BAREKENG: Journal of Mathematics and Its Applications
December 2026 Volume 20 Issue 4 Page 3213-3222
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol20iss4pp3213-3222

DIGRUNDY NUMBER OF DIRECTED STAR, BANANA TREE, FIREWORKS, AND COCONUT TREE GRAPHS

Raventino [1*], **Fransiskus Fran** [2]

[1,2]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Tanjungpura*
Jln. Prof. Dr. H. Hadari Nawawi, Pontianak, 78124, Indonesia

*Corresponding author's e-mail: * **raventino@fmipa.untan.ac.id***

Article Info

Article History:

Received: 7th October 2025
Revised: 6th May 2026
Accepted: 7th June 2026
Available online: 1st July 2026

Keywords:

Acyclic Digraphs:
Digrundy Coloring;
Maximum Number of Colors;
Vertex Coloring.

ABSTRACT

A digrundy coloring of a graph is a vertex coloring in which every vertex assigned a higher color is adjacent to vertices assigned all smaller colors. The maximum number of colors that can be realized in such a coloring of an acyclic directed graph is called the digrundy number. This paper determines the digrundy numbers of several classes of acyclic directed graphs, namely directed star graphs, directed banana tree graphs, directed fireworks graphs, and directed coconut tree graphs. The analysis is based on structural properties of the graphs and combinatorial arguments derived from digrundy coloring constraints. The results show that the digrundy number of directed star graphs $\overrightarrow{K_{1,a}}$ is 2 under orientations where the central vertex satisfies $d_{out}(v) \neq 0$ and $d_{in}(v) \neq 0$. For directed banana tree graphs $\overrightarrow{B_{2,s}}$ with a specified orientation H', the digrundy number is 2 for $s = 2$ and 3 for $s \geq 3$. Under arbitrary orientations, directed fireworks graphs $\overrightarrow{F(2,c)}$ have digrundy number 2, while for directed coconut tree graphs $\overrightarrow{CT(h,4)}$, the digrundy number is bounded by $2 \leq dG\left(\overrightarrow{CT(h,4)}\right) \leq 3$. These findings provide exact values of the digrundy number for the graph classes considered and highlight the role of structural constraints in governing digrundy coloring behavior.

How to cite this article:

Raventino and F. Fran, "DIGRUNDY NUMBER OF DIRECTED STAR, BANANA TREE, FIREWORKS, AND COCONUT TREE GRAPHS", *BAREKENG: J. Math. & App.*, vol. 20, no. 4, pp. 3213-3222, Dec, 2026.

1. INTRODUCTION

Graph coloring is a central topic in graph theory with wide applications in optimization, scheduling, and network analysis [1], [2], [3]. Graphs provide a mathematical framework for modeling complex systems through vertices and edges [4], [5], [6]. Within this framework, vertex coloring assigns colors to vertices such that adjacent vertices receive distinct colors, forming the basis for numerous theoretical developments [7], [8], [9]. Over time, this classical notion has been extended through various constrained coloring models to better capture structural properties of graphs [10], [11], [12].

Among these extensions, stronger coloring constraints have been introduced to enforce interactions between color classes. Variants such as injective and dynamic coloring impose additional adjacency requirements and have been investigated in different combinatorial contexts [13], [14]. Another important refinement is complete coloring, which requires adjacency relations between distinct color classes and has been studied in directed graph settings [15]. These developments naturally lead to more restrictive coloring frameworks that combine structural and ordering constraints.

A prominent example is digrundy coloring, in which colors are assigned with an inherent order such that each vertex assigned a higher color is adjacent to vertices assigned all smaller colors. This concept has been formalized for digraphs, where the maximum number of colors that can be realized defines the digrundy number [16]. Existing studies have primarily examined general properties, computational aspects, and variants of Grundy-type colorings [17], [18], [19], while further work highlights their combinatorial complexity and extensions [20].

Despite these developments, the behavior of the digrundy number for structured acyclic digraphs remains only partially understood. Several graph families have been investigated from structural and applied perspectives, including star graphs [21], which have been widely used as fundamental models in graph theory [22], as well as banana tree graphs [23], fireworks graphs [24], and coconut tree graphs [25], [26]. However, their digrundy numbers have not been systematically characterized, particularly when different edge orientations are considered. In parallel, recent results on diachromatic numbers show that coloring parameters in digraphs are highly sensitive to both graph structure and orientation [27]. Although the digrundy number is related to the diachromatic number, the two parameters do not generally coincide, and their relationship is not yet fully clarified [16].

These observations motivate a closer examination of how structural configurations and edge orientations influence digrundy colorings in acyclic digraphs. In particular, it becomes essential to determine whether consistent patterns arise for specific graph families and to what extent orientation affects the attainable number of colors.

Motivated by this direction, the present study determines the digrundy numbers of several classes of acyclic directed graphs, namely directed star graphs, directed banana tree graphs, directed fireworks graphs, and directed coconut tree graphs. The analysis distinguishes between specific and arbitrary orientations, enabling a precise characterization of their impact on digrundy coloring. The main contribution of this work is the derivation of exact digrundy numbers for these graph classes, providing a clearer understanding of the interplay between graph structure, orientation, and digrundy coloring constraints.

2. RESEARCH METHODS

This study is conducted within a rigorous theoretical framework based on the structural analysis of directed acyclic graphs and the properties of digrundy coloring. Let $\vec{G}$ be a digraph with a coloring function $f: V(\vec{G}) \to \{T_1, T_2, \dots, T_n\}$. A coloring is said to be greedy if, for every vertex assigned color T_j, there exists adjacent vertices colored T_i for all $i < j$, in accordance with the definition of digrundy coloring. The determination of the digrundy number is established through a unified analytical strategy that combines constructive realization and structural limitation. A lower bound is obtained by explicitly constructing a coloring that satisfies the greedy condition for a prescribed number of colors, ensuring the existence of such a coloring. In contrast, an upper bound is derived by demonstrating that any attempt to extend the coloring to a higher number of colors necessarily violates the greedy condition. This impossibility is rigorously justified by exploiting intrinsic structural constraints of the graph, particularly adjacency limitations and degree restrictions, as formalized in Lemma 1 and Lemma 2.

Accordingly, if a digrundy coloring with m colors exists while no valid digrundy coloring can be constructed with $m+1$ colors, then it follows that $dG(\vec{G}) = m$. This principle is applied consistently across all graph classes considered, ensuring that each result is derived as an exact value supported by both constructive evidence and structural impossibility. Furthermore, the analysis explicitly incorporates arbitrary orientations. The arguments are formulated independently of edge directions and rely solely on adjacency relationships, guaranteeing that the obtained results hold uniformly for all possible orientations of the underlying graph. This orientation-invariant approach strengthens the generality of the results and ensures their applicability beyond specific directional configurations.

3. RESULTS AND DISCUSSION

This section commences with the formal definitions of several graph classes that constitute the primary focus of the study, namely star graphs, banana tree graphs, fireworks graphs, and coconut tree graphs. Subsequently, foundational concepts are introduced, including the definitions of digrundy coloring and the digrundy number, along with associated lemmas derived from these definitions. The section concludes with the determination of the digrundy numbers for all the defined graph classes.

Definition 1. [21] The graph $K_{1,a}$ with $a \in N$ and $a \geq 2$, is a star graph whose vertex and edge sets are given, respectively, by

$$V(K_{1,a}) = \{v, u_i | 1 \leq i \leq a\}$$

and

$$E(K_{1,a}) = \{vu_i | 1 \leq i \leq a\}.$$

Definition 2. [23] The graph $B_{a,s}$ where $a \geq 1$ and $s \geq 2$, is a banana tree graph whose vertex and edge sets are defined, respectively, as

$$V(B_{a,s}) = \{v, b_i, v_{ij} | 1 \leq i \leq a, 1 \leq j \leq s-1\}$$

and

$$E(B_{a,s}) = \{b_i v_{ij} | 1 \leq i \leq a, 1 \leq j \leq s-1, s \geq 2\} \cup \{v_{i1} v | 1 \leq i \leq a\}.$$

Definition 3. [24] The graph $F(a,c)$ *with* $a, c \geq 2$, is a fireworks graph whose vertex and edge sets are defined, respectively, as

$$V(F(a,c)) = \{b_i, v_{ij} | 1 \leq i \leq a, 1 \leq j \leq c-1\}$$

and

$$E(F(a,c)) = \{b_i v_{ij} | 1 \leq i \leq a, 1 \leq j \leq c-1\} \cup \{v_{i1} v_{(i+1)1} | 1 \leq i \leq a-1\}.$$

Definition 4. [25] The graph $CT(h,a)$ where $h, a \in N$, is a coconut graph whose vertex and edge sets are given, respectively, by

$$V(CT(h,a)) = \{b_i, v_j \ | 1 \leq i \leq a, 1 \leq j \leq h\}$$

and

$$E(CT(h,a)) = \{b_i b_{i+1} | 1 \leq i \leq a-1\} \cup \{b_t v_j | t = max\,\{i\}, 1 \leq i \leq a, 1 \leq j \leq h\}.$$

Definition 5. [18] A digrundy coloring of a graph G is a vertex coloring $f: V(G) \to \{T_1, T_2, \dots, T_k\}$ such that, for every j and for each $i < j$, every vertex $v \in V(G)$ with $f(v) = T_j$ has a neighbor $u \in V(G)$ with $f(u) = T_i$.

Moreover, as shown in [18], a coloring f is a digrundy coloring if and only if f satisfies the greedy coloring condition. Hence, in the subsequent discussion, digrundy coloring is used interchangeably with greedy coloring.

Lemma 1. *Consider a directed graph $\vec{G}$, and let m represent the number of distinct colors used in a digrundy coloring of $\vec{G}$. Suppose $f: V(\vec{G}) \to T$ is a digrundy coloring function, where $V(\vec{G})$ denotes the set of vertices* *and T the set of color types. Under this coloring, the number of ordered color pairs ${}_mP_2$ must satisfy the inequality ${}_mP_2 \leq |A(\vec{G})|$, where $A(\vec{G})$ is the set of edge in $\vec{G}$.*

Proof. Let f be a digrundy coloring as in Definition 5. Then, for every j and each $i < j$, every vertex colored T_j has a neighbor colored T_i. Hence, for every pair of distinct color types T_i and T_j with $i < j$, there exists an arc whose endpoints receive these colors. Thus, all pairs of distinct color types are realized on adjacent vertices, implying that f is a complete coloring; see [18]. Since a complete coloring with m color types require at least one arc for each ordered pair of distinct colors, and the number of such pairs is ${}_mP_2 = m(m-1)$, it follows that ${}_mP_2 \leq |A(\vec{G})|$ in accordance with the bound for complete colorings in [17]. ■

Lemma 2. *Let $\vec{G}$ be a directed acyclic graph, and let $f: V(\vec{G}) \to \{T_1, T_2, T_3, \dots, T_n\}$ be a digrundy coloring, where $n \in N$. If a vertex $v \in V(\vec{G})$ satisfies $d_{out}(v) + d_{in}(v) < n - 1$, then it follows that $f(v) \neq T_n$.*

Proof. Let $v \in V(\vec{G})$ satisfy $d_{out}(v) + d_{in}(v) < n - 1$. Suppose that $f(v) = T_n$. Since f is a digrundy coloring as in Definition 5, for each $i < n$, there exists a vertex $u_i \in V(\vec{G})$ such that u_i is adjacent to v and $f(u_i) = T_i$. Thus, v is adjacent to at least $n - 1$ distinct vertices. In a directed graph, each such adjacency contributes to either $d_{out}(v)$ or $d_{in}(v)$, and therefore $d_{out}(v) + d_{in}(v) \geq n - 1$. This contradicts the initial condition $d_{out}(v) + d_{in}(v) < n - 1$. Hence, $f(v) \neq T_n$. ■

Definition 6. [18] The digrundy number of a graph $\vec{G}$, denoted by $dG(\vec{G})$, is defined as the maximum number of colors that can be assigned to the vertices of $\vec{G}$ using a greedy coloring.

Example 1. Given $\overrightarrow{CT(7,4)}$ be a directed coconut tree graph as defined in Definition 4, where we know the vertex set $V\left(\overrightarrow{CT(7,4)}\right) = \{b_1, b_2, b_3, b_4, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and edge set $A\left(\overrightarrow{CT(7,4)}\right) = \{b_1b_2, b_2b_3, b_3b_4, b_4v_1, v_2b_4, b_4v_3, v_4b_4, b_4v_5, v_6b_4, b_4v_7\}$. An illustration of this graph is provided in Fig. 1.

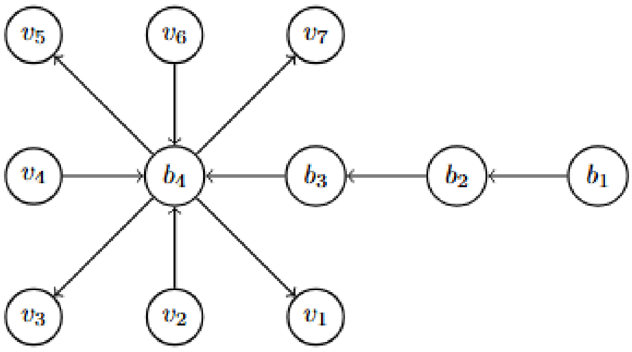

Figure 1. Coconut Tree Graph $\overrightarrow{CT(7,4)}$

We now determine the digrundy number of the directed coconut tree graph $\overrightarrow{CT(7,4)}$. First, let $T = \{T_1, T_2\}$. A digrundy coloring exists by assigning $g_1(b_4) = g_1(b_2) = T_2$ and all remaining vertices by T_1. Hence, $dG(\overrightarrow{CT(7,4)}) \geq 2$. Next, let $T' = \{T_1, T_2, T_3\}$, and suppose $g_2: V(\overrightarrow{CT(7,4)}) \to T'$ is a digrundy coloring. Without loss of generality (wlog), take $g_2(b_4) = T_3$. By Definition 5, a vertex colored T_3 must be adjacent to vertices colored T_1 and T_2. Thus, b_4 must have a neighbor colored T_2. For each leaf v_i, $i = 1,2, \dots, 7$, we have $d_{in}(v_i) + d_{out}(v_i) = 1 < 2$. By Lemma 2, $g_2(v_i) \neq T_3$. Moreover, since each v_i is adjacent only to b_4, assigning T_2 to v_i does not satisfy Definition 5, as no adjacent vertex provides color T_1. Hence, $g_2(v_i) = T_1$ for all i. Therefore, none of the leaves can supply a neighbor of b_4 with color T_2. The remaining vertices b_1, b_2, b_3 form a path and cannot fulfill this requirement without violating the greedy condition at some vertex. Thus, a digrundy coloring with three colors is not feasible. Hence, $dG(\overrightarrow{CT(7,4)}) = 2$. A visualization of this coloring is provided in Fig. 2.

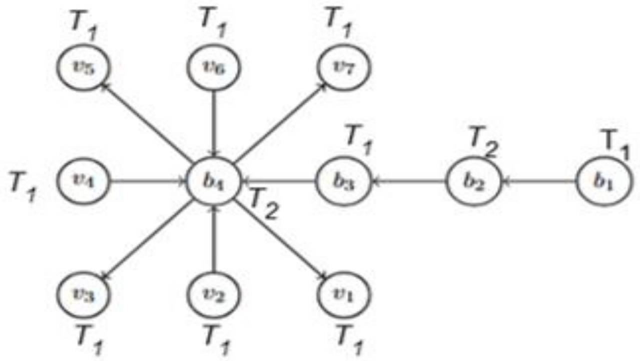

Figure 2. Illustrates the Coconut Tree Graph $\overrightarrow{CT(7,4)}$ Colored Using Two Distinct Colors According to The Digrundy Coloring Method

It is observed that the digrundy number of the coconut tree graph $\overrightarrow{CT(7,4)}$ is 2, while [26] earlier findings established that the diachromatic number of the same graph is 3. This discrepancy highlights the existence of a structural difference between the diachromatic number and the digrundy number, even when applied to the same graph. Motivated by this distinction, the present study focuses on determining the digrundy numbers of the star graph and the banana tree graph under specific orientations, and subsequently extends the analysis to the fireworks graph and coconut tree graph under arbitrary orientations.

Theorem 1. *Let $\overrightarrow{K_{1,a}}$ be a star graph as defined in Definition 1, with the center vertex v, and assume that the orientation satisfies $d_{out}(v) \neq 0$ and $d_{in}(v) \neq 0$. The digrundy number of the graph $\overrightarrow{K_{1,a}}$ is 2, $dG\left(\overrightarrow{K_{1,a}}\right) = 2$.*

***Proof*.** Consider the directed star graph $\overrightarrow{K_{1,a}}$. Since $d_{out}(v) \neq 0$ and $d_{in}(v) \neq 0$, there exist two leaves u_i, u_j such that $(v, u_i) \in A(\overrightarrow{K_{1,a}})$ and $(u_j, v) \in A(\overrightarrow{K_{1,a}})$. Hence, a digrundy coloring with two colors exists by assigning $f(v) = T_2$ and all leaves T_1. Thus, $dG(\overrightarrow{K_{1,a}}) \geq 2$. Next, consider $T = \{T_1, T_2, T_3\}$, and suppose $g: V(\overrightarrow{K_{1,a}}) \rightarrow T$ is a digrundy coloring. We examine the possibility of assigning color T_3. First, let $g(v) = T_3$. By Definition 5, v must be adjacent to vertices colored T_1 and T_2. Hence, there exists at least one leaf u with $g(u) = T_2$. However, every leaf has degree one, so $d_{in}(u) + d_{out}(u) = 1 < 2$. By Lemma 2, u cannot be assigned color T_2. Therefore, $g(v) \neq T_3$. Next, let $g(u) = T_3$ for some leaf u. By Definition 5, u must be adjacent to vertices colored T_1 and T_2. Since u is adjacent only to v, this requirement cannot be satisfied. Hence, no leaf can be assigned color T_3. Thus, no vertex in $\overrightarrow{K_{1,a}}$ can be assigned color T_3. Therefore, a digrundy coloring with three colors is not feasible, and it follows that $dG(\overrightarrow{K_{1,a}}) = 2$. ■

Let the orientation H' of the directed banana tree graph $\overrightarrow{B_{2,s}}$ be defined as follows. For the two central vertices b_1 and b_2, all incident edges are directed outward, so that $d_{out}(b_1) = d_{out}(b_2) = s - 1$ and $d_{in}(b_1) = d_{in}(b_2) = 0$. The root vertex v satisfies $d_{out}(v) = 2$ and $d_{in}(v) = 0$, with edges directed from v to v_{11} and v_{21}. For each $i = 1, 2$, the vertex v_{i1} satisfies $d_{out}(v_{i1}) = 0$ and $d_{in}(v_{i1}) = 2$, receiving edges from both v and b_i. Furthermore, for each $i = 1, 2$ and $j = 2, 3, \dots, s - 1$, the vertices v_{ij} satisfy $d_{out}(v_{ij}) = 0$ and $d_{in}(v_{ij}) = 1$, with edges directed from b_i to v_{ij}. This orientation is defined for $s \geq 2$ and is constructed to control the distribution of in-degrees and out-degrees, which is essential for analyzing the feasibility of digrundy colorings and determining the digrundy number.

Lemma 3. *Consider the directed banana tree graph $\overrightarrow{B_{2,s}}$ as defined in Definition 2, with orientation specified by H'. For $s \geq 3$, the digrundy number of $\overrightarrow{B_{2,s}}$ is found to be 3, $dG\left(\overrightarrow{B_{2,s}}\right) = 3$.*

***Proof*.** First, we show that a digrundy coloring with three colors exists. Let $T = \{T_1, T_2, T_3\}$, and define digrundy coloring g_1 by $g_1(v) = T_1$, $g_1(b_2) = T_2$, $g_1(b_1) = T_3$, $g_1(v_{11}) = T_2$, $g_1(v_{21}) = T_3$, and $g_1(v_{ij}) = T_1$ for $j \geq 2$. By Definition 5, each vertex colored T_3 is adjacent to vertices colored T_1 and T_2, and each vertex colored T_2 is adjacent to a vertex colored T_1. Hence, g_1 is a digrundy coloring and $dG(\overrightarrow{B_{2,s}}) \geq 3$. Next, suppose a digrundy coloring g_2 uses four colors $T' = \{T_1, T_2, T_3, T_4\}$. By Lemma 2, only vertices with total degree at least 3 can receive color T_4; thus, only b_1 or b_2 may be colored T_4. Wlog, let $g_2(b_1) = T_4$. By Definition 5, b_1 must be adjacent to vertices colored T_1, T_2, and T_3. However, all its neighbors v_{1j} have degree 1, so assigning them T_2 or T_3 violates the greedy condition. Hence, all v_{1j} must be colored T_1, and no neighbor of b_1 can provide colors T_2 or T_3. Thus, a digrundy coloring with four colors is not feasible. Therefore, $dG(\overrightarrow{B_{2,s}}) = 3$. ■

Lemma 4. *Consider the directed banana tree graph $\overrightarrow{B_{2,2}}$, defined according to Definition 2 and oriented as specified by H'. Under this orientation, the digrundy number of $\overrightarrow{B_{2,2}}$ is found to be 2, $dG\left(\overrightarrow{B_{2,2}}\right) = 2$.*

***Proof*.** Let $\overrightarrow{B_{2,2}}$ be as stated. Since $\left|A\left(\overrightarrow{B_{2,2}}\right)\right| = 4$, it follows from Lemma 1 that any digrundy coloring with m colors must satisfy ${}_mP_2 \le 4$. For $m = 3$, we obtain ${}_3P_2 = 6 > 4$, hence a digrundy coloring with three colors is not possible. Therefore, $dG\left(\overrightarrow{B_{2,2}}\right) \le 2$. Next, we show that a digrundy coloring with two colors exists. Let $T = \{T_1, T_2\}$ and define $g: V\left(\overrightarrow{B_{2,2}}\right) \to T$ by $g(b_1) = T_2, g(b_2) = T_2, g(v) = T_2$, and assign all remaining vertices the color T_1. Under this assignment, every vertex $v \in V\left(\overrightarrow{B_{2,2}}\right)$ with $g(v) = T_2$ has a neighbor $u \in V\left(\overrightarrow{B_{2,2}}\right)$ such that $g(u) = T_1$, in accordance with the definition of digrundy coloring. Meanwhile, vertices colored T_1 trivially satisfy the condition. Hence, g is a valid digrundy coloring using two colors, and therefore $dG\left(\overrightarrow{B_{2,2}}\right) \ge 2$. Therefore, the digrundy number of $\overrightarrow{B_{2,2}}$ is exactly two, that is, $dG\left(\overrightarrow{B_{2,2}}\right) = 2$. ■

Theorem 2. *Let $\overrightarrow{B_{2,s}}$ be the directed banana tree graph defined according to Definition 2, with orientation specified by H', where $s \ge 2$. The digrundy number of $\overrightarrow{B_{2,s}}$ is given by:*

$$dG\left(\overrightarrow{B_{2,s}}\right) = \begin{cases} 2 & ; s = 2, \\ 3 & ; s \ge 3. \end{cases}$$

***Proof*.** The result is an immediate consequence of Lemma 3 and Lemma 4. By Lemma 4, $dG\left(\overrightarrow{B_{2,2}}\right) = 2$. For $s \ge 3$, Lemma 3 yields $dG\left(\overrightarrow{B_{2,s}}\right) = 3$. These cases exhaust all admissible values of $s \ge 2$, thereby completely characterizing the digrundy number of $\overrightarrow{B_{2,s}}$ under the orientation H'.

Theorem 3. *Let $\overrightarrow{F(2,c)}$ be the directed fireworks graph as defined in Definition 3, with an arbitrary orientation. The digrundy number of $\overrightarrow{F(2,c)}$ is equal to 2, $dG\left(\overrightarrow{F(2,c)}\right) = 2$.*

***Proof*.** Let $\overrightarrow{F(2,c)}$ be arbitrarily oriented, with internal vertices b_1, b_2 and pendant vertices v_{1j}, v_{2j}, Each has a total degree of one. Define a coloring $g_1: V\left(\overrightarrow{F(2,c)}\right) \to \{T_1, T_2\}$ by assigning $g_1(b_1) = T_1$, $g_1(b_2) = T_2$, $g_1(v_{1j}) = T_2$, and $g_1(v_{2j}) = T_1$ for all j. Then every vertex colored T_2 is adjacent to a vertex colored T_1, so g_1 is a digrundy coloring. Hence, $dG\left(\overrightarrow{F(2,c)}\right) \ge 2$. Now suppose there exists a digrundy coloring $g_2: V\left(\overrightarrow{F(2,c)}\right) \to \{T_1, T_2, T_3\}$. By definition, every vertex colored T_3 must be adjacent to vertices colored T_1 and T_2. Wlog, if both b_1 and b_2 are assigned color T_3, then each pendant vertex, having only one neighbor, cannot support the existence of a vertex colored T_2 that is adjacent to a vertex colored T_1, so the greedy condition is not satisfied. On the other hand, if a vertex of degree one is assigned color T_3, then it cannot be adjacent simultaneously to vertices colored T_1 and T_2, which again violates the greedy condition. These situations represent all possibilities up to recoloring of vertices. Therefore, no digrundy coloring with three colors exists, and hence $dG\left(\overrightarrow{F(2,c)}\right) \le 2$. Consequently, $dG\left(\overrightarrow{F(2,c)}\right) = 2$. ■

Theorem 4. *Let $\overrightarrow{CT(h,4)}$ be the directed coconut tree graph as defined in Definition 4, with an arbitrary orientation. Then the digrundy number that satisfies $\overrightarrow{CT(h,4)}$ is $2 \le dG\left(\overrightarrow{CT(h,4)}\right) \le 3$.*

***Proof*.** First, we show that $dG\left(\overrightarrow{CT(h,4)}\right) \ge 2$. Since $\overrightarrow{CT(h,4)}$ contains at least one edge, there exist two adjacent vertices that can be assigned distinct colors T_1 and T_2, and the digrundy coloring condition for T_2 is satisfied. Hence, a digrundy coloring with two colors always exists. Next, we establish that $dG\left(\overrightarrow{CT(h,4)}\right) \le 3$. Suppose that there exists a digrundy coloring using four colors, that is, a function $f: V\left(\overrightarrow{CT(h,4)}\right) \to \{T_1, T_2, T_3, T_4\}$. Then any vertex assigned color T_4 must be adjacent to vertices colored T_1, T_2, and T_3. However, by the structural constraints of $\overrightarrow{CT(h,4)}$, no vertex has a neighborhood rich enough to support three distinct lower colors. In particular, leaf vertices have degree one and cannot realize colors beyond T_2, while backbone vertices have limited adjacency that prevents the simultaneous occurrence of three distinct lower colors in their neighborhoods. This violates the necessary condition for color T_4, and thus a digrundy coloring with four colors is not possible. Therefore, $dG\left(\overrightarrow{CT(h,4)}\right) \leq 3$. Finally, the exact value depends on the chosen orientation. In orientations where the adjacency structure allows a vertex to be adjacent to vertices colored T_1 and T_2, a digrundy coloring with three colors can be constructed, yielding $dG\left(\overrightarrow{CT(h,4)}\right) = 3$. Otherwise, the coloring is restricted to two colors, and $dG\left(\overrightarrow{CT(h,4)}\right) = 2$. Hence, for arbitrary orientation, $2 \leq dG\left(\overrightarrow{CT(h,4)}\right) \leq 3$. ■

4. CONCLUSION

This study determines the digrundy numbers of several classes of acyclic digraphs by exploiting structural properties and the constraints imposed by digrundy coloring. The results are obtained through a combination of explicit constructions and structural arguments, ensuring that each value is rigorously justified in accordance with the definition of digrundy coloring.

1. For directed star graphs $\overrightarrow{K_{1,a}}$ with $a \geq 2$, under orientations satisfying $d_{out}(v) \neq 0$ and $d_{in}(v) \neq 0$, the digrundy number is

$$dG\left(\overrightarrow{K_{1,a}}\right) = 2.$$

2. For directed banana tree graphs $\overrightarrow{B_{2,s}}$ with orientation H', the digrundy number is given by

$$dG\left(\overrightarrow{B_{2,s}}\right) = \begin{cases} 2, & s = 2, \\ 3, & s \geq 3. \end{cases}$$

3. For directed fireworks graphs $\overrightarrow{F(2,c)}$, the analysis shows that two colors are sufficient, while any extension violates the greedy condition, yielding

$$dG\left(\overrightarrow{F(2,c)}\right) = 2.$$

4. Finally, for directed coconut tree graphs $\overrightarrow{CT(h,4)}$, it is established that three colors can be realized through a valid digrundy coloring, whereas any attempt to extend to four colors fails, hence

$$2 \leq dG\left(\overrightarrow{CT(h,4)}\right) \leq 3.$$

These results demonstrate that the digrundy number is governed not merely by the size of the graph, but by the structural capacity of its adjacency configuration to sustain successive digrundy coloring levels. In particular, while branching components impose strong local constraints, backbone structures may enable the propagation of higher color levels, leading to nontrivial distinctions between graph families. The conclusions are restricted to the graph classes and orientations explicitly analyzed in this work. Further research is needed to extend these results to broader classes of acyclic digraphs and to develop a more comprehensive characterization of how structural parameters influence the digrundy number under varying orientations. This work suggests that the digrundy number serves as a sensitive structural invariant, capturing the extent to which local adjacency constraints permit the global escalation of digrundy coloring levels.

Author Contributions

Raventino contributed to the conceptualization of the research, the collection of relevant references, the development of problem-solving ideas, and the verification of all results. Fransiskus Fran was responsible for reviewing the accuracy of the findings, validating the outcomes, and ensuring originality through Turnitin checking. Both authors jointly discussed the results and contributed to the preparation and completion of the final manuscript.

Funding Statement

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Acknowledgment

The author extends sincere gratitude to all individuals who contributed to the development of this research.

Declarations

The authors declare no conflicts of interest to report study.

Declaration of Generative AI and AI-assisted Technologies

AI-assisted technology (ChatGPT) was used to support light paraphrasing and sentence restructuring for clarity. The authors confirm that the underlying ideas, arguments, data analyses, and conclusions are original and were not generated by AI. All AI-assisted edits were critically reviewed and validated by the authors.

REFERENCES

[1] M. Aslan, "A PERFORMANCE COMPARISON OF GRAPH COLORING ALGORITHMS," *Int. J. Intell. Syst. Appl. Eng.*, vol. 4, no. Special Issue-1, pp. 1–7, Dec. 2016. doi: https://doi.org/10.18201/ijisae.273053.

[2] K. R. Saoub, *GRAPH THEORY: AN INTRODUCTION TO PROOFS, ALGORITHMS, AND APPLICATIONS*, 1st ed. in Textbooks in mathematics. Boca Raton: CRC Press, 2021. doi: https://doi.org/10.1201/9781138361416.

[3] I. Gutman, V. R. Kulli, and I. R. zepovic, "IRREGULARITY SOMBOR INDEX," *Bull. Acad. Sci. Arts Repub. Serbia Cl. Math. Nat. Sci.*, vol. 156, pp. 31–37, 2023, doi: 10.5281/zenodo.18005956.

[4] F. Susanto, R. Simanjuntak, and E. T. Baskoro, "FURTHER RESULTS ON THE TOTAL VERTEX IRREGULARITY STRENGTH OF TREES," *Electron. J. Graph Theory Appl.*, vol. 13, no. 1, p. 123, Apr. 2025. doi: https://doi.org/10.1201/9781138361416.

[5] S. Wagner and H. Wang, *INTRODUCTION TO CHEMICAL GRAPH THEORY*. Boca Raton: CRC Press, Taylor & Francis Group, 2018. doi: https://zenodo.org/doi/10.5281/zenodo.17785567.

[6] V. Zverovich, *MODERN APPLICATIONS OF GRAPH THEORY*, 1st ed. Oxford: Oxford university press, 2021. doi: https://doi.org/10.1093/oso/9780198856740.001.0001.

[7] G. Ali, M. Bača, M. Lascsáková, A. Semaničová-Feňovčíková, A. ALoqaily, and N. Mlaiki, "MODULAR TOTAL VERTEX IRREGULARITY STRENGTH OF GRAPHS," *AIMS Math.*, vol. 8, no. 4, pp. 7662–7671, 2023. doi: https://doi.org/10.3934/math.2023384.

[8] V. Aparna, N. Mohanapriya, and S. Broumi, "SINGLE VALUED NEUTROSOPHIC R-DYNAMIC VERTEX COLORING OF GRAPHS," *Neutrosophic Sets Syst.*, vol. 48, pp. 306–317, 2022.

[9] J. Lin, "A REDUCTION BASED METHOD FOR COLORING VERY LARGE GRAPHS," *Int. Jt. Conf. Artif. Intell.*, pp. 517–523, 2017. doi: https://doi.org/10.24963/ijcai.2017/73.

[10] Md. S. Rahman, *BASIC GRAPH THEORY*. IN UNDERGRADUATE TOPICS IN COMPUTER SCIENCE. Cham: Springer International Publishing, 2017. doi: https://doi.org/10.1007/978-3-319-49475-3.

[11] R. Diestel, *GRAPH THEORY*, 5TH ED. IN GRADUATE TEXTS IN MATHEMATICS. Berlin, Heidelberg: Springer Berlin Heidelberg, 2017. doi: https://doi.org/10.1007/978-3-662-53622-3

[12] S. D. Pasham, "NETWORK TOPOLOGY OPTIMIZATION IN CLOUD SYSTEMS USING ADVANCED GRAPH COLORING ALGORITHMS," *Res. Anal. J.*, vol. 6, no. 11, pp. 01–25, 2023. doi: https://doi.org/10.18535/raj.v6i11.426.

[13] Z. Dvořák and L. Postle, "CORRESPONDENCE COLORING AND ITS APPLICATION TO LIST-COLORING PLANAR GRAPHS WITHOUT CYCLES OF LENGTHS 4 TO 8," *J. Comb. Theory Ser. B*, vol. 129, pp. 38–54, 2018. doi: https://doi.org/10.1016/j.jctb.2017.09.001.

[14] D. Cardoso, O. Cerdeira, C. Dominicc, and P. Cruz, "INJECTIVE EDGE COLORING OF GRAPHS," *Filomat*, vol. 33, no. 19, pp. 6411–6423, 2019. doi: https://doi.org/10.2298/FIL1919411C.

[15] S. Jahanbekam, J. Kim, S. O, and D. B. West, "ON R-DYNAMIC COLORING OF GRAPHS," *Discrete Appl. Math.*, vol. 206, pp. 65–72, 2016. doi: https://doi.org/10.1016/j.dam.2016.01.016.

[16] E. Andrews, D. Johnston, and P. Zhang, "ON TWIN EDGE COLORINGS IN TREES," *J Comb. Math Comb. Comput*, vol. 94, pp. 115–131, 2015.

[17] N. Kusumastuti, Raventino, and F. Fran, "THE DIACHROMATIC NUMBER OF DOUBLE STAR GRAPH," *J. Phys. Conf. Ser.*, vol. 2106, no. 1, p. 012024, Nov. 2021. doi: https://doi.org/10.1088/1742-6596/2106/1/012024.

[18] G. Araujo-Pardo, J. J. Montellano-Ballesteros, M. Olsen, and C. Rubio-Montiel, "THE DIGRUNDY NUMBER OF DIGRAPHS," *Discrete Appl. Math.*, vol. 317, pp. 117–123, 2022. doi: https://doi.org/10.1016/j.dam.2022.04.005.

[19] B. Effantin, U. de Lyon, and U. Lyon, "A NOTE ON GRUNDY COLORINGS OF CENTRAL GRAPHS," *Australas. J. Comb.*, vol. 68, no. 3, pp. 346–356, 2017.

[20] É. Bonnet, F. Foucaud, E. J. Kim, and F. Sikora, "COMPLEXITY OF GRUNDY COLORING AND ITS VARIANTS," *Discrete Appl. Math.*, vol. 243, pp. 99–114, 2018. doi: https://doi.org/10.1016/j.dam.2017.12.022.

[21] T. Fujita, "PLITHOGENIC LINE GRAPH, STAR GRAPH, AND REGULAR GRAPH," *J. Graph Struct. Appl.*, 2024. doi: https://doi.org/10.61356/j.plc.2025.4580.

[22] B. S. Panda and S. Verma, "ON PARTIAL GRUNDY COLORING OF BIPARTITE GRAPHS AND CHORDAL GRAPHS," *Discrete Appl. Math.*, vol. 271, pp. 171–183, 2019. doi: https://doi.org/10.1016/j.dam.2019.08.005.

[23] H. T. Gölpek, "VULNERABILITY OF BANANA TREES VIA CLOSENESS AND RESIDUAL CLOSENESS PARAMETERS," *Maltepe J. Math.*, vol. 4, no. 2, pp. 33–37, Oct. 2022. doi: https://doi.org/10.47087/mjm.1156370.

[24] A. Ali, H. Iqbal, W. Nazeer, and S. M. Kang, “ON TOPOLOGICAL INDICES FOR THE LINE GRAPH OF FIRECRACKER GRAPH,” *Int. J. Pure Appl. Math.*, vol. 116, no. 4, pp. 1035–1042, 2017. doi: https://zenodo.org/doi/10.5281/zenodo.17785284.

[25] R. Ponraj, A. Gayathri, and S. Somasundaram, “4−Remainder Cordial of Some Tree Related Graphs,” *Int. J. Math. Comb.*, vol. 2, pp. 56–71, 2022. doi: https://zenodo.org/doi/10.5281/zenodo.17785446.

[26] Raventino and S. Yeni, “DIACHROMATIC NUMBER OF SOME ACYCLIC DIGRAPHS,” *J. Indones. Math. Soc.*, vol. 31, no. 3, pp. 1710–1727, 2025. doi: https://doi.org/10.22342/jims.v31i3.1710.

[27] P. Aboulker, É. Bonnet, E. J. Kim, and F. Sikora, “GRUNDY COLORING AND FRIENDS, HALF-GRAPHS, BICLIQUES,” *Algorithmica*, vol. 85, no. 1, pp. 1–28, 2023. doi: https://doi.org/10.1007/s00453-022-01001-2.