra. The set of all such matrices of order $m \times n$ is denoted by $\mathbb{R}_\varepsilon^{m\times n}$.

The eigenvalues and the eigenvectors of a matrix are fundamental concepts in max-plus algebra, especially for square matrices. A communication graph is used to represent a square matrix in max-plus algebra [3]. Communication graphs are categorized into two distinct categories: strongly connected and not strongly connected. An irreducible matrix represents a strongly connected graph, whereas a reducible matrix represents a graph that is not strongly connected.

In max-plus algebra, the process of determining the eigenvalues differs between irreducible and reducible matrices. An irreducible matrix has one eigenvalue. A reducible matrix can have more than one eigenvalue. There are various approaches to computing the eigenvalues and eigenvectors of matrices over max-plus algebra. Three common methods used for finding the eigenvalues and eigenvectors of irreducible matrices over max-plus algebra are the Kleene Star Algorithm [4], [5], [6], the Modified Kleene Star Algorithm [7], [8], and the Power Algorithm [9], [10]. For reducible matrices, the process involves transforming the matrix into Frobenius Normal Form (FNF) [11], followed by solving it using the eigenmode method [12], [13] or the graphical method [14], [15].

In addition to max-plus algebra, min-plus algebra represents another structure worth examining [16], [17]. Min-plus algebra is an algebraic structure defined on the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$, where $\mathbb{R}$ denotes the set of all real numbers and the epsilon ($\varepsilon'$) represents positive infinity ($+\infty$). This structure incorporates two binary operations: minimum (denoted by $\oplus'$) and addition (denoted by $\otimes$). The algebraic system ($\mathbb{R}_{\varepsilon'}$,$\oplus'$,$\otimes$) is commonly denoted by $\mathbb{R}_{\varepsilon'}$ [18]. An isomorphic correspondence exists between min-plus algebra and max-plus algebra [19], [20]. Therefore, the determination of eigenvectors of reducible matrices in min-plus algebra can follow the concept applied to eigenvectors of reducible matrices in max-plus algebra. Several studies have investigated the eigenvalues and eigenvectors of matrices over min-plus algebra. The determination of eigenvalues and eigenvectors for irreducible matrices is presented in [21]. The evaluation of eigenvalues and eigenvectors for reducible matrices with eigenmode is discussed in [22].

Unlike previous studies [4] and [9] that discussed eigenvalues and eigenvectors of irreducible matrices over min-plus algebra, this study aims to determine the eigenvectors of reducible matrices over min-plus algebra using a Frobenius Normal Form and a graph-based method. Then, existing studies have not systematically addressed the determination of eigenvectors for reducible matrices in min-plus algebra using Frobenius Normal Form and condensation digraphs. To this day, most research has focused on irreducible matrices, whereas studies on reducible matrices have mainly employed the eigenmode method [22], without utilizing the Frobenius Normal Form and condensation digraph framework.

This study enhances the understanding of min-plus algebra by establishing a systematic framework for identifying the eigenvectors of reducible matrices through the use of Frobenius Normal Forms and condensation digraphs. Additionally, the research delineates the eigenvector basis for each spectral class and formulates criteria for the existence of finite eigenvectors. This study establishes a basis for subsequent research in min-plus algebra, specifically in the expansion of eigenvector analysis to interval min-plus matrices and its applications in optimization, scheduling, and network analysis.

## 2. RESEARCH METHODS

The research method is divided into three sections. Two sections consist of preliminary definitions and theorems about min-plus algebra. One section outlines the workflow of this research.

### 2.1 Min-Plus Algebra

Min-plus algebra is an algebraic structure constructed on the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$, with $\mathbb{R}$ being the set of all real numbers and the epsilon ($\varepsilon'$) represents positive infinity ($+\infty$). Min-plus algebra endowed with minimum ($\oplus'$) and addition ($\otimes$) [17]. Moreover, ($\mathbb{R}_{\varepsilon'}$,$\oplus'$,$\otimes$) is referred to as $\mathbb{R}_{\varepsilon'}$. Given $a, b \in \mathbb{R}_{\varepsilon'}$, then the basic operations in min-plus algebra are formulated as follows.

$$a \oplus' b = \min(a, b) \text{ and}$$
$$a \otimes b = a + b.$$

Since two operations ($\oplus'$) and ($\otimes$) are commutative and associative; it follows that we have

$$a \oplus' (b \oplus' c) = (a \oplus' b) \oplus' c,$$
$$a \otimes (b \otimes c) = (a \otimes b) \otimes c,$$
$$a \oplus' b = b \oplus' a,$$
$$a \otimes b = b \otimes a,$$

for all $a, b, c \in \mathbb{R}_{\varepsilon'}$.

In $\mathbb{R}_{\varepsilon'}$, the identity element over the operation $\oplus'$ is $\varepsilon' = +\infty$, we possess

$$a \oplus' \varepsilon' = \varepsilon' \oplus' a = \min(a, +\infty) = a,$$

and the identity element over the operation $\otimes$ is $e = 0$, we possess

$$e \otimes a = a \otimes e = 0 + a = a.$$

The operation $\otimes$ is distributive over $\oplus$:

$$(a \otimes b) \oplus' (a \otimes c) = a \otimes (b \oplus' c).$$

The identity element $\varepsilon' = +\infty$ behaves in the role of an absorbing element for $\otimes$ in relation to $\oplus'$:

$$a \otimes \varepsilon' = \varepsilon' \otimes a = \varepsilon'.$$

The operation $\oplus'$ is idempotent:

$$a \oplus' a = \min\{a, a\} = a.$$

Given $\mathbb{R}_{\varepsilon'}^{m \times n}$ as the set of all $m \times n$ matrices in which every entry belongs to $\mathbb{R}_{\varepsilon'}$, where $m$ and $n$ are positive integers. The matrix $I_n \in \mathbb{R}_{\varepsilon'}^{n \times n}$ is defined such that $(I_n)_{ii} = 0$ for all $i$ and $(I_n)_{ij} = \varepsilon'$ for all $i, j$ with $i \neq j$. Following the definition in [18], several operations on $\mathbb{R}_{\varepsilon'}^{m \times n}$ are introduced as follows.

a. Given $A, B \in \mathbb{R}_{\varepsilon'}^{m \times n}$ then

$$[A \oplus' B]_{ij} = a_{ij} \oplus' b_{ij} = \min\{a_{ij}, b_{ij}\}.$$

b. Given $A \in \mathbb{R}_{\varepsilon'}^{m \times k}$ and $B \in \mathbb{R}_{\varepsilon'}^{k \times n}$ then

$$[A \otimes B]_{ij} = \oplus'^{k}_{l=1} \left(a_{il} \otimes b_{lj}\right) = \min_{l=1,2,..,k}\{a_{il} + b_{lj}\}.$$

c. Given $A \in \mathbb{R}_{\varepsilon'}^{m \times n}$ then

$$[k \otimes A]_{ij} = k \otimes a_{ij}.$$

d. Given $A \in \mathbb{R}_{\varepsilon'}^{m \times n}$, we define its transpose $A^T \in \mathbb{R}_{\varepsilon'}^{n \times m}$ from $A$ by $[A^T]_{ij} = a_{ji}$.

### 2.2 The Eigenvalues of Reducible Matrices over Min-Plus Algebra

This section discusses the eigenvalues of reducible matrices over min-plus algebra.

**Definition 1.** Let $A \in \mathbb{R}_{\varepsilon'}^{n \times n}$ be a square matrix. If $\lambda \in \mathbb{R}_{\varepsilon'}$ is a scalar and $x \in \mathbb{R}_{\varepsilon'}^{n}$ is a vector that includes a minimum of a finite element, so that

$$A \otimes x = \lambda \otimes x,$$

with $\lambda$ is known as an eigenvalue of $A$ with associated eigenvector $x$.

**Definition 2.** Let $\sigma$ denote the collection of every cycle of a graph $G(A)$. Then

$$\lambda = \min_{p\in\sigma}\frac{|p|_w}{|p|_l}.$$

**Definition 3.** Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and $\lambda$ known as in Definition 2, then matrix $A_\lambda$ can be interpreted as follows

$$A_\lambda = \lambda(A)^{-1} \otimes A = a_{ij} - \lambda,$$

matrix $\Gamma(A_\lambda)$ can be written as

$$\Gamma(A_\lambda) = A_\lambda \oplus' A_\lambda^{\otimes 2} \oplus' \dots \oplus' A_\lambda^{\otimes n}.$$

Based on [4] and [15], we can give several definitions below based on the isomorphic properties between max-plus algebra and min-plus algebra.

**Definition 4.** Suppose $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and $\lambda \in \mathbb{R}_{\varepsilon'}$. We will interpret

$$\begin{aligned} V(A,\lambda) &= \{x \in \mathbb{R}_{\varepsilon'} | A \otimes x = \lambda \otimes x\},\\ \Lambda(A) &= \{\lambda \in \mathbb{R}_{\varepsilon'} | V(A,\lambda) \neq \{\varepsilon'\}\},\\ V(A) &= \bigcup_{\lambda\in\Lambda(A)} V(A,\lambda),\\ V^+(A,\lambda) &= V(A,\lambda) \cap \mathbb{R}^n,\\ V^+(A) &= V(A) \cap \mathbb{R}^n. \end{aligned}$$

To put it another way, $V(A,\lambda)$ means the collection of eigenvectors of $A$ that correspond to the value $\lambda$, $\Lambda(A)$ means the collection of all potential eigenvalues of $A$, and $V(A)$ is the collection of $A'$s eigenvectors. Furthermore, the collection of finite eigenvectors of $A$ that correspond to the value $\lambda$ is denoted by $V^+(A,\lambda)$ and the collection of all finite eigenvectors of $A$ is denoted by $V^+(A)$.

**Definition 5.** Every matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ may be changed to a Frobenius Normal Form (FNF) in linear time by permuting the columns and rows concurrently, i.e.,

$$P^{-1} \otimes A \otimes P = \begin{pmatrix} A_{11} & \varepsilon' & \cdots & \varepsilon' \\ A_{21} & A_{22} & \cdots & \varepsilon' \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{pmatrix}$$

where $P$ is a permutation matrix and $A_{11}, \dots, A_{rr}$ are irreducible square submatrices of $A$.

The permutation matrix $P$ is constructed based on the decomposition of the communication graph associated with matrix $A$. The diagonal blocks are uniquely determined up to a simultaneous permutation of their rows and columns. However, their order is not uniquely determined. This is because each form is essentially determined by the strongly connected components of the digraph $A$.

**Definition 6.** Suppose $A$ becomes FNF, defines the condensation digraph of $A$, i.e.,

$$C(A) = (\{N_1, N_2, \dots, N_r\}, \{(N_i, N_j) | \exists k \in N_i, \exists l \in N_j \ s.t. \, a_{lk} < \varepsilon\}).$$

Keep in mind that a condensation graph can consist of multiple initial and final classes. The vertices of $C(A)$ without any inbound arcs are known as the initial classes. On the other hand, the vertices of $C(A)$ without any outbound arcs are known as the final classes.

**Theorem 1.** *Suppose $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ becomes FNF. So*

$$\Lambda(A) = \{\lambda(A_{jj}) | \lambda(A_{jj}) = \min_{N_i \to N_j} \lambda(A_{ii})\}.$$

**Definition 7.** Suppose $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ becomes FNF. If

$$\lambda(A_{jj}) = \min_{N_i \to N_j} \lambda(A_{ii})$$

then $A_{jj}$ will be called as spectral.

**Corollary 1.** *Every initial class of $C(A)$ is spectral.*

**Example 1.** Consider the following matrix

$$K = \begin{pmatrix} 1 & +\infty & +\infty & 2 & +\infty & +\infty \\ +\infty & 3 & 3 & 4 & +\infty & +\infty \\ +\infty & 4 & 5 & +\infty & +\infty & +\infty \\ 1 & +\infty & +\infty & 2 & +\infty & +\infty \\ +\infty & 5 & 4 & +\infty & 2 & +\infty \\ 1 & +\infty & +\infty & 6 & +\infty & 0 \end{pmatrix}$$

The matrix $K$ above can be transformed into FNF through permutations along rows and columns as follows.

$$P^{-1} \otimes K \otimes P = \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & 0 & +\infty & +\infty \\ +\infty & +\infty & 0 & +\infty & +\infty & +\infty \\ +\infty & 0 & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & 0 \\ +\infty & +\infty & +\infty & +\infty & 0 & +\infty \end{pmatrix} \otimes \begin{pmatrix} 1 & +\infty & +\infty & 2 & +\infty & +\infty \\ +\infty & 3 & 3 & 4 & +\infty & +\infty \\ +\infty & 4 & 5 & +\infty & +\infty & +\infty \\ 1 & +\infty & +\infty & 2 & +\infty & +\infty \\ +\infty & 5 & 4 & +\infty & 2 & +\infty \\ 1 & +\infty & +\infty & 6 & +\infty & 0 \end{pmatrix}$$
$$\otimes \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & 0 & +\infty & +\infty \\ +\infty & +\infty & 0 & +\infty & +\infty & +\infty \\ +\infty & 0 & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & 0 \\ +\infty & +\infty & +\infty & +\infty & 0 & +\infty \end{pmatrix}$$

We can obtain the matrix $K$ that becomes FNF, i.e.,

$$K' = \begin{pmatrix} 1 & 2 & +\infty & +\infty & +\infty & +\infty \\ 1 & 2 & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & 5 & 4 & +\infty & +\infty \\ +\infty & 4 & 3 & 3 & +\infty & +\infty \\ 1 & 6 & +\infty & +\infty & 0 & +\infty \\ +\infty & +\infty & 4 & 5 & +\infty & 2 \end{pmatrix}$$

Because $K'$ becomes the FNF, then we may conclude that

$$K'_{11} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \qquad K'_{22} = \begin{pmatrix} 5 & 4 \\ 3 & 3 \end{pmatrix}, \qquad K'_{33} = (0), \qquad K'_{44} = (2),$$

with $\lambda(K'_{11}) = 1,\ \lambda(K'_{22}) = 3,\ \lambda(K'_{33}) = 0$, and $\lambda(K'_{44}) = 2$.

The representation of the condensation digraph of the matrix $K'$ is as follows.

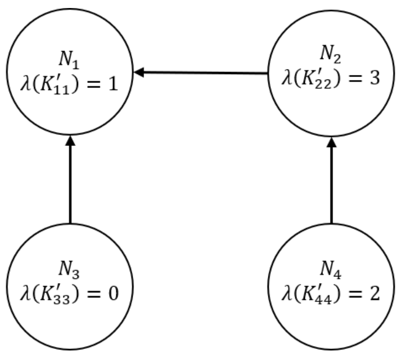


**Figure 1. The Condensation Digraph of the Matrix $K'$**

It follows that $N_4$ is spectral because it is the initial class. Additionally, $N_3$ is spectral since

$$\lambda(K'_{33}) = \min_{r=1,\ldots,4} \lambda(K'_{rr}).$$

Since $N_1$ may be reached from $N_3$ and $\lambda(K'_{33}) < \lambda(K'_{11})$, then $N_1$ is not a spectral class. Additionally, $N_2$ may be reached from $N_4$ and $\lambda(K'_{44}) < \lambda(K'_{22})$, then $N_2$ is not a spectral class.

Thus, we have $\Lambda(K) = \{0, 2\}$.

### 2.3 Research Methods

The method employed in this study begins with a literature review of sources, including books and articles [16], [17], [18], as well as other writings on min-plus algebra, matrices over min-plus algebra, and the eigenvalues of reducible matrices over min-plus algebra. Furthermore, this study also incorporates literature addressing eigenvalues and eigenvectors over max-plus algebra, like [2] and [4].

In this study, the workflow for a conceptual study about the eigenvectors of reducible matrices over min-plus algebra is described as follows.

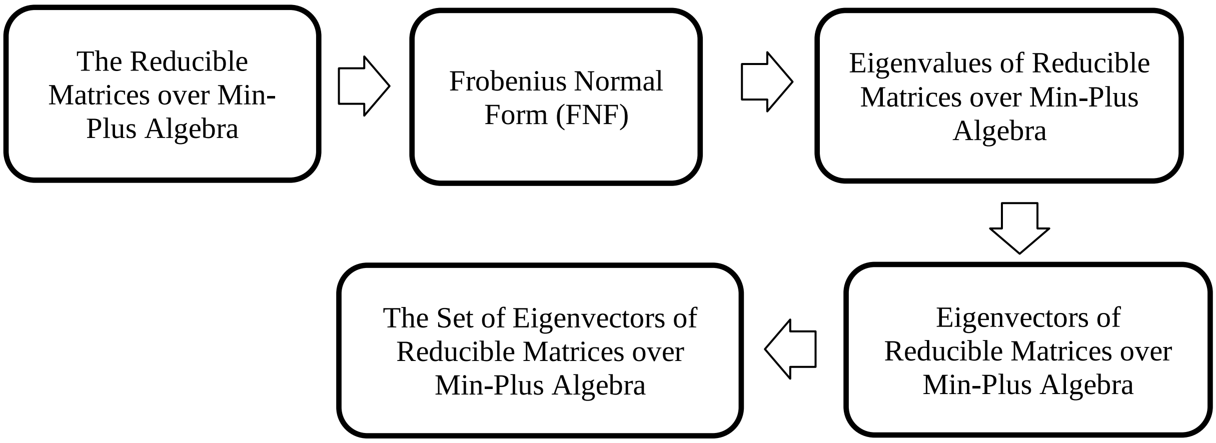


**Figure 2. Workflow for The Conceptual Study of Eigenvectors of Reducible Matrices over Min-Plus Algebra**

Based on the above workflow, we obtain the following algorithm.

1. Identify the reducible matrix in min-plus algebra.
2. Transform the matrix into Frobenius Normal Form (FNF) and build the condensation digraph to reveal its block structure.
3. Identify spectral classes and determine the eigenvalues for each class using the graphical method.
4. Compute the corresponding eigenvectors.
5. Form the set of eigenvectors as a min-plus combination of all obtained eigenvectors.

## 3. RESULTS AND DISCUSSION

This section discusses the determination of eigenvectors of reducible matrices over min-plus algebra by investigating whether the concept of eigenvectors in max-plus algebra can be stated within min-plus algebra. In this study [2], [4], and [15], there are definitions and theorems about eigenvectors in max-plus algebra. A related concept is employed by altering the coefficients and the min-plus operations. Therefore, some definitions and theorems share similarities with [2], [4], and [15]. However, there is a distinction in terms of the matrix space.

**Definition 8.** Suppose $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ becomes FNF. Denote its classes by $N_1, N_2, \dots, N_r$ and let $R = \{1, 2, \dots, r\}$. Let $\lambda \in \Lambda(A)$ is a fixed eigenvalue and $\lambda < \varepsilon'$.

$$I(\lambda) = \{i \in R \mid \lambda(N_i) = \lambda, N_i \text{ spectral}\}$$

and

$$E(\lambda) = \bigcup_{i \in I(\lambda)} E(A_{ii}) = \left\{ j \in N \mid g_{jj} = 0, j \in \bigcup_{i \in I(\lambda)} N_i \right\}$$

with $\Gamma(\lambda^{-1} \otimes A) = (g_{jj})$.

**Definition 9.** Suppose $i, j \in E(\lambda)$, then nodes $i$ and $j$ are considered $\lambda$-equivalent when $i$ and $j$ lie on the same cycle of cycle mean $\lambda$. This equivalence is represented by $i \sim_\lambda j$.

Note that if $\lambda = \lambda(A)$ then $\sim_\lambda$ is similar to $\sim$. Furthermore, we give a theorem on the basis of eigenvectors.

**Theorem 2.** *Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and $\lambda \in \Lambda(A)$ with $\lambda < \varepsilon'$.Then, for every $j \in E(\lambda)$, we have $g_j \in \mathbb{R}_{\varepsilon'}^{n}$. By choosing one representative $g_j$ from each $\sim_\lambda$ equivalence class, we obtain a basis of $V(A,\lambda)$.*

**Proof**. We write $M = \bigcup_{i\in I(\lambda)} N_i$. Using the definition of FNF and W.L.O.G, we assume that $A$ can be transformed into the form

$$\begin{pmatrix} * & \varepsilon' \\ * & A[M] \end{pmatrix}.$$

Hence, $\Gamma(\lambda^{-1} \otimes A)$ is

$$\begin{pmatrix} * & \varepsilon' \\ * & C \end{pmatrix}$$

with $C = \Gamma(\lambda(A[M]))^{-1} \otimes A[M])$, so that $\lambda = \lambda(A[M])$. Therefore, $\sim_\lambda$ equivalence for $A$ corresponds to the $\sim$-equivalence for $A[M]$. ■

The above theorem shows that the eigenvectors associated with a particular eigenvalue $\lambda$ of a reducible matrix can be completely characterized by focusing on the submatrix $A[M]$, which contains the relevant spectral classes. The following is a corollary of the theorem on the basis of eigenvectors.

**Corollary 2.** *For every $\lambda \in \Lambda(A)$ such that $\lambda < \varepsilon'$, a basis of $V(A,\lambda)$ can be computed in $O(k^3)$ operations, with k equal to $|I(\lambda)|$. Moreover, we have*

$$V(A,\lambda) = \{\Gamma(\lambda^{-1} \otimes A) \otimes z | z \in \mathbb{R}_{\varepsilon'}^{n},\ z_j = \varepsilon' for\ all\ j \notin E(\lambda)\}.$$

**Corollary 3.** *Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and $\lambda \in \Lambda(A)$. The dimension of $V(A,\lambda)$ is $r_\lambda$ then there is a column $\mathbb{R}$-astic matrix $G_\lambda \in \mathbb{R}_{\varepsilon'}^{n\times r_\lambda}$ such that*

$$V(A,\lambda) = \{G_\lambda \otimes z; z \in \mathbb{R}_{\varepsilon'}^{n\times r_\lambda}\}.$$

**Proof**. The eigenvectors $V(A,\lambda)$ can be found as follows

If $I(\lambda) = \{j\}$ then define

$$M_2 = \bigcup_{N_i \to S} N_i, \quad M_1 = N - M_2.$$

Hence

$$V(A,\lambda) = \{x; x[M_1] = \varepsilon', x[M_2] \in V^+(A[M_2])\}.$$

If the set $I(\lambda)$ consists of more than one index, then the same process has to be repeated for each nonempty subset of $I(\lambda)$, that is for each $J \subseteq I(\lambda), J \neq \emptyset$ we set $S = \bigcup_{j\in J} N_j$ and

$$M_2 = \bigcup_{N_i \to S} N_i, \quad M_1 = N - M_2.$$

This method is not practical for computing all eigenvectors, as it involves examining every subset, which is computationally infeasible. However, it does introduce another criterion for the existence of finite eigenvectors. Here is a theorem on finite eigenvectors.

**Theorem 3.** *Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$. The set $V^+(A)$ is nonempty if and only if $\lambda(A)$ is the eigenvalue of every final class (across all superblocks).*

**Proof**. By using the Spectral Theorem (Theorem 1), it is obtained that if the case where $I(\lambda)$ comprises several indices, for each $J \subseteq I(\lambda),\ J \neq \emptyset$, can be formed a set $S = \bigcup_{j\in J} N_j$ and

$$M_2 = \bigcup_{N_i \to S} N_i, \quad M_1 = N - M_2.$$

The set $M_1$ must be an empty set for a finite eigenvector to exist. This implies that every class in $S$ must be accessible from all classes within the superblock. Such a condition can only be satisfied if $S$ is composed entirely of final classes, ensuring that no additional classes are reachable from them.

Again, if all final classes share the same eigenvalue $\lambda(A)$, then $\lambda = \lambda(A)$. In this case, the set $S$ coincides with the collection of all final classes, which are reachable from every class in the superblock. Consequently, $M_1$ becomes empty, and a finite eigenvector is obtained. ■

The theorem states that the existence of finite eigenvectors in terms of the structure of final classes. The set $V^+(A)$ is nonempty only if all final classes have the same eigenvalue $\lambda(A)$. If there exists a final class that has an eigenvalue different from $\lambda(A)$, then there is no subset $S \subseteq I(\lambda)$ such that all classes in the superblock can reach $S$. This means that $M_1 \neq \emptyset$. Consequently, no finite eigenvector can be constructed. Conversely, if all final classes equal the eigenvalue $\lambda(A)$, then the set S coincides with the collection of all final classes, which are reachable from every class in the superblock. This ensures that $M_1 = \emptyset$. Hence, a finite eigenvector exists.

**Corollary 4.** *The set $V^+(A) = \emptyset$ if and only if there exists at least one final class has an eigenvalue greater than $\lambda(A)$.*

**Proposition 1.** *Suppose $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ becomes the FNF with classes $N_1, N_2, \dots, N_r$, and let $R = \{1, 2, \dots, r\}$. Define $K = \{1, 2, \dots, k\}$ and $\Lambda(A) = \{\lambda_i | i \in K\}$, where $k \leq r$. Then, we can construct $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(k)}$ such that for every $i \in K$,*

$$Im\left(\Gamma^{(i)}\right) = V(A, \lambda_i).$$

**Proof**. Because the cardinality of the set of eigenvalues is bounded above by the number of classes, we immediately have $k \leq r$. Suppose $\lambda_i \in \Lambda(A)$, Corollary 1 gives the definition

$$V(A, \lambda_i) = \left\{ \Gamma\left(\lambda_i^{-1} \otimes A\right) \otimes z \mid z \in \mathbb{R}_{\varepsilon'}^{n},\ z_j = \varepsilon' \text{ for all } j \notin E(\lambda) \right\}.$$

Let $E(\lambda_i) = \{e_i, e_2, \dots, e_l\}$ and suppose $\Gamma^{(i)} = \left(g_{ij}\right)$, where the columns are $g_1, g_2, \dots, g_l$. If $g_m$ is corresponds to the $e_m^{th}$ column of $\Gamma\left(\lambda_i^{-1} \otimes A\right)$, then we obtain

$$Im\left(\Gamma^{(i)}\right) = V(A, \lambda_i). ■$$

**Example 2.** Observe the matrix shown below

$$L = \begin{pmatrix} 2 & +\infty & +\infty & +\infty & 3 & +\infty & +\infty & +\infty \\ +\infty & 2 & +\infty & +\infty & +\infty & +\infty & 5 & +\infty \\ +\infty & +\infty & 1 & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & 1 & +\infty & 5 & +\infty & 3 \\ 3 & +\infty & 3 & +\infty & 2 & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & 1 & 2 & 2 & +\infty & 2 \\ +\infty & 3 & 1 & +\infty & 4 & +\infty & 2 & +\infty \\ +\infty & +\infty & 4 & 7 & +\infty & 0 & +\infty & 3 \end{pmatrix}.$$

The matrix $L$ above may be changed into FNF by permuting the columns and rows of $L$ concurrently. We can set

$$P = \begin{pmatrix} +\infty & 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & 0 & +\infty & +\infty & +\infty \\ 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & 0 & +\infty & +\infty \\ +\infty & +\infty & 0 & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & 0 & +\infty \\ +\infty & +\infty & +\infty & 0 & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & 0 \end{pmatrix}$$

$$\text{and } P^{-1} = \begin{pmatrix} +\infty & +\infty & 0 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & 0 & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & 0 & +\infty \\ +\infty & 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & 0 & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & 0 & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & 0 \end{pmatrix}.$$

Furthermore, we can obtain the matrix $L$ in Frobenius Normal Form (FNF). This is done by rearranging the matrix into block upper triangular form, where each diagonal block corresponds to a strongly connected component, i.e.,

$$L' = P^{-1} \otimes L \otimes P = \begin{pmatrix} 1 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & 2 & 3 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 3 & 3 & 2 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 1 & +\infty & 4 & 2 & 3 & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & 5 & 2 & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & 1 & 5 & 3 \\ +\infty & +\infty & 2 & +\infty & +\infty & 1 & 2 & 2 \\ 4 & +\infty & +\infty & +\infty & +\infty & 7 & 0 & 3 \end{pmatrix}.$$

Because $L'$ becomes the FNF, then we may conclude that

$$L'_{11} = (1), \qquad L'_{22} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \qquad L'_{33} = \begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix}, \qquad L'_{44} = \begin{pmatrix} 1 & 5 & 3 \\ 1 & 2 & 2 \\ 7 & 0 & 3 \end{pmatrix}$$

with $\lambda(L'_{11}) = 1,\ \lambda(L'_{22}) = 2,\ \lambda(L'_{33}) = 2$, and $\lambda(L'_{44}) = 1$.

The representation of the condensation digraph of the matrix $L'$ is as follows.

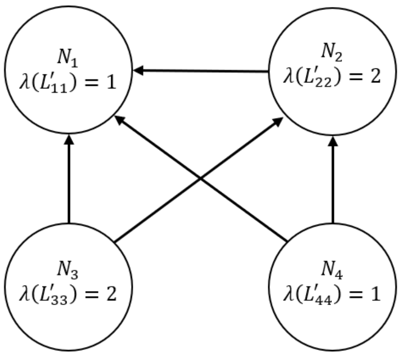


**Figure 3. The Condensation Digraph of the Matrix $L'$**

It follows that $N_3$ and $N_4$ are spectral as they are initial classes. Additionally, $N_1$ is spectral since

$$\lambda(L'_{11}) = \min_{r=1,\dots,4} \lambda(L'_{rr}).$$

Since $N_2$ may be reached from $N_4$ and $\lambda(L'_{44}) < \lambda(L'_{22})$, then $N_2$ is not a spectral class. Therefore, we have $\Lambda(L) = \{1, 2\}$. We denote $\lambda_1 = 1$ and $\lambda_2 = 2$. Next, we determine all the eigenvectors of each eigenvalue.

First, we choose $\lambda = \lambda_1 = 1$ so that we get $I(\lambda_1) = \{1, 4\}$ and $E(\lambda_1) = \{1, 6, 7, 8\}$. Furthermore, we calculate $\Gamma(\lambda_1^{-1} \otimes L) = \Gamma(L_{\lambda_1})$. We get the calculation results of $L_{\lambda_1}, L_{\lambda_1}^2, \dots, L_{\lambda_1}^8$ as follows.

$$L_{\lambda_1} = \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & 1 & 2 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 2 & 2 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & +\infty & 3 & 1 & 2 & +\infty & +\infty & \infty \\ +\infty & +\infty & +\infty & 4 & 1 & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & 0 & 4 & 2 \\ +\infty & +\infty & 1 & +\infty & +\infty & 0 & 1 & 1 \\ 3 & +\infty & +\infty & +\infty & +\infty & 6 & -1 & 2 \end{pmatrix},$$

$$L_{\lambda_1}^2 = \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ 4 & 2 & 3 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 2 & 3 & 2 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & 5 & 4 & 2 & 3 & +\infty & +\infty & +\infty \\ 4 & +\infty & 7 & 5 & 2 & +\infty & +\infty & +\infty \\ 5 & +\infty & 5 & +\infty & +\infty & 0 & 1 & 2 \\ 3 & 3 & 2 & +\infty & +\infty & 0 & 0 & 2 \\ 3 & +\infty & 0 & +\infty & +\infty & -1 & 0 & 0 \end{pmatrix},$$

$$L_{\lambda_1}^3 = \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ 4 & 3 & 4 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 2 & 4 & 3 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & 6 & 5 & 3 & 4 & +\infty & +\infty & +\infty \\ 4 & 9 & 8 & 6 & 3 & +\infty & +\infty & +\infty \\ 5 & 7 & 2 & +\infty & +\infty & 0 & 1 & 2 \\ 3 & 4 & 1 & +\infty & +\infty & 0 & 1 & 1 \\ 2 & 2 & 1 & +\infty & +\infty & -1 & -1 & 1 \end{pmatrix},$$

$$L_{\lambda_1}^4 = \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ 4 & 4 & 5 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 2 & 5 & 4 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & 7 & 6 & 4 & 5 & +\infty & +\infty & +\infty \\ 4 & 10 & 9 & 7 & 4 & +\infty & +\infty & +\infty \\ 4 & 4 & 2 & +\infty & +\infty & 0 & 1 & 2 \\ 3 & 3 & 2 & +\infty & +\infty & 0 & 0 & 2 \\ 2 & 3 & 0 & +\infty & +\infty & -1 & 0 & 0 \end{pmatrix},$$

$$L_{\lambda_1}^5 = \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ 4 & 5 & 6 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 2 & 6 & 5 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & 8 & 7 & 5 & 6 & +\infty & +\infty & +\infty \\ 4 & 11 & 10 & 8 & 5 & +\infty & +\infty & +\infty \\ 4 & 4 & 2 & +\infty & +\infty & 0 & 1 & 2 \\ 3 & 4 & 1 & +\infty & +\infty & 0 & 1 & 1 \\ 2 & 2 & 1 & +\infty & +\infty & -1 & -1 & 1 \end{pmatrix},$$

$$L_{\lambda_1}^6 = \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ 4 & 6 & 7 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 2 & 7 & 6 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & 9 & 8 & 6 & 7 & +\infty & +\infty & +\infty \\ 4 & 12 & 11 & 9 & 6 & +\infty & +\infty & +\infty \\ 4 & 4 & 2 & +\infty & +\infty & 0 & 1 & 2 \\ 3 & 3 & 2 & +\infty & +\infty & 0 & 0 & 2 \\ 2 & 3 & 0 & +\infty & +\infty & -1 & 0 & 0 \end{pmatrix},$$

$$L_{\lambda_1}^7 = \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ 4 & 7 & 8 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 2 & 8 & 7 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & 10 & 9 & 7 & 8 & +\infty & +\infty & +\infty \\ 4 & 13 & 12 & 10 & 7 & +\infty & +\infty & +\infty \\ 4 & 4 & 2 & +\infty & +\infty & 0 & 1 & 2 \\ 3 & 4 & 1 & +\infty & +\infty & 0 & 1 & 1 \\ 2 & 2 & 1 & +\infty & +\infty & -1 & -1 & 1 \end{pmatrix}, \text{and}$$

$$L_{\lambda_1}^8 = \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ 4 & 8 & 9 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 2 & 9 & 8 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & 11 & 10 & 8 & 9 & +\infty & +\infty & +\infty \\ 4 & 14 & 13 & 11 & 8 & +\infty & +\infty & +\infty \\ 4 & 4 & 2 & +\infty & +\infty & 0 & 1 & 2 \\ 3 & 3 & 2 & +\infty & +\infty & 0 & 0 & 2 \\ 2 & 3 & 0 & +\infty & +\infty & -1 & 0 & 0 \end{pmatrix}.$$

Next, we calculate $\Gamma(\lambda_1^{-1} \otimes L) = L_{\lambda_1} \oplus' L_{\lambda_1}^2 \oplus' \dots \oplus' L_{\lambda_1}^8$. We get the matrix as follows.

$$\Gamma(\lambda_1^{-1} \otimes L) = \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ 4 & 1 & 2 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 2 & 2 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & 5 & 3 & 1 & 2 & +\infty & +\infty & +\infty \\ 4 & 9 & 7 & 4 & 1 & +\infty & +\infty & +\infty \\ 4 & 4 & 2 & +\infty & +\infty & 0 & 1 & 2 \\ 3 & 3 & 1 & +\infty & +\infty & 0 & 0 & 1 \\ 2 & 2 & 0 & +\infty & +\infty & -1 & -1 & 0 \end{pmatrix}.$$

From the matrix $\Gamma(\lambda_1^{-1} \otimes L)$, we get $\Gamma^{(1)} = \left(g_1^{(1)}, g_2^{(1)}, g_3^{(1)}\right)$, where $g_1^{(1)}$ is the 1st column, $g_2^{(1)}$ is the 6th column, and $g_2^{(1)}$ is the 7th column of $\Gamma(\lambda_1^{-1} \otimes L)$. Next, we obtain the matrix as follows.

$$\Gamma^{(1)} = \begin{pmatrix} 0 & +\infty & +\infty \\ 4 & +\infty & +\infty \\ 2 & +\infty & +\infty \\ 0 & +\infty & +\infty \\ 4 & +\infty & +\infty \\ 4 & 0 & 1 \\ 3 & 0 & 0 \\ 2 & -1 & -1 \end{pmatrix}.$$

Next, we choose $\lambda = \lambda_2 = 2$ so that we get $I(\lambda_2) = \{3\}$ and $E(\lambda_2) = \{4, 5\}$. Furthermore, we calculate $\Gamma(\lambda_2^{-1} \otimes L) = \Gamma\left(L_{\lambda_2}\right)$. We get the calculation results of $L_{\lambda_2}, L_{\lambda_2}^2, \dots, L_{\lambda_2}^8$ as follows.

$$L_{\lambda_2} = \begin{pmatrix} -1 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & 0 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 1 & 1 & 0 & +\infty & +\infty & +\infty & +\infty & +\infty \\ -1 & +\infty & 2 & 0 & 1 & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & 3 & 0 & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & -1 & 3 & 1 \\ +\infty & +\infty & 0 & +\infty & +\infty & -1 & 0 & 0 \\ 2 & +\infty & +\infty & +\infty & +\infty & 5 & -2 & 1 \end{pmatrix},$$

$$L_{\lambda_2}^2 = \begin{pmatrix} -2 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ 2 & 0 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty \\ 0 & 1 & 0 & +\infty & +\infty & +\infty & +\infty & +\infty \\ -2 & 3 & 2 & 0 & 1 & +\infty & +\infty & +\infty \\ 2 & +\infty & 5 & 3 & 0 & +\infty & +\infty & +\infty \\ 3 & +\infty & 3 & +\infty & +\infty & -2 & -1 & 0 \\ 1 & 1 & 0 & +\infty & +\infty & -2 & -2 & 0 \\ 1 & +\infty & -2 & +\infty & +\infty & -3 & -2 & -2 \end{pmatrix},$$

$$L_{\lambda_2}^{3}=\begin{pmatrix}-3 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty\\ 1 & 0 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -1 & 1 & 0 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -3 & 3 & 2 & 0 & 1 & +\infty & +\infty & +\infty\\ 1 & 6 & 5 & 3 & 0 & +\infty & +\infty & +\infty\\ 2 & 4 & -1 & +\infty & +\infty & -3 & -2 & -1\\ 0 & 1 & -2 & +\infty & +\infty & -3 & -2 & -2\\ -1 & -1 & -2 & +\infty & +\infty & -4 & -4 & -2\end{pmatrix},$$

$$L_{\lambda_2}^{4}=\begin{pmatrix}-4 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty\\ 0 & 0 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -2 & 1 & 0 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -4 & 3 & 2 & 0 & 1 & +\infty & +\infty & +\infty\\ 0 & 6 & 5 & 3 & 0 & +\infty & +\infty & +\infty\\ 0 & 0 & -2 & +\infty & +\infty & -4 & -3 & -2\\ -1 & -1 & -2 & +\infty & +\infty & -4 & -4 & -2\\ -2 & -1 & -4 & +\infty & +\infty & -5 & -4 & -4\end{pmatrix},$$

$$L_{\lambda_2}^{5}=\begin{pmatrix}-5 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty\\ -1 & 0 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -3 & 1 & 0 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -5 & 3 & 2 & 0 & 1 & +\infty & +\infty & +\infty\\ -1 & 6 & 5 & 3 & 0 & +\infty & +\infty & +\infty\\ -1 & -1 & -3 & +\infty & +\infty & -5 & -4 & -3\\ -2 & -1 & -4 & +\infty & +\infty & -5 & -4 & -4\\ -3 & -3 & -4 & +\infty & +\infty & -6 & -6 & -4\end{pmatrix},$$

$$L_{\lambda_2}^{6}=\begin{pmatrix}-6 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty\\ -2 & 0 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -4 & 1 & 0 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -6 & 3 & 2 & 0 & 1 & +\infty & +\infty & +\infty\\ -2 & 6 & 5 & 3 & 0 & +\infty & +\infty & +\infty\\ -2 & -2 & -4 & +\infty & +\infty & -6 & -5 & -4\\ -3 & -3 & -4 & +\infty & +\infty & -6 & -6 & -4\\ -4 & -3 & -6 & +\infty & +\infty & -7 & -6 & -6\end{pmatrix},$$

$$L_{\lambda_2}^{7}=\begin{pmatrix}-7 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty\\ -3 & 0 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -5 & 1 & 0 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -7 & 3 & 2 & 0 & 1 & +\infty & +\infty & +\infty\\ -3 & 6 & 5 & 3 & 0 & +\infty & +\infty & +\infty\\ -3 & -3 & -5 & +\infty & +\infty & -7 & -6 & -5\\ -4 & -3 & -6 & +\infty & +\infty & -7 & -6 & -6\\ -5 & -5 & -6 & +\infty & +\infty & -8 & -8 & -6\end{pmatrix}, \text{and}$$

$$L_{\lambda_2}^{8}=\begin{pmatrix}-8 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty\\ -4 & 0 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -6 & 1 & 0 & +\infty & +\infty & +\infty & +\infty & +\infty\\ -8 & 3 & 2 & 0 & 1 & +\infty & +\infty & +\infty\\ -4 & 6 & 5 & 3 & 0 & +\infty & +\infty & +\infty\\ -4 & -4 & -6 & +\infty & +\infty & -8 & -7 & -6\\ -5 & -5 & -6 & +\infty & +\infty & -8 & -8 & -6\\ -6 & -5 & -8 & +\infty & +\infty & -9 & -8 & -8\end{pmatrix}.$$

Next, we calculate $\Gamma(\lambda_2^{-1}\otimes L)=L_{\lambda_2}\oplus' L_{\lambda_2}^{2}\oplus' \dots \oplus' L_{\lambda_2}^{8}$. We get the matrix as follows.

$$\Gamma(\lambda_2^{-1} \otimes L) = \begin{pmatrix} -8 & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ -4 & 0 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty \\ -6 & 1 & 0 & +\infty & +\infty & +\infty & +\infty & +\infty \\ -8 & 3 & 2 & 0 & 1 & +\infty & +\infty & +\infty \\ -4 & 6 & 5 & 3 & 0 & +\infty & +\infty & +\infty \\ -4 & -4 & -6 & +\infty & +\infty & -8 & -7 & -6 \\ -5 & -5 & -6 & +\infty & +\infty & -8 & -8 & -6 \\ -6 & -5 & -8 & +\infty & +\infty & -9 & -8 & -8 \end{pmatrix}$$

From the matrix $\Gamma(\lambda_2^{-1} \otimes L)$, we get $\Gamma^{(1)} = \left(g_1^{(2)}, g_2^{(2)}\right)$, where $g_1^{(2)}$ is the 4th column and $g_2^{(2)}$ is the 5th column of $\Gamma(\lambda_2^{-1} \otimes L)$. Next, we obtain the matrix as follows.

$$\Gamma^{(2)} = \begin{pmatrix} +\infty & +\infty \\ +\infty & +\infty \\ +\infty & +\infty \\ 0 & 1 \\ 3 & 0 \\ +\infty & +\infty \\ +\infty & +\infty \\ +\infty & +\infty \end{pmatrix}.$$

Using Proposition 1, we can determine the set of all eigenvectors. We do this by converting it into the task of determining the image of the following mapping

$$x \to \Gamma^{(i)} \otimes x, i = 1, 2, 3, \ldots$$

so that we get the set of an image is $Im\left(\Gamma^{(i)}\right) = V(A, \lambda_i)$, for $i = 1, 2, 3, \ldots$ . Next, we determine the set of all eigenvectors of each eigenvalue of the matrix $L$. For $\lambda_1 = 1$, we get a mapping

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \to \begin{pmatrix} 0 & +\infty & +\infty \\ 4 & +\infty & +\infty \\ 2 & +\infty & +\infty \\ 0 & +\infty & +\infty \\ 4 & +\infty & +\infty \\ 4 & 0 & 1 \\ 3 & 0 & 0 \\ 2 & -1 & -1 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, c_1, c_2, c_3 \in \mathbb{R}_{\varepsilon'}.$$

Thus, we obtain

$$\begin{aligned} V(L, \lambda_1) = \{ c_1 \otimes (0 \quad 4 \quad 2 \quad 0 \quad 4 \quad 4 \quad 3 \quad 2)^T \oplus' c_2 \\ \otimes (+\infty \quad +\infty \quad +\infty \quad +\infty \quad +\infty \quad 0 \quad 0 \quad -1)^T \oplus' c_3 \\ \otimes (+\infty \quad +\infty \quad +\infty \quad +\infty \quad +\infty \quad 1 \quad 0 \quad -1)^T | \, c_1, c_2, c_3 \in \mathbb{R}_{\varepsilon'} \}. \end{aligned}$$

For $\lambda_2 = 2$, we get a mapping

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \to \begin{pmatrix} +\infty & +\infty \\ +\infty & +\infty \\ +\infty & +\infty \\ 0 & 1 \\ 3 & 0 \\ +\infty & +\infty \\ +\infty & +\infty \\ +\infty & +\infty \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, c_1, c_2 \in \mathbb{R}_{\varepsilon'}.$$

Thus, we obtain

$$\begin{aligned} V(L, \lambda_2) = \{ c_1 \otimes (+\infty \quad +\infty \quad +\infty \quad 0 \quad 3 \quad +\infty \quad +\infty \quad +\infty)^T \oplus' c_2 \\ \otimes (+\infty \quad +\infty \quad +\infty \quad 1 \quad 0 \quad +\infty \quad +\infty \quad +\infty)^T | \, c_1, c_2, c_3 \in \mathbb{R}_{\varepsilon'} \}. \end{aligned}$$

## 4. CONCLUSION

Based on the results and discussion, a method for determining eigenvectors through graph has been derived. The reducible matrices in the Frobenius Normal Form (FNF) can be used to find the eigenvalues. The determination of eigenvectors for each eigenvalue can be achieved through an $O(k^3)$ operations, which is $V(A,\lambda)=\left\{\Gamma(\lambda^{-1}\otimes A)\otimes z | z\in\mathbb{R}^n_{\varepsilon'},\ z_j=\varepsilon' \text{ for all } j\notin E(\lambda)\right\}$. Furthermore, the set of all eigenvectors for each eigenvalue is obtained through a mapping $x\rightarrow\Gamma^{(i)}\otimes x, i=1,2,3,\ldots$ . From this mapping, it can be obtained that $Im(\Gamma^{(i)})=V(A,\lambda_i)$, for $i=1,2,3,\ldots$ . Readers interested in this topic may continue the research on the eigenvalues and eigenvectors for reducible matrices over min-plus interval algebra.

### Author Contributions

The first author was responsible for conceptualization, methodology, drafting the original manuscript, and validation. The second author contributed to draft preparation. All authors participated in discussions of the results and contributed to the final version of the manuscript.

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### Declarations

The authors report no known financial involvement, personal, or professional conflicts of interest that could have appeared to influence the research process, interpretation of the results, or the preparation and publication of this study.

### Declaration of Generative AI and AI-assisted Technologies

The authors declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, including writing, editing, data analysis, or the creation of tables and figures.

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