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DYNAMICAL ANALYSIS OF TWO LOAN CATEGORIES IN A BANKING SYSTEM WITH NON-PERFORMING LOANS

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ABSTRACT

The ability of banks to extend credit and meet their obligations to depositors can be significantly affected by the presence of non-performing loans (NPL). The intermediation process, in which deposits are transformed into loans, can be represented through a predator–prey modeling framework. In this study, we formulate and analyze a system of three nonlinear ordinary differential equations representing deposits as the prey and two categories of loans, individual loans and company loans, as predators within a single banking system. The model incorporates the effect of NPLs as factors that reduce effective loan performance and influence system interactions. The analytical results show that the system possesses five equilibrium points. The equilibrium corresponding to the absence of deposits and loans is unstable, while the remaining four equilibria are globally asymptotically stable under specific parameter conditions, particularly those related to loan growth rates and NPL levels. The stability analysis indicates that higher NPL rates tend to reduce the stability region of equilibria and may destabilize the banking system. Furthermore, numerical simulations are conducted to support and illustrate the analytical findings. These results provide insight into the long-term dynamic behavior of deposits and loans, emphasizing the role of NPLs in determining banking system stability.

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1. INTRODUCTION

The stability and resilience of banking systems are fundamental to sustaining modern economies [1]. Banks function as key financial intermediaries, transforming deposits into loans that support investment, consumption [2], and long-term economic growth [3]. Through this intermediation, banks facilitate capital flows, reduce transaction costs, and enhance economic efficiency [4]. However, this process is often disrupted by the presence of non-performing loans (NPL), which represent a persistent vulnerability within the financial sector [5], [6]. NPL, defined as loans in which the borrower has failed to make scheduled payments for a specified period [7], [8], resulting in reduced income and deteriorated asset quality for banks.

High levels of NPL pose serious risks to banking performance and financial stability. They reduce profitability by lowering interest income and increasing provisioning costs [9], [10], while also limiting banks' ability to provide new credit to borrowers [11], [12], [13]. This restriction on lending weakens investment and economic activity, erodes confidence, and ultimately threatens both bank solvency and the stability of the financial system [14]. These challenges underscore the need for effective risk management and strong regulatory measures to control NPL growth and safeguard the stability of banking intermediation in supporting economic development.

These concerns have motivated extensive research on mathematical models of banking system dynamics. In recent years, mathematical modeling has been increasingly applied to study the dynamics of banking systems, particularly to better capture the interaction between deposits, loans, and systemic risks. A widely used approach is the predator-prey framework, which treats deposits as the "prey" serving as the resource base, and loans as the "predators" competing for these funds. This perspective provides a systematic way to analyze how lending activities evolve and how external shocks, such as non-performing loans (NPL), shape system stability. Several variations of this approach have been explored. Several studies have applied this approach in different ways. A loan-deposit interaction model was first introduced using the classical Lotka–Volterra predator-prey framework [15]. Later, other researchers developed and extended this model, which subsequently became an important reference for many studies in this field [16], [17]. Building on this, other researchers developed a bank balance sheet model that explicitly incorporates non-performing loans, emphasizing the impact of credit risk [18]. In the realm of predator-prey studies, a two-predator one-prey model was proposed to capture the competition between two loan types for a deposit [19]. A subsequent study extended this framework by modeling two loan categories within a single bank and incorporating mechanisms such as non-performing loans and withdrawals [20]. Nevertheless, the explicit dynamical impact of non-performing loans on system stability remains insufficiently investigated. Therefore, further investigation is needed to understand how non-performing loan dynamics influence the stability and long-term behavior of multi-loan banking systems.

Despite these developments, most existing models focus either on a single loan category within a predator-prey structure or on multiple loan types without explicitly analyzing how non-performing loans alter the qualitative stability structure of the system. In particular, the combined effect of loan competition and NPL-induced performance deterioration on equilibrium configuration and stability regions has not been systematically investigated. Consequently, it remains unclear whether introducing multiple loan categories together with explicit NPL dynamics generates new stability patterns compared to single-loan or non-competitive frameworks.

To address this gap, this paper contributes to the existing literature by developing a mathematical framework that explicitly incorporates two categories of loans, individual and company, within a single banking institution. First, unlike single-loan models, the proposed framework captures the competitive and interactive effects of individual and company loans on deposit utilization and system performance. Second, it explicitly incorporates non-performing loans (NPL) into the model and rigorously analyzes their impact on equilibrium behavior and stability of the banking system. Finally, through analytical results supported by numerical simulations, the study demonstrates how increasing NPL levels may destabilize the system and constrain lending capacity, thereby providing insights that may support credit risk management and policy formulation in the banking sector.

The structure of the paper is organized as follows: Section 2 presents the construction of the mathematical model, followed by an examination of the existence, uniqueness, non-negativity, and boundedness of the model's solutions. Section 3 provides a comprehensive dynamical analysis, including the identification of equilibrium points, their local and global stability, and numerical simulations that verify the results and illustrate the impact of NPL on deposits and loans. Finally, Section 4 presents the conclusion of the study.

2. RESEARCH METHODS

In this paper, the deposit-loan model [15], which consists of deposit volume (D) and loan volume (L) is developed by dividing the loan volume into two categories: loans for individuals (L_1) and loans for companies (L_2) utilized in line with [20]. In [18], the effect of non-performing loans on loan volumes was considered; thus, this model incorporates the non-performing loan parameter (n_i) for both loan types [18]. The model for deposits and two types of loans with the influence of non-performing loans is as follows:

$$\begin{aligned}\frac{dD}{dt} &= \gamma D\left(1-\frac{D}{k}\right)-\frac{p_1 D L_1}{b_1 D+1}-\frac{p_2 D L_2}{b_2 D+1},\\ \frac{dL_1}{dt} &= \frac{p_1 D L_1}{b_1 D+1}-(\sigma_1+n_1)L_1-c_1(1-\mu_1)L_1-\beta_1 L_1 L_2-h_1 L_1^2,\\ \frac{dL_2}{dt} &= \frac{p_2 D L_2}{b_2 D+1}-(\sigma_2+n_2)L_2-c_2(1-\mu_2)L_2-\beta_2 L_1 L_2-h_2 L_2^2,\end{aligned} \tag{1}$$

with $\boldsymbol{D(0), L_1(0), L_2(0) \geq 0}$. The first equation describes deposit dynamics, where deposits grow logistically in the absence of lending and decrease due to loan allocation to individual and company sectors. The second and third equations describe the evolution of individual and company loans, whose growth depends on deposit availability as the main funding source. The non-performing loan parameters represent credit deterioration that reduces effective loan growth, while the interaction and quadratic terms capture competition effects and internal limiting mechanisms within the banking system. The parameter definition is presented in Table 1, where $\boldsymbol{i = 1}$ for individual loans and $\boldsymbol{i = 2}$ for company loans. Some analyses follow the studies in [19] and [20], while extending their framework by explicitly incorporating non-performing loan effects into the two-loan system.

Table 1. Parameter Definitions

Parameter	Definition
b_i	Saturation coefficient representing borrowing constraints
c_i	Return payment rate on loan funds
h_1	Competition rate among individual loans
h_2	Competition rate among company loans
k	Maximum deposit capacity of the banking system
n_i	Loan deterioration rate due to non-performing loans
p_i	Maximum effective lending rate from deposits to loans
β_1	Competition rate of two loans affecting individual loan
β_2	Competition rate of two loans affecting company loan
γ	The deposit inflow rate driven by interest rate
μ_i	Proportion of Non-Performing Loans (NPL)
σ_i	Interest rate of loan

3. RESULTS AND DISCUSSION

In this section, the existence, uniqueness, non-negativity, and boundedness of the solution are given to confirm the feasibility of the model. The dynamical behaviour around each equilibrium point will be determined through local and global stability analysis.

3.1 The Existence and Uniqueness of the Solution

The existence and uniqueness of the solution are first established to ensure that the proposed model is mathematically well-posed and reliable for further analysis.

We establish the existence and uniqueness of solutions to system (5.1) in the region $\Omega \times [0, \infty)$, where

$$\Omega = \{(D, L_1, L_2) \in \mathbb{R}_+^3 \cup \{\vec{0}\}: \max\{|D|, |L_1|, |L_2|\} \leq M\},$$

for some sufficiently large constant M. The existence of such a bound will follow from the boundedness of the solutions, which is proven in the subsequent subsection. In addition, we assume that the state variables remain nonnegative, i.e., $D(t), L_1(t), L_2(t) \geq 0$ for all $t \geq 0$; this property will also be verified in the next subsection.

Let $X = (D, L_1, L_2)$ and $\bar{X} = (\bar{D}, \overline{L_1}, \overline{L_2})$. The vector field $F(X) = \big(F_1(X), F_2(X), F_3(X)\big)$, where

$$\begin{aligned}
F_1(X) &= \gamma D\left(1 - \frac{D}{k}\right) - \frac{p_1 D L_1}{b_1 D + 1} - \frac{p_2 D L_2}{b_2 D + 1}, \\
F_2(X) &= \frac{p_1 D L_1}{b_1 D + 1} - (\sigma_1 + n_1)L_1 - c_1(1 - \mu_1)L_1 - \beta_1 L_1 L_2 - h_1 L_1^2, \\
F_3(X) &= \frac{p_2 D L_2}{b_2 D + 1} - (\sigma_2 + n_2)L_2 - c_2(1 - \mu_2)L_2 - \beta_2 L_1 L_2 - h_2 L_2^2.
\end{aligned}$$

Then, for any $X \in \Omega$, we have

$$\begin{aligned}
\|F(X) - F(\bar{X})\| &= |F_1(X) - F_1(\bar{X})| + |F_2(X) - F_2(\bar{X})| + |F_3(X) - F_3(\bar{X})| \\
&\leq \left(\gamma + \frac{2\gamma M}{k} + 2p_1 M + 2p_2 M\right)|D - \bar{D}| \\
&\quad +(2p_1 M + H_5 + \beta_1 M + 2h_1 M + \beta_2 M)|L_1 - \overline{L_1}| \\
&\quad +(2p_2 M + H_6 + \beta_1 M + 2h_2 M + \beta_2 M)|L_2 - \overline{L_2}| \\
&= \mathcal{K}_1|D - \bar{D}| + \mathcal{K}_2|L_1 - \overline{L_1}| + \mathcal{K}_3|L_2 - \overline{L_2}| \\
&= \mathcal{K}(|D - \bar{D}| + |L_1 - \overline{L_1}| + |L_2 - \overline{L_2}|) = \mathcal{K}\|X - \bar{X}\|,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_1 &= \gamma + \frac{2\gamma M}{k} + 2p_1 M + 2p_2 M, \\
\mathcal{K}_2 &= H_5 + M(2p_1 + \beta_1 + 2h_1 + \beta_2), \\
\mathcal{K}_3 &= H_6 + M(2p_2 + \beta_1 + 2h_2 + \beta_2), \\
\mathcal{K} &= max\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}, \\
H_5 &= \sigma_1 + n_1 + c_1(1 - \mu_1), \\
H_6 &= \sigma_2 + n_2 + c_2(1 - \mu_2).
\end{aligned}$$

Referring to a lemma in [21], it can be established that the function $F(X)$ satisfies the Lipschitz condition, thereby guaranteeing the existence and uniqueness of solutions of Eq. (1) exist and are unique in $\Omega \times [0, \infty)$.

3.2 The Non-Negativity and Boundedness Solution

To establish the mathematical and economic consistency of the proposed banking model in Eq. (1), it is essential to demonstrate that all state variables remain non-negative and ultimately bounded. In a financial context, deposit and loan volumes cannot take negative values, and their growth is naturally constrained by market capacity and regulatory frameworks.

It will be shown that if the initial conditions satisfy $D(0) \geq 0$, $L_1(0) \geq 0$, and $L_2(0) \geq 0$, then the solutions remain nonnegative for all $t > 0$; that is, $D(t) \geq 0$, $L_1(t) \geq 0$, and $L_2(t) \geq 0$. Assuming that $D(t)$ is allowed to be negative, it follows that there exists $t_1 > 0$ such that $D(t) > 0$ for $0 < t < t_1$, equals zero at $t = t_1$, and becomes negative for $t > t_1$. From Eq. (1), it follows that

$$\left.\frac{dD}{dt}\right|_{t=t_1} = 0$$

which indicates that the deposit fund exhibits no rate of increase at $t = t_1$. This result contradicts the earlier assumption that $D(t) < 0$ for $t > t_1$. Hence, the supposition is false, and it can be concluded that $D(t) \geq 0$ for all $t > 0$. By applying the same reasoning, it can similarly be established that $L_1(t) \geq 0$ and $L_2(t) \geq 0$ for any $t > 0$. Economically, this confirms that the banking activities described by the model will never result in negative fund volumes, maintaining the realism of the financial intermediation process.

Next, we establish the boundedness of the solutions to ensure that the system remains well-defined for all $t \geq 0$. Define $V(t) = D(t) + L_1(t) + L_2(t)$. The first derivative of $V(t)$ is then given by

$$\begin{aligned}\frac{dV}{dt} &= \gamma D\left(1-\frac{D}{k}\right)-\frac{p_1DL_1}{b_1D+1}-\frac{p_2DL_2}{b_2D+1}\ +\frac{p_1DL_1}{b_1D+1}-(\sigma_1+n_1)L_1-c_1(1-\mu_1)L_1\\ &\quad -\beta_1L_1L_2-h_1L_1^2\ +\frac{p_2DL_2}{b_2D+1}-(\sigma_2+n_2)L_2-c_2(1-\mu_2)L_2-\beta_2L_1L_2-h_2L_2^2\\ &= \gamma D-\frac{\gamma D^2}{k}+(c_1\mu_1-c_1-\sigma_1-n_1)L_1+(c_2\mu_2-c_2-\sigma_2-n_2)L_2\\ &\quad -(\beta_1+\beta_2)L_1L_2-h_1L_1^2-h_2L_2^2\\ &\le \gamma D-\frac{\gamma D^2}{k}+(c_1\mu_1-c_1-\sigma_1-n_1)L_1+(c_2\mu_2-c_2-\sigma_2-n_2)L_2.\end{aligned}$$

Then, for any positive constant ξ, it follows that

$$\begin{aligned}\frac{dV}{dt}+\xi V &\le \gamma D-\frac{\gamma D^2}{k}+(c_1\mu_1-c_1-\sigma_1-n_1)L_1\ +\ (c_2\mu_2-c_2-\sigma_2-n_2)L_2+\xi(D+L_1+L_2)\\ &= (\gamma+\xi)D-\frac{\gamma D^2}{k}+(\xi+c_1\mu_1-c_1-\sigma_1-n_1)L_1\ +(\xi+c_2\mu_2-c_2-\sigma_2-n_2)L_2.\end{aligned}$$

By choosing $\xi < \min\{\, c_1(1-\mu_1)+\sigma_1+n_1, c_2(1-\mu_2)+\sigma_2+n_2\}$, we obtain

$$\frac{dV}{dt}+\xi V \le (\gamma+\xi)D-\frac{\gamma D^2}{k}\ = \frac{k(\gamma+\xi)^2}{4\gamma}\ ,$$
$$\frac{dV}{dt} \le \frac{k(\gamma+\xi)^2}{4\gamma}-\xi V.$$

It is evident that the solution of the corresponding first-order differential inequality is given by

$$V(t) \le \frac{k(\gamma+\xi)^2}{4\xi\gamma}+\left(V(0)-\frac{k(\gamma+\xi)^2}{4\xi\gamma}\right)e^{-\xi t}.$$

It follows that $V(t) \to \frac{k(\gamma+\xi)^2}{4\xi\gamma}$, as $t \to \infty$. Therefore, the region of Eq. (1) is

$$\Omega = \left\{\{D(t), L_1(t), L_2(t)\} \in R_+^3 \cup \{\vec{0}\}: V(t) \le \frac{k(\gamma+\xi)^2}{4\xi\gamma}\right\}.$$

Hence, all solutions of Eq. (1) are non-negative and ultimately bounded for any non-negative initial condition $D(0), L_1(0)$, and $L_2(0)$.

From an economic perspective, boundedness represents the bank's carrying capacity. A banking institution cannot expand its deposits or credit offerings infinitely due to market saturation (represented by the capacity parameter k), capital adequacy constraints, and the inherent risks associated with non-performing loans (NPL). The existence of this region Ω confirms that the system is mathematically well-posed and stays within a domain that reflects the finite nature of financial resources and banking infrastructure.

3.3 The Existence of The Equilibrium Points

The equilibrium points of Eq. (1) are determined when

$$\frac{dD}{dt} = \frac{dL_1}{dt} = \frac{dL_2}{dt} = 0 \tag{2}$$

By solving the system in Eq. (2), we obtain five equilibrium points representing different operational conditions of the banking system, including the absence of activity, partial lending activity, and full banking operation, as follows:

1. The equilibrium point without deposit and loan activities, $E_0 = (0,0,0)$, that always exists.
2. The loan-free equilibrium point, $E_1 = (k,0,0)$, that always exists.
3. The individual loan-free equilibrium point, $E_2 = \left(D_{(2)}, 0, L_{2_{(2)}}\right)$ where

$$L_{2_{(2)}} = \frac{1}{h_2}\left(\frac{p_2D_{(2)}}{1+b_2D_{(2)}}-\left(\sigma_2+n_2+c_2(1-\mu_2)\right)\right),$$

$$H_{4_{(2)}} = \frac{p_2 D_{(2)}}{1 + b_2 D_{(2)}},$$
$$H_6 = \sigma_2 + n_2 + c_2(1 - \mu_2).$$

Since $L_{2_{(2)}}$ includes $D_{(2)}$ and $D_{(2)}$ are the roots of the following cubic equation

$$D_{(2)}^3 + 3B_1 D_{(2)}^2 + 3B_2 D_{(2)} + B_3 = 0\,,$$

where

$$B_1 = -\frac{h_2 b_2 \gamma (k b_2 - 2)}{3\gamma h_2 b_2^2},$$
$$B_2 = \frac{-\gamma h_2 (2k b_2 - 1) + k p_2 (p_2 - H_6 b_2)}{3\gamma h_2 b_2^2},$$
$$B_3 = -\frac{k(\gamma h_2 + p_2 H_6)}{\gamma h_2 b_2^2}.$$

Therefore, based on Cardan's criterion [22], the existence of D_2 can be guaranteed as in [20]. So, the third equilibrium point E_2 exist if $H_{4_{(2)}} > H_6$ and satisfies one of Cardan's criteria.

4. The company's loan-free equilibrium point, $E_3 = \left(D_{(3)}, L_{1_{(3)}}, 0\right)$. The existence of this equilibrium point also uses Cardan's criterion [22] in the same way as E_2. So that, E_3 exist if $H_{3_{(3)}} > H_5$ and satisfies one of Cardan's criteria where

$$L_{1_{(3)}} = \frac{1}{h_1}\left(\frac{p_1 D_{(3)}}{1 + b_1 D_{(3)}} - \left(\sigma_1 + n_1 + c_1(1 - \mu_1)\right)\right),$$
$$H_{3_{(3)}} = \frac{p_1 D_{(3)}}{1 + b_1 D_{(3)}},$$
$$H_5 = \sigma_1 + n_1 + c_1(1 - \mu_1).$$

5. The equilibrium point with deposit and loan activities, $E^* = (D^*, L_1^*, L_2^*)$, where

$$L_1^* = \frac{p_1 h_2 (D^* + b_2 D^{*2}) - \beta_1 p_2 (D^* + b_1 D^{*2}) + (1 + b_1 D^*)(1 + b_2 D^*)[\beta_1 H_6 - H_5 h_2]}{(1 + b_1 D^*)(1 + b_2 D^*)(h_1 h_2 - \beta_1 \beta_2)},$$

and

$$L_2^* = \frac{p_2 h_1 (D^* + b_1 D^{*2}) - \beta_2 p_1 (D^* + b_2 D^{*2}) + (1 + b_1 D^*)(1 + b_2 D^*)[\beta_2 H_5 - H_6 h_1]}{(1 + b_1 D^*)(1 + b_2 D^*)(h_1 h_2 - \beta_1 \beta_2)}.$$

L_1^* and L_2^* exist if they satisfy either Eq. (3) or Eq. (4) given below.

a. If $h_1 h_2 > \beta_1 \beta_2$, then it must satisfy

$$\frac{h_2}{\beta_1} > \frac{H_4^* - H_6}{H_3^* - H_5} > \frac{\beta_2}{h_1}. \tag{3}$$

b. If $h_1 h_2 < \beta_1 \beta_2$, then it must satisfy

$$\frac{h_2}{\beta_1} < \frac{H_4^* - H_6}{H_3^* - H_5} < \frac{\beta_2}{h_1}. \tag{4}$$

The existence of D^*reduces to finding positive real roots of the following polynomial equation:

$$G_0 D^{*5} + G_1 D^{*4} + G_2 D^{*3} + G_3 D^{*2} + G_4 D^* + G_5 = 0, \tag{5}$$

where

$$G_0 = -\gamma(h_1 h_2 - \beta_1 \beta_2) b_1^2 b_2^2,$$
$$G_1 = \gamma(h_1 h_2 - \beta_1 \beta_2) b_1 b_2 [k b_1 b_2 - 2(b_1 + b_2)],$$
$$\begin{aligned} G_2 = {} & \gamma(h_1 h_2 - \beta_1 \beta_2)[2k b_1 b_2 (b_1 + b_2) - (b_1^2 + b_2^2 + 4 b_1 b_2)] \\ & + k p_1 b_2 [b_1 b_2 (h_2 H_5 - \beta_1 H_6) + b_1 p_2 (\beta_1 + \beta_2) - p_1 b_2 h_2] \\ & + k p_2 b_1 [b_1 b_2 (h_1 H_6 - \beta_2 H_5) - p_2 b_1 h_1], \end{aligned}$$

$$
\begin{aligned}
G_3 &= \gamma(h_1h_2-\beta_1\beta_2)[k(b_1^2+b_2^2+4b_1b_2)-2(b_1+b_2)] \\
&\quad +kp_1[(2b_1b_2+b_2^2)(h_2H_5-\beta_1H_6)+p_2(\beta_1+\beta_2)(b_1+b_2)-2p_1b_2h_2] \\
&\quad +kp_2[(2b_1b_2+b_1^2)(h_1H_6-\beta_2H_5)-2p_2b_1h_1], \\
G_4 &= \gamma(h_1h_2-\beta_1\beta_2)[2k(b_1+b_2)-1] \\
&\quad +kp_1[(b_1+2b_2)(h_2H_5-\beta_1H_6)+p_2(\beta_1+\beta_2)-p_1h_2] \\
&\quad +kp_2[(2b_1+b_2)(h_1H_6-\beta_2H_5)-p_2h_1], \\
G_5 &= \gamma k(h_1h_2-\beta_1\beta_2)+k[p_1(h_2H_5-\beta_1H_6)+p_2(h_1H_6-\beta_2H_5)].
\end{aligned}
$$

By numerical simulation in sub section 3.6, Eq. (5) will have at least one positive real root, thereby confirming the existence of the equilibrium point E^*.

Since the model describes banking quantities, only equilibria with non-negative components are economically meaningful. The positivity conditions will be further examined in the stability and numerical analysis sections.

3.4 Local Stability Analysis of The Equilibrium Points

In the banking context, local stability means that small deviations in deposits or loan volumes will naturally return to their equilibrium points. This property can be analyzed mathematically by linearizing Eq. (1) around an equilibrium point. The resulting Jacobian matrix is obtained by evaluating the partial derivatives of Eq. (1) with respect to the state variables D, L_1, and L_2, as given below.

$$
J(D,L_1,L_2)=\begin{pmatrix} J_1 & -H_3 & -H_4 \\ H_1 & J_2 & -\beta_1L_1 \\ H_2 & -\beta_2L_2 & J_3 \end{pmatrix},
$$

where

$$
J_1=\gamma\left(1-\frac{2D}{k}\right)-H_1-H_2\,,\qquad J_2=H_3-H_5-\beta_1L_2-2h_1L_1\,,\qquad J_3=H_4-H_6-\beta_2L_1-2h_2L_2,
$$
$$
H_1=\frac{p_1L_1}{(b_1D+1)^2},\qquad H_2=\frac{p_2L_2}{(b_2D+1)^2},\qquad H_3=\frac{p_1D}{b_1D+1},\qquad H_4=\frac{p_2D}{b_2D+1},
$$
$$
H_5=\sigma_1+n_1+c_1(1-\mu_1),\qquad H_6=\sigma_2+n_2+c_2(1-\mu_2).
$$

Theorem 1. *For the system in Eq. (1), the equilibrium point without deposit and loan activities E_0 is always unstable.*

Proof. The Jacobian matrix at E_0 is given by

$$
J(0,0,0)=\begin{pmatrix} \gamma & 0 & 0 \\ 0 & -\sigma_1-n_1-c_1(1-\mu_1) & 0 \\ 0 & 0 & -\sigma_2-n_2-c_2(1-\mu_2) \end{pmatrix}.
$$

It follows that the eigenvalues are

$$
\begin{aligned}
\lambda_1 &= \gamma>0\,, \\
\lambda_2 &= -\sigma_1-n_1-c_1(1-\mu_1)<0\,. \\
\lambda_3 &= -\sigma_2-n_2-c_2(1-\mu_2)<0.
\end{aligned}
$$

Consequently, the equilibrium point E_0 without deposit and loan activities remains unstable under all conditions.

Theorem 2. *For the system in Eq. (1), the loan-free equilibrium point E_1 is locally asymptotically stable if $\frac{p_1k}{b_1k+1}<H_5$ and $\frac{p_2k}{b_2k+1}<H_6$.*

Proof. The Jacobian matrix at E_1 is given by

$$
J(k,0,0)=\begin{pmatrix} -\gamma & -\dfrac{p_1k}{b_1k+1} & -\dfrac{p_2k}{b_2k+1} \\ 0 & \dfrac{p_1k}{b_1k+1}-H_5 & 0 \\ 0 & 0 & \dfrac{p_2k}{b_2k+1}-H_6 \end{pmatrix}.
$$

It is clear that the eigenvalues are $\lambda_1 = -\gamma < 0$, $\lambda_2 = \frac{p_1 k}{b_1 k+1} - H_5$, and $\lambda_3 = \frac{p_2 k}{b_2 k+1} - H_6$. When the conditions $\frac{p_1 k}{b_1 k+1} < H_5$ and $\frac{p_2 k}{b_2 k+1} < H_6$ are satisfied, both λ_2 and λ_3 become negative, implying that E_1 is locally asymptotically stable. Otherwise, E_1 is an unstable point. This result indicates that the loan-free equilibrium remains stable when loan growth rates are insufficient to overcome the corresponding loss and repayment rates.

Theorem 3. *For the system in Eq. (1), the individual loan-free equilibrium point E_2 is locally asymptotically stable if $H_{3_{(2)}} - H_5 < \beta_1 L_{2_{(2)}}$ and $\frac{\gamma}{k} > H_{2_{(2)}} b_2$ where $H_{2_{(2)}}$ and $H_{3_{(2)}}$ are stated in the proof.*

Proof. The Jacobian matrix at E_2 can be written as

$$J\left(D_{(2)}, 0, L_{2_{(2)}}\right) = \begin{pmatrix} D_{(2)}\left(H_{2_{(2)}} b_2 - \frac{\gamma}{k}\right) & -H_{3_{(2)}} & -H_{4_{(2)}} \\ 0 & H_{3_{(2)}} - H_5 - \beta_1 L_{2_{(2)}} & 0 \\ H_{2_{(2)}} & -\beta_2 L_{2_{(2)}} & -h_2 L_{2_{(2)}} \end{pmatrix},$$

where

$$H_{2_{(2)}} = \frac{p_2 L_{2_{(2)}}}{\left(b_2 D_{(2)} + 1\right)^2}, \qquad H_{3_{(2)}} = \frac{p_1 D_{(2)}}{b_1 D_{(2)} + 1}, \qquad H_{4_{(2)}} = \frac{p_2 D_{(2)}}{b_2 D_{(2)} + 1},$$
$$H_5 = \sigma_1 + n_1 + c_1(1-\mu_1), \qquad H_6 = \sigma_2 + n_2 + c_2(1-\mu_2).$$

The characteristic equation of the Jacobian matrix $J(E_2)$ is given by

$$\left(H_{3_{(2)}} - H_5 - \beta_1 L_{2_{(2)}} - \lambda\right)\left|J_{(2)} - \lambda I\right| = 0,$$

where

$$J_{(2)} = \begin{pmatrix} D_{(2)}\left(H_{2_{(2)}} b_2 - \frac{\gamma}{k}\right) & -H_{4_{(2)}} \\ H_{2_{(2)}} & -h_2 L_{2_{(2)}} \end{pmatrix}.$$

The first eigenvalue is $\lambda_1 = H_{3_{(2)}} - H_5 - \beta_1 L_{2_{(2)}}$ which will be negative if $H_{3_{(2)}} - H_5 < \beta_1 L_{2_{(2)}}$. Furthermore, if $\frac{\gamma}{k} > H_{2_{(2)}} b_2$, then $tr\left(J_{(2)}\right) < 0$ and $\left|J_{(2)}\right| > 0$ which means the other two eigenvalues $\lambda_2 < 0$ and $\lambda_3 < 0$, so that E_2 is locally asymptotically stable.

Theorem 4. *For the system in Eq. (1), the company loan-free equilibrium point E_3 is locally asymptotically stable, if $H_{4_{(3)}} - H_6 < \beta_2 L_{1_{(3)}}$ and $\frac{\gamma}{k} > H_{1_{(3)}} b_1$, where*

$$H_{1_{(3)}} = \frac{p_1 L_{1_{(3)}}}{\left(b_1 D_{(3)} + 1\right)^2}, \qquad H_{3_{(3)}} = \frac{p_1 D_{(3)}}{b_1 D_{(3)} + 1}, \qquad H_{4_{(3)}} = \frac{p_2 D_{(3)}}{b_2 D_{(3)} + 1},$$
$$H_5 = \sigma_1 + n_1 + c_1(1-\mu_1), \qquad H_6 = \sigma_2 + n_2 + c_2(1-\mu_2).$$

The proof follows the same procedure as in Theorem 3 by evaluating the Jacobian matrix at E_3 and analyzing the eigenvalues using the trace–determinant conditions.

Theorem 5. *For the system in Eq. (1), the equilibrium point with deposit and loan activities E^* is locally asymptotically stable if $a_1 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$ where a_1, a_2 and a_3 are stated in the proof.*

Proof. The Jacobian matrix at E^* is respresented by

$$J(D^*, L_1^*, L_2^*) = \begin{pmatrix} D^*\left(H_1^* b_1 + H_2^* b_2 - \frac{\gamma}{k}\right) & -H_3^* & -H_4^* \\ H_1^* & -h_1 L_1^* & -\beta_1 L_1^* \\ H_2^* & -\beta_2 L_2^* & -h_2 L_2^* \end{pmatrix},$$

where

$$H_1^* = \frac{p_1 L_1^*}{(b_1 D^* + 1)^2}, \qquad H_2^* = \frac{p_2 L_2^*}{(b_2 D^* + 1)^2}, \qquad H_3^* = \frac{p_1 D^*}{b_1 D^* + 1}, \qquad H_4^* = \frac{p_2 D^*}{b_2 D^* + 1},$$

$$H_5 = \sigma_1 + n_1 + c_1(1-\mu_1), \qquad H_6 = \sigma_2 + n_2 + c_2(1-\mu_2).$$

The characteristic equation of the matrix $J(E^*)$ is derived by evaluating $|J(E^*) - \lambda I| = 0$ as follow.

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{6}$$

where

$$\begin{aligned}
a_1 &= j_2 + j_5 - j_1,\\
a_2 &= j_2 j_5 + H_2^* H_4^* + H_1^* H_3^* - j_1 j_2 - j_1 j_5 - j_3 j_4\,,\\
a_3 &= j_1 j_3 j_4 + j_2 H_2^* H_4^* + j_5 H_1^* H_3^* - j_1 j_2 j_5 - j_3 H_2^* H_3^* - j_4 H_1^* H_4^*\,,\\
j_1 &= D^*\left(H_1^* b_1 + H_2^* b_2 - \frac{\gamma}{k}\right),\\
j_2 &= h_1 L_1^* > 0, \quad j_3 = \beta_1 L_1^* > 0, \quad j_4 = \beta_2 L_2^* > 0, \quad j_5 = h_2 L_2^* > 0.
\end{aligned}$$

Applying the Routh–Hurwitz criterion [23], one concludes that Eq. (6) has roots with negative real parts if and only if $a_1 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$. Hence, E^* is locally asymptotically stable.

3.5 Global Stability Analysis of The Equilibrium Points

The global stability analysis is performed in the positively invariant region Ω defined in the previous subsection. To establish global stability, we construct Lyapunov functions of logarithmic type. This class of functions is widely used in nonlinear dynamic systems because it is positive definite with respect to the equilibrium point and naturally defined in the nonnegative region. Moreover, its structure allows the time derivative to be expressed in quadratic form, which facilitates the application of LaSalle's invariance principle. The global stability properties of each equilibrium point are presented in the following theorems.

Theorem 6. *Global asymptotic stability (GAS) of E_1 holds if $p_1 D_{(1)} \le H_5$ and $p_2 D_{(1)} \le H_6$.*

Proof. We define the Lyapunov function as follows

$$\mathcal{U} = D - D_{(1)} - D_{(1)} \ln\left(\frac{D}{D_{(1)}}\right) + L_1 + L_2.$$

The first derivative of $\mathcal{U}$ with respect to t is obtained as

$$\begin{aligned}
\frac{d\mathcal{U}}{dt} &= (D - D_{(1)})\left(\frac{\gamma D_{(1)}}{k} - \frac{\gamma D}{k} - \frac{p_1 L_1}{b_1 D + 1} - \frac{p_2 L_2}{b_2 D + 1}\right) + \left(\frac{p_1 D L_1}{b_1 D + 1} - H_5 L_1 - \beta_1 L_1 L_2 - h_1 L_1^2\right)\\
&\quad + \left(\frac{p_2 D L_2}{b_2 D + 1} - H_6 L_2 - \beta_2 L_1 L_2 - h_2 L_2^2\right)\\
&= -\frac{\gamma}{k}(D - D_{(1)})^2 + \left(\frac{p_1 D_{(1)}}{b_1 D + 1} - H_5\right) L_1 + \left(\frac{p_2 D_{(1)}}{b_2 D + 1} - H_6\right) L_2 - (\beta_1 + \beta_2) L_1 L_2 - h_1 L_1^2 - h_2 L_2^2\\
&\le -\frac{\gamma}{k}(D - D_{(1)})^2 + (p_1 D_{(1)} - H_5) L_1 + (p_2 D_{(1)} - H_6) L_2 - (\beta_1 + \beta_2) L_1 L_2 - h_1 L_1^2 - h_2 L_2^2.
\end{aligned}$$

If $p_1 D_{(1)} \le H_5$ and $p_2 D_{(1)} \le H_6$, then $\frac{d\mathcal{U}}{dt} \le 0$ for all $(D, L_1, L_2) \in \Omega$ with equality $\frac{d\mathcal{U}}{dt} = 0$ occuring only at $(D, L_1, L_2) = (D_{(1)}, 0,0)$. By applying LaSalle's invariance principle [24], it follows that E_1 is GAS.

Theorem 7. *E_2 is GAS under the conditions $\frac{\gamma}{k} > p_2 b_2 L_{2(2)}$ and $p_1 D_{(2)} + v_1 \beta_2 L_{2(2)} \le H_5$.*

Proof. Consider the following Lyapunov function.

$$\mathcal{V}_1 = D - D_{(2)} - D_{(2)} \ln\left(\frac{D}{D_{(2)}}\right) + L_1 + v_1\left(L_2 - L_{2(2)} - L_{2(2)} \ln\left(\frac{L_2}{L_{2(2)}}\right)\right)$$

where $v_1 = b_2 D_{(2)} + 1$. We obtain the first derivative of $\mathcal{V}_1$ as follows.

$$\begin{aligned}
\frac{d\mathcal{V}_1}{dt} &= (D - D_{(2)})\left(\frac{\gamma D_{(2)}}{k} + \frac{p_2 L_{2_{(2)}}}{b_2 D_{(2)} + 1} - \frac{\gamma D}{k} - \frac{p_1 L_1}{b_1 D + 1} - \frac{p_2 L_2}{b_2 D + 1}\right)\\
&\quad + \left(\frac{p_1 D L_1}{b_1 D + 1} - H_5 L_1 - \beta_1 L_1 L_2 - h_1 L_1^2\right)\\
&\quad + v_1\left(L_2 - L_{2_{(2)}}\right)\left(\frac{p_2 D}{b_2 D + 1} - \frac{p_2 D_{(2)}}{b_2 D_{(2)} + 1} + h_2 L_{2_{(2)}} - \beta_2 L_1 - h_2 L_2\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\gamma}{k}\left(D - D_{(2)}\right)^2 - v_1 h_2 \left(L_2 - L_{2_{(2)}}\right)^2 + \frac{p_2 b_2 L_{2_{(2)}} \left(D - D_{(2)}\right)^2}{\left(b_2 D_{(2)} + 1\right)\left(b_2 D + 1\right)} \\
&\quad + \frac{p_2\left(D - D_{(2)}\right)\left(L_2 - L_{2_{(2)}}\right)\left[v_1 - \left(b_2 D_{(2)} + 1\right)\right]}{\left(b_2 D_{(2)} + 1\right)\left(b_2 D + 1\right)} - (v_1\beta_2 + \beta_1) L_1 L_2 \\
&\quad + \left(\frac{p_1 D_{(2)}}{b_1 D + 1} - H_5 + v_1 \beta_2 L_{2_{(2)}}\right) L_1 - h_1 L_1^2 \\
&\le -\frac{\gamma}{k}\left(D - D_{(2)}\right)^2 - v_1 h_2 \left(L_2 - L_{2_{(2)}}\right)^2 + p_2 b_2 L_{2_{(2)}} \left(D - D_{(2)}\right)^2 \\
&\quad + \frac{p_2\left(D - D_{(2)}\right)\left(L_2 - L_{2_{(2)}}\right)\left[v_1 - \left(b_2 D_{(2)} + 1\right)\right]}{\left(b_2 D_{(2)} + 1\right)\left(b_2 D + 1\right)} - (v_1\beta_2 + \beta_1) L_1 L_2 \\
&\quad + \left(p_1 D_{(2)} - H_5 + v_1 \beta_2 L_{2_{(2)}}\right) L_1 - h_1 L_1^2.
\end{aligned}
$$

Recalling that $v_1 = b_2 D_{(2)} + 1$, it can be expressed as:

$$
\begin{aligned}
\frac{d\mathcal{V}_1}{dt} \le &-\left(\frac{\gamma}{k} - p_2 b_2 L_{2_{(2)}}\right)\left(D - D_{(2)}\right)^2 - v_1 h_2 \left(L_2 - L_{2_{(2)}}\right)^2 - (v_1\beta_2 + \beta_1) L_1 L_2 \\
&+ \left(p_1 D_{(2)} - H_5 + v_1 \beta_2 L_{2_{(2)}}\right) L_1 - h_1 L_1^2.
\end{aligned}
$$

If $\frac{\gamma}{k} > p_2 b_2 L_{2_{(2)}}$ and $p_1 D_{(2)} + v_1 \beta_2 L_{2_{(2)}} \le H_5$, it follows that $\frac{d\mathcal{V}_1}{dt} \le 0$ for all $(D, L_1, L_2) \in \Omega$ with equality $\frac{d\mathcal{V}_1}{dt} = 0$ holding at $(D, L_1, L_2) = \left(D_{(2)}, 0, L_{2_{(2)}}\right)$. Consequently, according to the LaSalle invariance principle in [24], E_2 is GAS.

Theorem 8. *E_3 is GAS if $\frac{\gamma}{k} > p_1 b_1 L_{1(3)}$ and $p_2 D_{(3)} + v_2 \beta_1 L_{1(3)} \le H_6$ where $v_2 = b_1 D_{(3)} + 1$.*

Proof. The proof follows the same Lyapunov construction and steps as in Theorem 7, and is therefore omitted for simplicity.

Theorem 9. *Global asymptotic stability (GAS) of the equilibrium E^* holds under the conditions $\frac{\gamma}{k} > p_1 b_1 L_1^* + p_2 b_2 L_2^*$ and $0 < \frac{\beta_1}{2h_2 - \beta_2} < \frac{v_4}{v_3} < \frac{2h_1 - \beta_1}{\beta_2}$, where v_3 and v_4 are defined in the proof.*

Proof. Take the Lyapunov function as follows

$$
\mathcal{L} = D - D^* - D^* \ln\left(\frac{D}{D^*}\right) + v_3\left(L_1 - L_1^* - L_1^* \ln\left(\frac{L_1}{L_1^*}\right)\right) + v_4\left(L_2 - L_2^* - L_2^* \ln\left(\frac{L_2}{L_2^*}\right)\right),
$$

where $v_3 = b_1 D^* + 1$ and $v_4 = b_2 D^* + 1$. The first order derivative of $\mathcal{L}$ is

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &= (D - D^*)\left(-\frac{\gamma}{k}(D - D^*) - \frac{p_1\left(L_1(b_1 D^* + 1) - L_1^*(b_1 D + 1)\right)}{(b_1 D + 1)(b_1 D^* + 1)}\right) \\
&\quad -(D - D^*)\left(\frac{p_2\left(L_2(b_2 D^* + 1) - L_2^*(b_2 D + 1)\right)}{(b_2 D + 1)(b_2 D^* + 1)}\right) \\
&\quad +v_3(L_1 - L_1^*)\left(\frac{p_1(D - D^*)}{(b_1 D + 1)(b_1 D^* + 1)} - \beta_1(L_2 - L_2^*) - h_1(L_1 - L_1^*)\right) \\
&\quad +v_4(L_2 - L_2^*)\left(\frac{p_2(D - D^*)}{(b_2 D + 1)(b_2 D^* + 1)} - \beta_2(L_1 - L_1^*) - h_2(L_2 - L_2^*)\right) \\
&= -\frac{\gamma}{k}(D - D^*)^2 + (D - D^*)\left(\frac{p_1[b_1 L_1^*(D - D^*) - (L_1 - L_1^*)(b_1 D^* + 1)]}{(b_1 D + 1)(b_1 D^* + 1)}\right) \\
&\quad +(D - D^*)\left(\frac{p_2[b_2 L_2^*(D - D^*) - (L_2 - L_2^*)(b_2 D^* + 1)]}{(b_2 D + 1)(b_2 D^* + 1)}\right) + \frac{v_3 p_1 (D - D^*)(L_1 - L_1^*)}{(b_1 D + 1)(b_1 D^* + 1)} \\
&\quad + \frac{v_4 p_2 (D - D^*)(L_2 - L_2^*)}{(b_2 D + 1)(b_2 D^* + 1)} - v_3 h_1 (L_1 - L_1^*)^2 - v_4 h_2 (L_2 - L_2^*)^2 \\
&\quad -(v_3\beta_1 + v_4\beta_2)(L_1 - L_1^*)(L_2 - L_2^*)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\gamma}{k}(D-D^*)^2 + \frac{p_1 b_1 L_1^*(D-D^*)^2}{(b_1 D+1)(b_1 D^*+1)} + \frac{p_2 b_2 L_2^*(D-D^*)^2}{(b_2 D+1)(b_2 D^*+1)} \\
&\quad + \frac{p_1(D-D^*)(L_1-L_1^*)[v_3-(b_1 D^*+1)]}{(b_1 D+1)(b_1 D^*+1)} + \frac{p_2(D-D^*)(L_2-L_2^*)[v_4-(b_2 D^*+1)]}{(b_2 D+1)(b_2 D^*+1)} \\
&\quad - v_3 h_1(L_1-L_1^*)^2 - v_4 h_2(L_2-L_2^*)^2 - (v_3\beta_1+v_4\beta_2)(L_1-L_1^*)(L_2-L_2^*) \\
&\le -\frac{\gamma}{k}(D-D^*)^2 + p_1 b_1 L_1^*(D-D^*)^2 + p_2 b_2 L_2^*(D-D^*)^2 + \frac{p_1(D-D^*)(L_1-L_1^*)[v_3-(b_1 D^*+1)]}{(b_1 D+1)(b_1 D^*+1)} \\
&\quad + \frac{p_2(D-D^*)(L_2-L_2^*)[v_4-(b_2 D^*+1)]}{(b_2 D+1)(b_2 D^*+1)} - v_3 h_1(L_1-L_1^*)^2 - v_4 h_2(L_2-L_2^*)^2 \\
&\quad - (v_3\beta_1+v_4\beta_2)(L_1-L_1^*)(L_2-L_2^*) \\
&= -\left(\frac{\gamma}{k} - p_1 b_1 L_1^* - p_2 b_2 L_2^*\right)(D-D^*)^2 - v_3 h_1(L_1-L_1^*)^2 - v_4 h_2(L_2-L_2^*)^2 \\
&\quad - (v_3\beta_1+v_4\beta_2)(L_1-L_1^*)(L_2-L_2^*).
\end{aligned}
$$

Suppose

$$
P = \sqrt{\frac{v_3\beta_1+v_4\beta_2}{2}}(L_1-L_1^*) \text{ and } Q = \sqrt{\frac{v_3\beta_1+v_4\beta_2}{2}}(L_2-L_2^*),
$$

it follows that

$$
P^2 + Q^2 - (P+Q)^2 = -2PQ = -(v_3\beta_1+v_4\beta_2)(L_1-L_1^*)(L_2-L_2^*),
$$

and consequently

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &\le -\left(\frac{\gamma}{k} - p_1 b_1 L_1^* - p_2 b_2 L_2^*\right)(D-D^*)^2 - v_3 h_1(L_1-L_1^*)^2 - v_4 h_2(L_2-L_2^*)^2 \\
&\quad + \frac{v_3\beta_1+v_4\beta_2}{2}(L_1-L_1^*)^2 + \frac{v_3\beta_1+v_4\beta_2}{2}(L_2-L_2^*)^2 \\
&\quad - \left(\sqrt{\frac{v_3\beta_1+v_4\beta_2}{2}}(L_1-L_1^*) + \sqrt{\frac{v_3\beta_1+v_4\beta_2}{2}}(L_2-L_2^*)\right)^2 \\
&\le -\left(\frac{\gamma}{k} - p_1 b_1 L_1^* - p_2 b_2 L_2^*\right)(D-D^*)^2 - v_3 h_1(L_1-L_1^*)^2 - v_4 h_2(L_2-L_2^*)^2 \\
&\quad + \frac{v_3\beta_1+v_4\beta_2}{2}(L_1-L_1^*)^2 + \frac{v_3\beta_1+v_4\beta_2}{2}(L_2-L_2^*)^2 \\
&= -\left(\frac{\gamma}{k} - p_1 b_1 L_1^* - p_2 b_2 L_2^*\right)(D-D^*)^2 - \left(v_3 h_1 - \frac{v_3\beta_1+v_4\beta_2}{2}\right)(L_1-L_1^*)^2 \\
&\quad - \left(v_4 h_2 - \frac{v_3\beta_1+v_4\beta_2}{2}\right)(L_2-L_2^*)^2.
\end{aligned}
$$

Under the conditions $\frac{\gamma}{k} > p_1 b_1 L_1^* + p_2 b_2 L_2^*$ and $0 < \frac{\beta_1}{2h_2-\beta_2} < \frac{v_4}{v_3} < \frac{2h_1-\beta_1}{\beta_2}$, it holds that $\frac{d\mathcal{L}}{dt} \le 0$ for all $(D, L_1, L_2) \in \Omega$, with the equality $\frac{d\mathcal{L}}{dt} = 0$ only at $(D, L_1, L_2) = (D^*, L_1^*, L_2^*)$. By LaSalle's invariance principle in [24], the equilibrium E^* is GAS.

3.6 Numerical Simulation

The outcomes of numerical simulations for the model in Eq. (1) are presented in this section. Since empirical data are unavailable, all parameter values were assigned hypothetically. To compute the solutions, the classical fourth-order Runge-Kutta method was employed, a scheme widely recognized in the literature for analyzing dynamical system models. All numerical simulations were performed using MATLAB software. Its effectiveness in solving first-order ordinary differential equations has been highlighted in several previous studies [25], [26], and in this work, it is applied to validate and illustrate the theoretical results.

Table 2. **Parameters Values**

Parameter	γ	k	p_1	p_2	n_1	n_2	b_1	b_2	σ_1	σ_2	c_1	c_2	β_1	β_2	h_1	h_2	μ_1	μ_2
Simulation 1	0.4	3	0.15	0.2	0.17	0.23	0.8	0.9	0.2	0.2	0.2	0.25	0.5	0.4	0.1	0.1	0.15	0.2
Simulation 2	0.4	1.75	0.25	0.8	0.17	0.2	0.8	0.7	0.2	0.1	0.2	0.2	0.5	0.4	0.1	0.1	0.15	0.1
Simulation 3	0.4	1.75	0.8	0.35	0.17	0.23	0.8	0.9	0.1	0.15	0.2	0.25	0.5	0.4	0.1	0.1	0.15	0.2
Simulation 4	0.4	1.75	0.8	0.85	0.17	0.2	0.9	0.95	0.2	0.15	0.15	0.2	0.1	0.15	0.3	0.25	0.15	0.2

Table 2 provides the specific parameter values employed in the model in Eq. (1) . For the simulations, three distinct sets of initial conditions were considered, $(D(0), L_1(0), L_2(0)) = \{(7,3,2), (10,4,6), (14,5,7)\}$. Using the parameters listed in the "simulation 1" column, the first numerical experiment demonstrates that the system's solution converges asymptotically to the loan-free equilibrium point $E_1 = (3,0,0)$. This behavior is illustrated in Fig. 1, which presents the phase portrait of the system and simultaneously reveals the existence of the trivial equilibrium point E_0 noted in the preceding analysis. Moreover, this numerical experiment serves as a verification of Theorem 6. The parameter set employed in simulation 1 satisfies $p_1 D_{(1)} = 0.45 < 0.54 = H_5$ and $p_2 D_{(1)} = 0.6 < 0.63 = H_6$, thereby guaranteeing global asymptotic stability. This confirms that the convergence toward the loan-free equilibrium point E_1 is fully consistent with the theoretical prediction. From an economic perspective, this equilibrium represents an undesirable state for banks, since the absence of loan activity eliminates interest income, thereby limiting their ability to meet operational costs and to remunerate depositors.

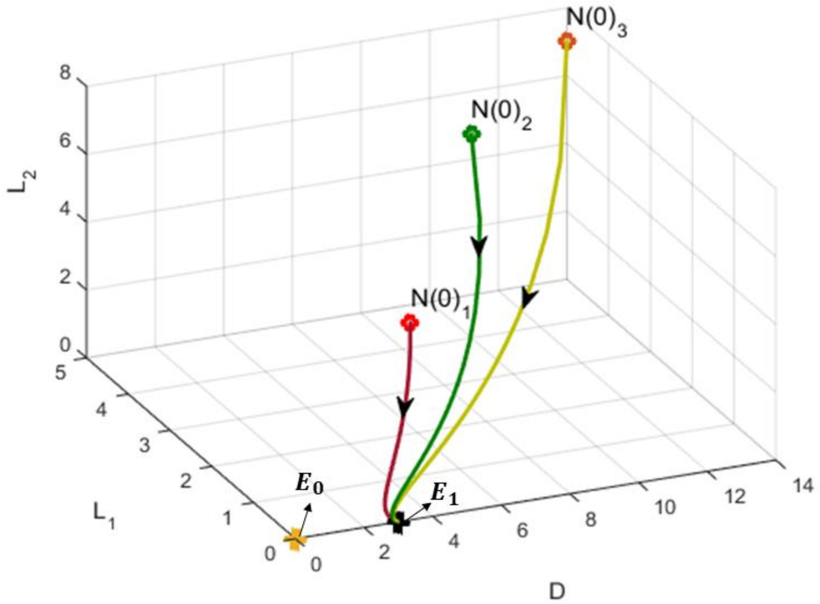


Figure 1. **Numerical Simulation for E_1**

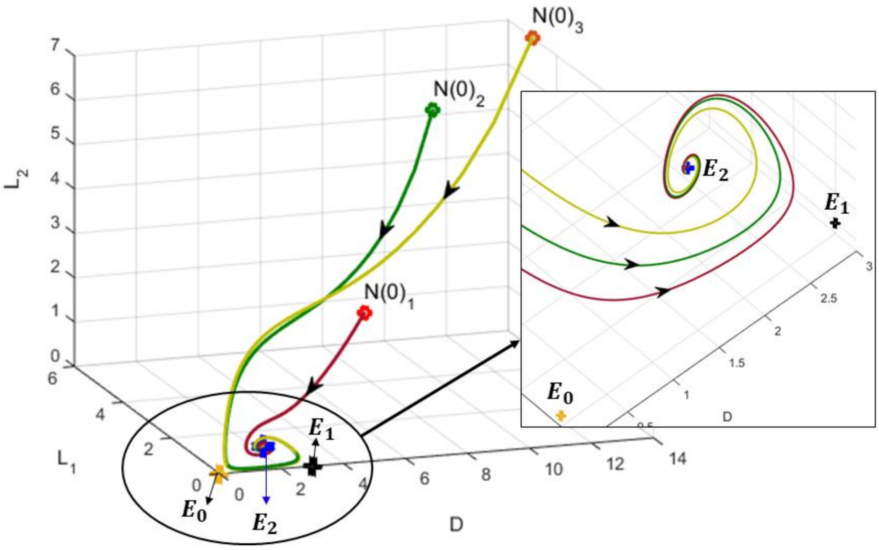


Figure 2. Numerical Simulation for E_2

The subsequent simulations provide further insight into the model's dynamical behavior under varying parameter conditions listed in Table 2. In the second simulation, which utilized the parameters in the simulation 2 column, the system's trajectory converges asymptotically to the individual loan-free equilibrium point, $E_2 = (1.1544, 0, 0.3077)$. This outcome is illustrated in Fig. 2, where both the trivial equilibrium E_0 and the loan-free equilibrium E_1 also remain present. This simulation also confirms the analytical result of Theorem 7. The parameter set in simulation 2 fulfills the following stability conditions required for global stability,

$$\frac{\gamma}{k} = 0.23 > 0.17 = p_2 b_2 L_{2(2)} \text{ and } p_1 D_{(2)} + v_1 \beta_2 L_{2(2)} = 0.51 < 0.54 = H_5.$$

Hence, the convergence toward the equilibrium point E_2 is not only numerically observed but also analytically confirmed to be globally asymptotically stable. Proceeding to the third simulation, based on the parameters from the simulation 3 column, the results indicate asymptotic stability at the company loan-free equilibrium point $E_3 = (1.1259, 0.3389, 0)$, as illustrated in Fig. 3. This figure also highlights the coexistence of E_0 and E_1, in addition to the stability of E_3. This behavior is consistent with the analytical result established in Theorem 8. Substituting the parameter values from simulation 3 into the derived stability criteria confirms that the parameter set fulfills the following conditions.

$$\frac{\gamma}{k} = 0.22 > 0.21 = p_1 b_1 L_{1(3)} \text{ and } p_2 D_{(3)} + v_2 \beta_1 L_{1(3)} = 0.54 < 0.58 = H_6$$

Consequently, the company's loan-free equilibrium point E_3 is analytically verified to be globally asymptotically stable, in agreement with the numerical findings.

Finally, the last simulation, corresponding to the simulation 4 column in Table 2, demonstrates that the system approaches the coexistence equilibrium $E^* = (1.8597, 0.2322, 0.2405)$, where deposits and both types of loans remain active. Fig. 4 confirms the global asymptotic stability of this equilibrium. This simulation also reveals the simultaneous existence of all four equilibria E_0, E_1, E_2, and E_3. This result is consistent with the analytical conclusion stated in Theorem 9. By substituting the parameter values from simulation 4 into the derived stability criteria, it can be verified that the parameter set fulfills the following conditions.

$$\frac{\gamma}{k} = 0.22 > 0.09 = p_1 b_1 L_1^* + p_2 b_2 L_2^* \text{ and } 0 < 0.29 = \frac{\beta_1}{2h_2 - \beta_2} < 1.03 = \frac{v_4}{v_3} < 3.33 = \frac{2h_1 - \beta_1}{\beta_2}.$$

Consequently, the coexistence equilibrium E^* is analytically confirmed to be globally asymptotically stable, in agreement with the numerical simulation. Economically, this case reflects the most desirable scenario for banks, where lending to both individuals and companies is sustained. The findings indicate that a strong capacity to convert deposits into loans ensures stability, making the equilibrium E^* the optimal state of the banking system.

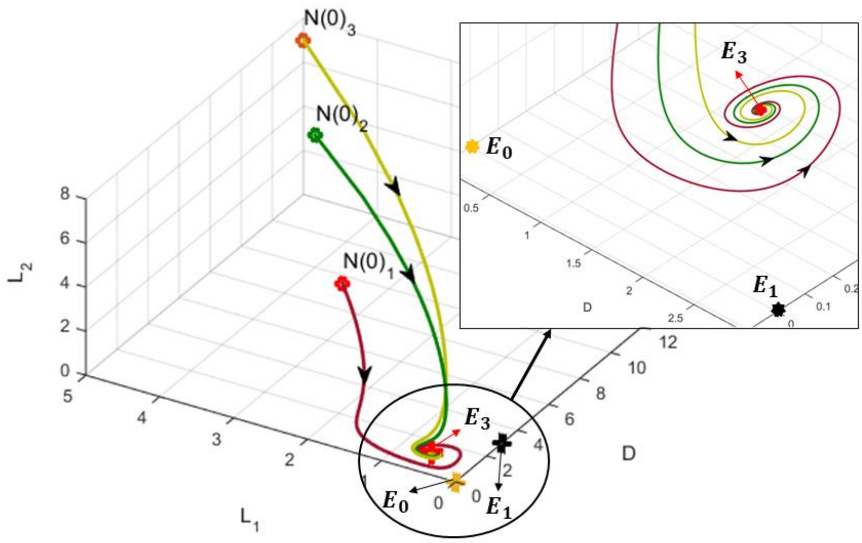


Figure 3. Numerical Simulation for E_3

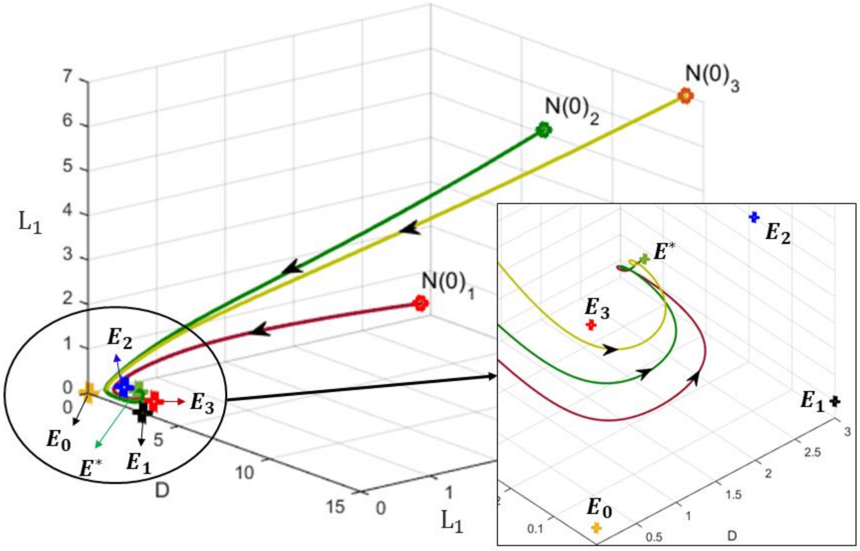
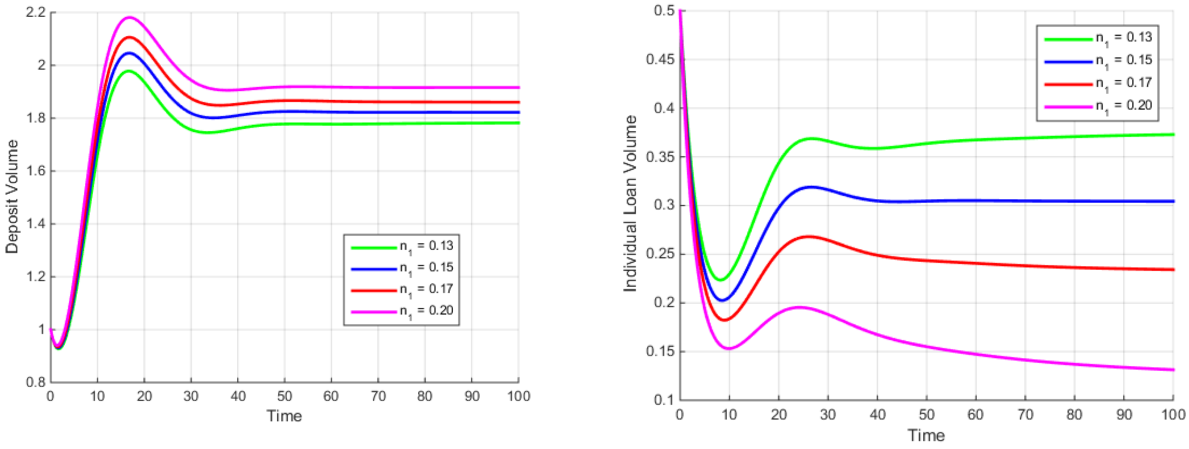
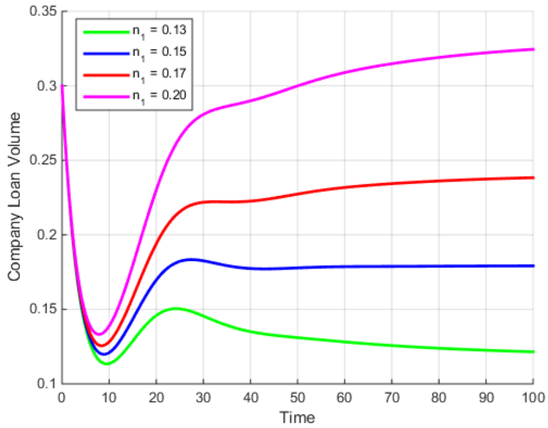


Figure 4. Numerical Simulation for E^*



(a) **(b)**

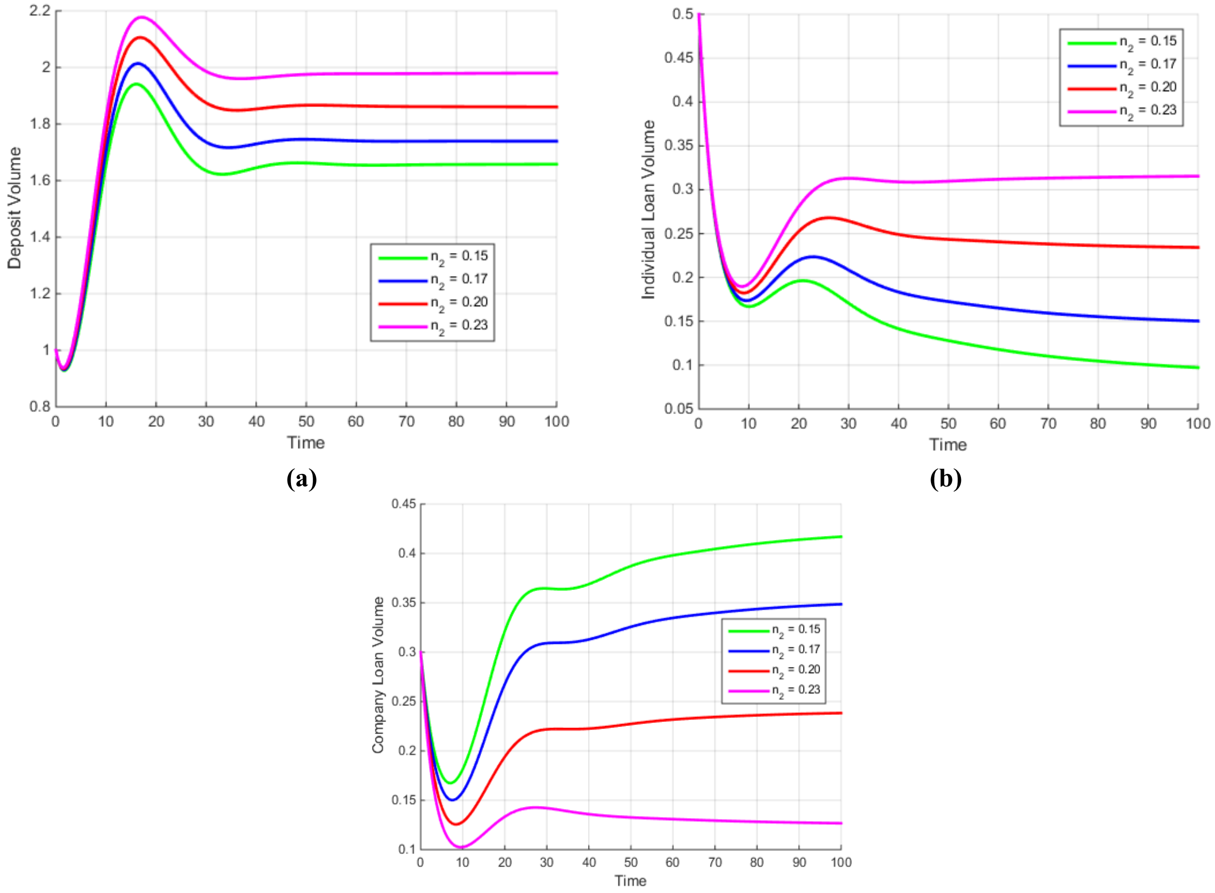


(c)

Figure 5. The Impact of Fluctuations in Individuals' NPL Rates (n_1) on the Volumes of (a) Deposit, (b) Individual Loan, and (c) Company Loan

Fig. 5 presents an analysis of the impact of fluctuations in individual NPL rates (n_1) on the volume of (a) deposits, (b) individual loans, and (c) company loans. From graph (a), it is evident that an increase in the individual NPL rate (n_1) leads to a decrease in deposit volume. For instance, at $n_1 = 0.13$ (green line), deposit volume reaches its highest level, whereas at $n_1 = 0.20$ (magenta line), deposit volume is the lowest after reaching an equilibrium point. Furthermore, in graph (b), which illustrates the volume of individual loans, an increase in n_1 correlates with a decrease in the volume of individual loans. The volume of individual loans is highest when $n_1 = 0.13$ and lowest when $n_1 = 0.20$. This behavior is consistent with the structure of the model, where the parameter n_1 appears as a reduction factor in the effective growth term of individual loans, thereby lowering their equilibrium level. Finally, graph (c) depicts the impact of individual NPL on the volume of company loans. In contrast to the previous two graphs, an increase in n_1 leads to an increase in the volume of company loans. This occurs because both loan types interact through shared deposits in the model, so a reduction in individual loan growth allows a greater proportion of deposits to be utilized by company loans.

Fig. 6 illustrates how variations in the company's NPL rate (n_2) influence the dynamics of (a) deposit volume, (b) individual loan volume, and (c) company loan volume. Graph (a) reveals that a higher company NPL rate correlates with a decrease in the overall deposit volume. For instance, the green line ($n_2 = 0.15$) shows the highest steady-state deposit volume, while the magenta line ($n_2 = 0.23$) indicates the lowest deposit volume as n_2 increases. Moving to graph (b), which depicts individual loan volume, an upward trend in n_2 corresponds to an increase in the volume of individual loans. This effect arises from the competitive interaction terms in the model, where a reduction in company loan effectiveness due to higher n_2 indirectly allows individual loans to utilize more of the available deposits. Finally, graph (c) examines the impact of n_2 on company loan volume. As the company's NPL rate (n_2) rises, there is a clear reduction in the volume of company loans. This result follows directly from the model structure, in which higher NPL reduces the effective loan growth, leading to a lower equilibrium point. In summary, fluctuations in NPL significantly affect deposit and loan volumes, as reflected in the mathematical structure of the model, where NPL parameters act as reduction factors in loan growth and influence the equilibrium distribution of deposits between loan types.



(a) (b)

(c)

Figure 6. The Impact of Fluctuations in Company's NPL Rates (n_2) on the Volumes of (a) Deposit, (b) Individual Loan, and (c) Company Loan

4. CONCLUSION

This paper presents a mathematical model constructed to analyze two distinct types of loans, those for individuals and those for companies, within a single banking institution. The model draws an analogy to a two-predator-one-prey system. We have rigorously established the existence and uniqueness of the model's solution, alongside demonstrating the non-negativity and boundedness of its outcomes. Our analysis reveals the presence of five distinct equilibrium points within the model. These include a point where no deposit or loan activities occur, a scenario where loans are absent, and specific points for the absence of individual loans, company loans, and a final point representing active deposit and loan operations. Significantly, the equilibrium point characterized by the absence of both deposit and loan activities is consistently found to be unstable, indicating that such a condition is not sustainable in real banking systems. In contrast, the remaining four equilibrium points demonstrate the potential for stability under specific conditions, suggesting that the balance between loan distribution and non-performing loan dynamics plays an important role in maintaining banking stability. These results provide insight into how different loan structures and credit risk conditions may influence the long-term performance of a banking institution. However, this model has limitations, as it is developed based on theoretical assumptions and has not yet been calibrated using empirical banking data. Therefore, future research may focus on empirical calibration using real banking data, particularly individual and company non-performing loan data, to enhance the model's practical applicability and validation.

Author Contributions

Onik Febria Damayanti: Conceptualization, Formal Analysis and Investigation, Writing–Original Draft Preparation, Resources, Visualization. Isnani Darti: Conceptualization, Writing–Review and Editing, Supervision, Validation. Agus Suryanto: Conceptualization, Methodology, Writing–Review and Editing, Supervision. Nur Shofianah: Conceptualization, Writing–Review and Editing, Supervision. Trisilowati:

Methodology, Writing-Review and Editing, Visualization. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare that there are no conflict of interests regarding the publication of this paper.

Declaration of Generative AI and AI-assisted Technologies

Generative AI tools (e.g., ChatGPT) were used solely for language refinement, including grammar, spelling, and clarity. The scientific content, analysis, interpretation, and conclusions were developed entirely by the authors. All final text was reviewed and approved by the authors.

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