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THE LOCATING RAINBOW EDGE CONNECTION NUMBERS OF SOME GENERALIZED SUN GRAPHS

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ABSTRACT

Throughout this paper, G denotes a finite, simple, connected, and undirected graph. The concept of the locating rainbow edge connection number is motivated by the concept of the locating rainbow connection number in which the coloring is assigned to edges while preserving the locating property. The coloring focuses on determining the smallest natural number k such that there exists a locating rainbow edge k-coloring of a graph such that every edge has a distinct rainbow code. In this paper, we define standard generalized sun graphs $Sun(n,p)$ and leveled generalized sun graphs Sun_n^l. We determine their locating rainbow edge connection numbers. The results are obtained through a theoretical approach and observations of graph structures with rigorous mathematical proofs. The results of Sun(n,p) and Sun_n^l depend on their cycle order and the number of bridges.

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1. INTRODUCTION

Throughout this paper, G denotes a finite, simple, connected, and undirected graph. For some $k \in \mathbb{N}$, the definition of a *vertex coloring* of a graph can be seen in [1]. A variation of vertex coloring called the *partition dimension* was introduced in 2000 [2] . An ordered partition $\Pi = \{S_1, S_2, \dots, S_k\}$ of $V(G)$ assigns to each vertex v. A representation of vertex v with respect to Π is $r(v|\Pi) = \big(d(v,S_1), d(v,S_2), \dots, d(v,S_k)\big)$, where $d(v,S_i) = min\{\, d(v,u) |\, u \in S_i\,\}$. The partition Π is called a *resolving k-partition* if every vertex of G has distinct representation. The smallest k such that there exists a resolving k-partition is called the *partition dimension* of G, denoted by $pd(G)$. Several results on the partition dimension of cycle-related graphs were presented in [3].

Besides vertex coloring and its variations, graph coloring can also be studied from the perspective of edge coloring. Chartrand et al. introduced the concept of rainbow connection in 2008. An edge coloring c is called a *rainbow coloring* if every pair of vertices in G is joined by a *rainbow path* whose edges have distinct colors. If k colors are used, then c is called a rainbow k-coloring. The smallest k such that there exists a rainbow k-coloring is called the *rainbow connection number* of G, denoted by $rc(G)$ [4], [5]. Several studies have been devoted to rainbow connection and related rainbow coloring parameters. The rainbow connection number has been determined for several graph classes, including origami and pizza graphs [6] , flower graphs [7], and n-crossed prism graphs and their corona products [8]. Further developments include rainbow connection under graph operations [9], and rainbow connection of graphs with diameter two [10]. Another studies related on concept of rainbow coloring is the strong 3-rainbow index of edge-amalgamation and edge-comb product graphs [11], [12].

Moreover, another relevant concept is a rainbow vertex coloring introduced by Krivelevich and Yuster. A vertex coloring c is said to be a rainbow vertex k-coloring if every pair of vertices in G is joined by a rainbow vertex path which is a path where every internal vertex has a distinct color. The smallest k such that there exists a rainbow vertex k-coloring is called the rainbow vertex connection number, denoted by $rvc(G)$ [13]. Further results on $rvc(G)$ have been obtained for star fan graphs [14], comb product graphs [15], star-cycle graphs [16], and pencil graphs [17].

In [18], Bustan et al. introduced a new concept using rainbow vertex coloring. The concept combines the idea of the partition dimension and a rainbow vertex coloring of a graph, called *the locating rainbow coloring*. A vertex coloring c is called a *locating rainbow coloring* if c is a rainbow vertex coloring and every vertex has a distinct rainbow code. The locating rainbow connection number has been studied for several graph classes, including amalgamations of complete graphs [19], edge-comb products of graphs [20], trees and regular bipartite graphs [21].

Motivated by the previous concept, in 2025 Mukayis et al. [22] introduced the locating rainbow edge coloring. This concept is a modification from the locating rainbow coloring concept that focus on vertex coloring become an edge coloring called the locating rainbow edge coloring.

For two edges $e_1 = a_1b_1$ and $e_2 = a_2b_2$, the edge distance is given by $d(e_1,e_2) = min\{d(a_1,a_2), d(a_1,b_2), d(a_2,b_1), d(b_1,b_2)\} + 1$ if $e_1 \neq e_2$ and $d(e_1,e_2) = 0$ if $e_1 = e_2$. Now let R_i denote the set of edges colored with i for each $i \in \{1, 2, \dots, k\}$ and define the ordered partition $\Pi = \{R_1, R_2, \dots, R_k\}$. The *rainbow code* of an edge $e \in E(G)$ with respect to Π is given by $rc_\Pi(e) = (d(e,R_1), d(e,R_2), \dots, d(e,R_k))$, where $d(e,R_i) = min\{d(e,f) | f \in R_i\}$. A rainbow coloring $c: E(G) \to \{1, 2, \dots, k\}$ is called a *locating rainbow edge k-coloring* (LREC-k) if c is a rainbow coloring and all edges have a distinct rainbow code. The *locating rainbow edge connection number* of G, denoted by $recl(G)$, is the smallest k for which an $LREC$-k exists for G.

One of the known results of the locating rainbow edge connection numbers concerns about cycle graphs [22]. On the other hand, bridges are also important since they affect the locating rainbow edge connection numbers of a graph. This motivates the study of generalized sun graphs. A generalized sun graph is formed from a cycle where every vertex is attached with some bridge edges. This makes generalized sun graph suitable for observing how the locating rainbow edge connection number changes when a cycle is extended by bridges structure. In this paper, we investigate two types of generalized sun graphs, namely the standard generalized sun graph $Sun(n,p)$ and the leveled generalized sun graph Sun_n^l. Furthermore, we determine their locating rainbow edge connection number.

2. RESEARCH METHODS

This research uses a theoretical literature-based method. We begin by reviewing concepts, known results, lemmas, and theorems related to the locating rainbow edge connection number of graphs. In particular, attention is given to previous results on cycle graphs and to structural properties involving bridges, since both aspects are closely related to generalized sun graphs.

The proof of each result is developed through two complementary steps. First, a lower bound is done using contradiction argument and previously established lemmas. Second, an upper bound is obtained by constructing an edge-coloring and verifying that it is a locating rainbow edge coloring. When the lower and upper bounds are equal, we obtain the exact value of the locating rainbow edge connection number. The results are then presented as the main theorems of this paper.

Before proving the main results, we recall two lemmas that will be used in our proofs

Lemma 1. [5] *Let G be a connected graph of order n containing two bridges e and f. For each integer k with $2 \leq k \leq n$, every rainbow k-coloring of G must assign distinct colors to e and f.*

The following lemma gives a restriction on color repetitions in unicyclic graphs.

Lemma 2. [23] *Let G_r be a unicyclic graph and $\{c_1, c_2, \dots, c_t\}$ be the set of colors used in a rainbow coloring of G_r. Then, for each tree component T_i attached to the cycle, the colors used on $E(T_i)$ for $1 \leq i \leq r$ appear at most twice in the entire graph G_r.*

3. RESULTS AND DISCUSSION

We begin this section by defining two classes of generalized sun graphs that will be the focus of our study. The first is the standard generalized sun graph, which extends the classical sun graph by allowing multiple leaves attached to each vertex of the cycle. The second is the leveled generalized sun graph, which organizes the leaves into distinct levels.

Definition 1. *Let n and p be two integers with $n \geq 3$ and $p \geq 1$. The standard generalized sun graph (n, p), denoted by $Sun(n, p)$, is a graph with vertex and edge sets defined, respectively, as follows:*

$$V(Sun(n,p)) = \{ v_i, v_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq p\}$$

and

$$E(Sun(n,p)) = \{ e_i = v_i\, v_{i+1} \mid 1 \leq i \leq n-1 \} \cup \{v_1 v_n\} \cup \{ e_{i,j} = v_i\, v_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq p\}.$$

Furthermore, if n is odd, the graph $Sun(n, p)$ is called an *odd generalized sun*, while if n is even, it is called an *even generalized sun*. Another generalized sun graph discussed in this paper is the leveled generalized sun graph, which is defined as follows.

Definition 2. For some integer $n \geq 3$ and $l \geq 1$, The leveled generalized sun graph (n, l), denoted by Sun_n^l, is a graph with vertex and edge sets defined respectively as follows:

$$V\left(Sun_n^l\right) = \{v_i \mid 1 \leq i \leq n\} \cup \bigcup_{i=1}^{n} \{v_{i,j} \mid 1 \leq j \leq 1 + (i-1)l \}$$

and

$$E\left(Sun_n^l\right) = \{e_i = v_i v_{i+1} \mid 1 \leq i \leq n-1\} \cup \{v_1 v_n\} \cup \left\{e_{i,1} = v_i v_{i,1} \middle| 1 \leq i \leq n\right\} \cup \bigcup_{i=1}^{n} \{e_{i,j} = v_{i,j} v_{i,j+1} \mid 1 \leq j \leq (i-1)l \}.$$

For illustration, the following Figs. 1 and 2 illustrate the graphs $Sun(5,3)$ and Sun_4^2.

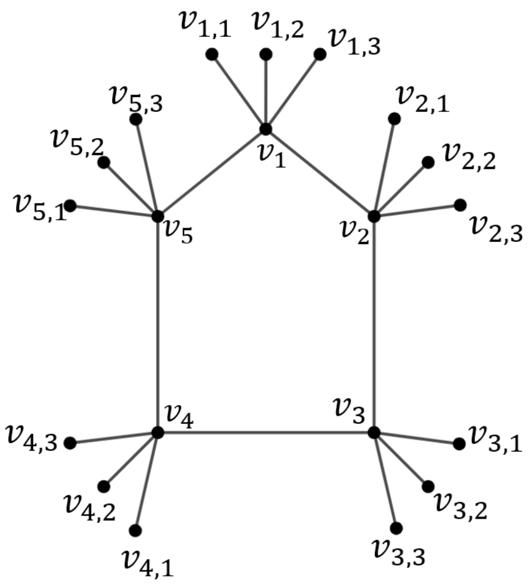


Figure 1. A Standard Generalized Sun Graph $Sun(5,3)$

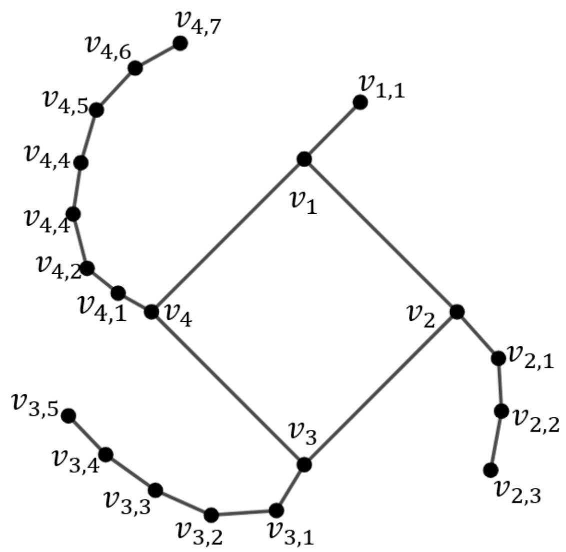


Figure 2. A Leveled Generalized Sun Graph Sun_4^2

For the purpose of proofs, we define two canonical paths between any pair of vertices $v_a, v_b \in V(R)$, where R is a subgraph of a unicyclic graph that is isomorphic to a cycle graph. The path $P_1(v_a, v_b)$ denotes the f*orward path along the cycle*, consisting of the sequence $(v_a, v_{a+1}, \dots, v_{b-1}, v_b)$ while $P_2(v_a, v_b)$ denotes the *backward path along the cycle*, consisting of the sequence $(v_a, v_{a-1}, \dots, v_{b+1}, v_b)$. Note that, $P_1(v_a, v_b)$is equal to $P_2(v_b, v_a)$. Fig. 3 illustrates $P_1(v_5, v_8)$ in blue, and $P_2(v_4, v_9)$ in red.

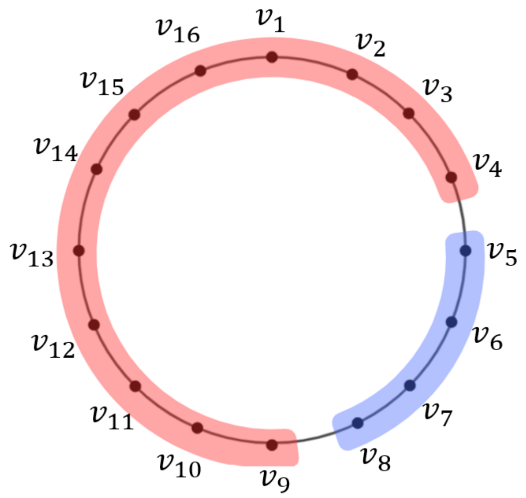


Figure 3. An Illustration of $P_1(v_5, v_8)$ and $P_2(v_4, v_9)$

This paper establishes the exact value of the locating rainbow edge coloring numbers for standard generalized sun graphs and leveled generalized sun graphs.

3.1 Standard Generalized Sun Graphs

The focus of this subsection is to determine the locating rainbow edge connection numbers of the standard generalized sun graph. We will show that $\boldsymbol{recl(Sun(n,p))}$ depends on the values of $\boldsymbol{n}$ and $\boldsymbol{p}$, with a special case for some condition. The main result for this subsection is established in the theorem below.

Theorem 1. *Let n and p be two positive integers with $n \geq 3$. Then the locating rainbow edge connection number of standard generalized sun graph (n,p) is*

$$recl(Sun(n,p)) = \begin{cases} 4, & for\, n = 3 \;\; and \;\; p = 1; \\ n+1, & for\, even\, n \;\; and \;\; p = 1; \\ np, & otherwise. \end{cases}$$

Proof. We consider three cases.

Case 1. For $n = 3$ and $p = 1$

Suppose $recl(Sun(3,1)) = 3$, then there exists a coloring c such an $LREC$-3 of $Sun(3,1)$. By Lemma 1, we have all bridges have a distinct color. WLOG, let $c'(e_{i,1}) = i$ for $1 \leq i \leq 3$. Thus, to ensure the existence of a rainbow path from $v_{1,1}$ to $v_{2,1}$, the edge v_1v_2 must receive color 3. Applying the same reasoning to all pair of bridges, we get edges $rc_{\Pi}(v_1v_2) = rc_{\Pi}(e_{3,1}) = (1,1,0)$. Therefore, it contradicts with c' is an $LREC$-3. Hence, $recl(Sun(3,1)) > 3$.

For the upper bound, we define $c: E(Sun(3,1)) \rightarrow [1,4]$ as illustrated in Fig. 4:

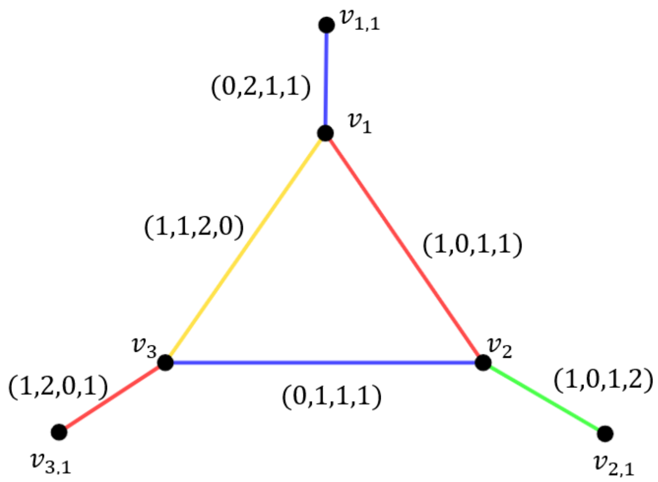


Figure 4. An LREC−4 of $Sun(3,1)$

By construction, c is an LREC-4. Thus, we conclude that, $recl(Sun(3,1)) = 4$.

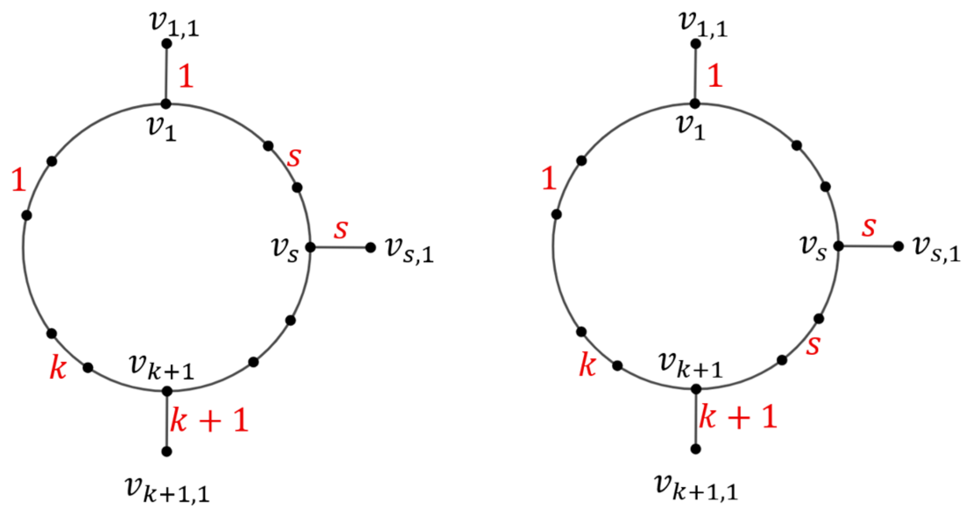
Case 2. For even n and $p = 1$

For the lower bound, suppose there exists an edge coloring c' such an LREC-n of $Sun(n, 1)$. Note that G contains n bridges. By Lemma 1, WLOG, let $c'(e_{i,1}) = i$ for all $1 \leq i \leq n$. By Lemma 2, we know that each color used on $e_{i,1}$ can be reused at most once on the cycle edges. Since the cycle contains n edges, each color used on a bridge must appear exactly once on the cycle.

Let $n = 2k$. Note that $c'(e_{1,1}) = 1$ and $c'(e_{k+1,1}) = k + 1$. Based on Lemma 2, each of these colors appear exactly once more on the cycle. To ensure that c' satisfies the rainbow coloring, both colors must appear either on the $P_1(v_1, v_{k+1})$ or $P_2(v_1, v_{k+1})$. WLOG, let colors 1 and $k + 1$ assign on the $P_2(v_1, v_{k+1})$.

Define the color set $C = \{ c \mid 2 \leq c \leq k \}$. Then $|C| = k - 1$. Since path $P_2(v_1, v_{k+1})$ has k edges and two of them are colored with 1 and $k + 1$, there remain $k - 2$ edges on $P_2(v_1, v_{k+1})$ to be colored. Hence, by pigeonhole principle at least one color from C must be assigned to an edge on $P_1(v_1, v_{k+1})$ namely color s. Note that vertex v_s lies on $P_1(v_1, v_{k+1})$. Now, consider the following two conditions.

1. If color s is assigned to edge e_t with $s > t$, then no rainbow path connecting $v_{s,1}$ and $v_{1,1}$. See Fig. 5 (a).
2. If color s is assigned to edge e_t with $s \leq t$, then no rainbow path connecting $v_{s,1}$ and $v_{k+1,1}$. See Fig. 5 (b).



(a) (b)

Figure 5. Illustrations for a contradiction in the proof of Case 2

This contradicts the assumption that $c'$is an LREC-n. Therefore, we conclude that $recl(Sun(n,1)) > n$.

For the upper bound, the edge coloring function $c : E(Sun(n,1)) \to \{1,2,\dots,n+1\}$ is defined as follows. For every $1 \le i \le n$, assign $c(e_{i,1}) = i$. This ensures that each bridge edge receives a unique color, using exactly n colors. Next, for the cycle edges, the coloring is defined as follows:

$$c(e_l) = \begin{cases} k+l, & \text{for } 1 \le l \le k-1; \\ l+1-k, & \text{for } k \le l \le 2k-2; \\ n+1, & \text{for } l = 2k-1; \\ k, & \text{for } l = 2k. \end{cases}$$

We show that c is a locating rainbow edge $(n+1)$-coloring. First, we begin to show that c is a rainbow coloring. For any $x, y \in V(Sun(n,1))$, we will show that there is a rainbow path from x to y.

1. Let $x = v_i$ and $y = v_j$ with $i \neq j$. For $x, y \in \{v_i \mid 1 \le i \le n\}$, the subgraph C_n is a cycle whose edges are colored distinctly. Hence, any path between x and y is a rainbow path.

2. Let $x = v_{i,1}$ and $y = v_j$ with $1 \le i, j \le n$. Observe that, $c(e_{j,1}) = j$ which appears exactly once on the cycle, either along $P_1(v_i, v_j)$ or $P_2(v_i, v_j)$. If it appears on $P_1(v_i, v_j)$, then the rainbow path between x and y is $x - P_2(v_i, y)$. Otherwise the rainbow path through $x - P_1(v_i, y)$.

3. Let $x = v_{i,1}$ and $y = v_{j,1}$ with $1 \le i, j \le n$. Observe that both $c(e_{i,1}) = i$ and $c(e_{j,1}) = j$ which appear on the cycle exactly once, either along $P_1(v_i, v_j)$ or $P_2(v_i, v_j)$. If they appear on $P_1(v_i, v_j)$, then a rainbow path between x and y is $v_{i,1} - P_2(v_i, v_j) - v_{j,1}$. Otherwise, the rainbow path through $v_{i,1} - P_1(v_i, v_j) - v_{j,1}$.

Hence, the coloring c is rainbow coloring.

Next, we prove that every edge in $Sun(n,1)$ has a distinct rainbow code. A pair of edges with distinct colors, their rainbow codes are distinguished by their own color classes. So that we only consider pair of edges that are the same color. For any $1 \le i \le n$, color i appears at most on two edges $e_{i,1}$ and e_{i+k-1}. Consider their distances to the color class R_{i+1}. Since $d(e_{i,1}, R_{i+1}) = 2$ and $d(e_{i+k-1}, R_{i+1}) = 1$, we have $rc_\Pi(e_{i,1}) \neq rc_\Pi(e_{i+k-1})$.

Since these holds for all $1 \le i \le n$, we conclude that all edges have distinct rainbow codes. Therefore, c is an LREC-$(n+1)$. Thus, $recl(Sun(n,1)) \le n+1$. Combining both upper and lower bounds, we obtain $recl(Sun(n,1)) = n+1$.

Case 3. For (odd n with $p \ge 1$) or (even n with $p \ge 2$)

Since $Sun(n,p)$ contains np pendant edges and each pendant edge is a bridge, by Lemma 1, each edge must be assigned a distinct color in any locating rainbow edge coloring. Therefore, coloring bridges require at least np distinct colors. Thus, we have $recl(Sun(n,p)) \ge np$.

The edge coloring function $c : E(Sun(n,p)) \to \{1,2,\dots,np\}$ is defined as follows. For each $1 \le i \le n$ and $1 \le j \le p$, assign $c(v_i v_{i,j}) = (i-1)p + j$. This ensures that each bridge edge receives a unique color, using exactly np colors. Now, we consider the cycle edges. For odd n, let $k = \frac{n-1}{2}$. Assign $c(e_{i,1}) = c(e_{i+k})$. While for even n, with $n = 2k$ and $p \ge 2$, assign

$$c(e_l) = \begin{cases} (k+l-1)p+1, & \text{for } 1 \le l \le k-1; \\ (l-k)p+1, & \text{for } k \le l \le 2k-2; \\ (k-1)p+2, & \text{for } l = 2k-1; \\ k, & \text{for } l = 2k. \end{cases}$$

For any $x, y \in V(Sun(n,p))$, there is a rainbow path from x to y. If x and y both are vertices on the cycle, then any path $x - y$ is a rainbow path because the color of every edge in the cycle is distinct. Now, we consider at least one of x or y is pendant vertex.

1. Let $x = v_{i,s}$ and $y = v_j$ with $1 \le i,j \le n$ and $1 \le s \le p$. Observe that, if $s \ne 1$, then the color of $v_i v_{i,s}$ is not used in cycle edges. Hence, $x - P_1(v_i, y)$ is a rainbow path. Otherwise, there are only 2 different paths that connecting x and y which are $x - P_1(v_1 y)$ or $x - P_2(v_1 y)$. If color of $c(v_i v_{i,1})$ appear on $P_1(v_1 y)$, then the rainbow path through $x - P_2(v_1 y)$. Otherwise, the rainbow path through $x - P_1(v_1 y)$.
2. Let $x = v_{i,s}$ and $y = v_{j,t}$ with $1 \le i,j \le n$. Observe that, if $s,t \ne 1$, then the colors of $v_i v_{i,s}$ and $v_j v_{j,t}$ are not used in cycle edges. Hence, $x - P_1(v_i, v_j) - y$ is a rainbow path. Otherwise, there are only two different paths that connecting x and y which are $x - P_1(v_1 v_j) - y$ or $x - P_2(v_1 v_j) - y$. By coloring c, the colors $c(v_i v_{i,s})$ and $c(v_j v_{j,t})$ both are can only appear either on $P_1(v_1 v_j)$ or $P_2(v_1 v_j)$. If appear on $P_1(v_1 v_j)$, then the rainbow path through $x - P_2(v_1 v_j) - y$. Otherwise, the rainbow path through $x - P_1(v_1 v_j) - y$.

Hence, c is a rainbow coloring. Next, we verify that the rainbow codes of all edges are distinct. A pair of edges with distinct colors, their rainbow codes are distinguished by their own color classes. So that we only consider edges with same color classes. Note that, under the coloring c, the only colors that may appear more than once are those assigned to the edges $e_{i,1} = v_i v_{i,1}$ for $1 \le i \le n$. For each i, let e_i' denote the edge on the cycle such that $c(e_i') = c(e_{i,1})$. To distinguish their rainbow code, consider the color class R_{i+p}. Observe that, by coloring c we have $d(e_{i,1}, R_{i+p}) = 2$ and $d(e_i', R_{i+p}) = 1$. Hence, $rc_\Pi(e_{i,1}) \ne rc_\Pi(e_i')$.

This guarantees that the associated rainbow codes are distinct. Thus, the coloring c satisfies locating rainbow edge coloring. We conclude that c is an LREC-np, and hence $recl(Sun(n,p)) \le np$. Combining both upper and lower bounds, we have $recl(Sun(n,p)) = np$.

The following figures illustrate the coloring c for each case from the proof. Fig. 6 corresponds to Case 2 (even n and $p = 1$), Fig. 7 corresponds to Case 3 (odd n and $p \ge 1$), and Fig. 8 corresponds to Case 3 (even n and $p \ge 2$).

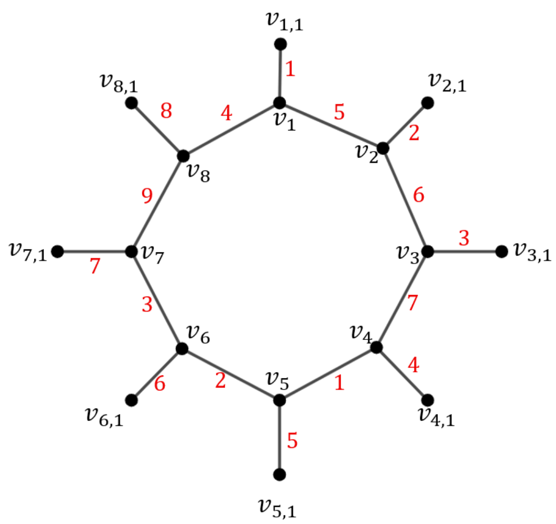


Figure 6. An LREC-9 of $Sun(8,1)$

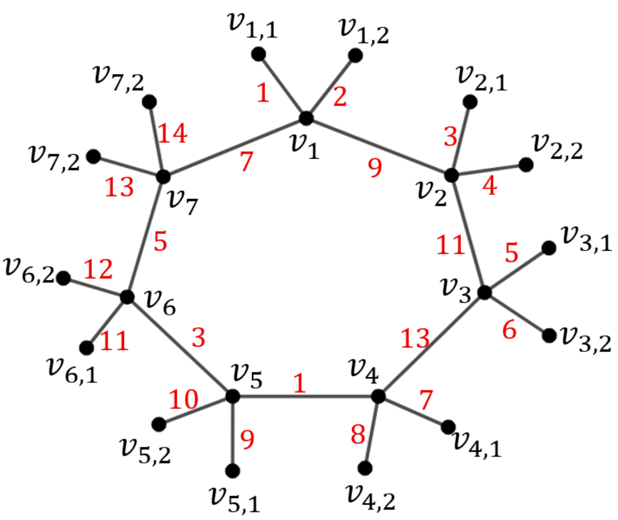


Figure 7. An LREC-14 of $Sun(7,2)$

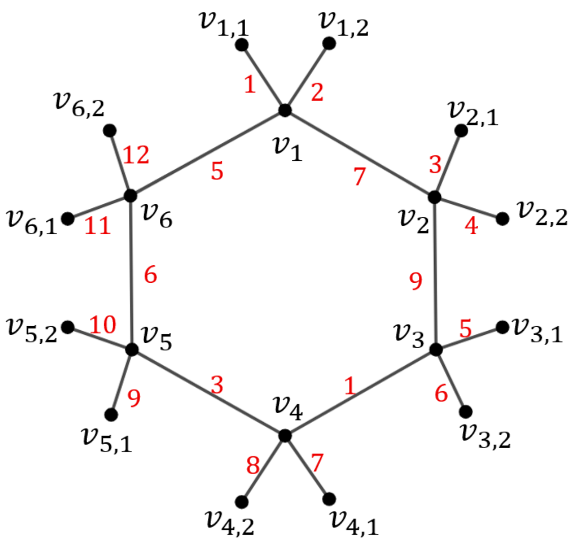


Figure 8. An LREC-12 of $Sun(6,2)$

3.2 Leveled Generalized Sun Graphs

The focus of this subsection is to determine the locating rainbow edge connection numbers of the leveled generalized sun graphs. To prove the main result, we first present the following lemma.

Lemma 3. *Let n and l be two positive integers with $n \geq 3$ and let Sun_n^l be a leveled generalized sun graph (n, l). If c be a locating rainbow edge coloring of Sun_n^l, then for every maximal connected subgraph of Sun_n^l that contains no cycle edge, the number of colors used in this subgraph that can be reused for the cycle edges is at most one and can be reused once.*

Proof. By Lemma 1, for every bridge must have a distinct color. Let H be a maximal connected subgraph of Sun_n^l that contains no cycle edge. Let $v \in V(H)$ be a vertex whose distance to the cycle is maximum and $w \in V(H)$ be a vertex whose distance to the cycle equal to zero. Suppose H contains two edges whose colors are reused on the cycle edges (say c_1 and c_2). Then there exists a vertex u on the cycle such that the two distinct paths on the cycle from w to u contain edges colored c_1 and c_2, respectively. Since every path from v to w passes through edges colored c_1 and c_2, it follows that every path from v to u contains at least two edges with the same color. Hence, there exists no rainbow path connecting u and v. It leads to condradict with the rainbow coloring. By Lemma 2, any color from subgraph that reused on cycle edge at most appear twice in Sun_n^l.

■

Fig. 9 below illustrates the contradiction in the above lemma.

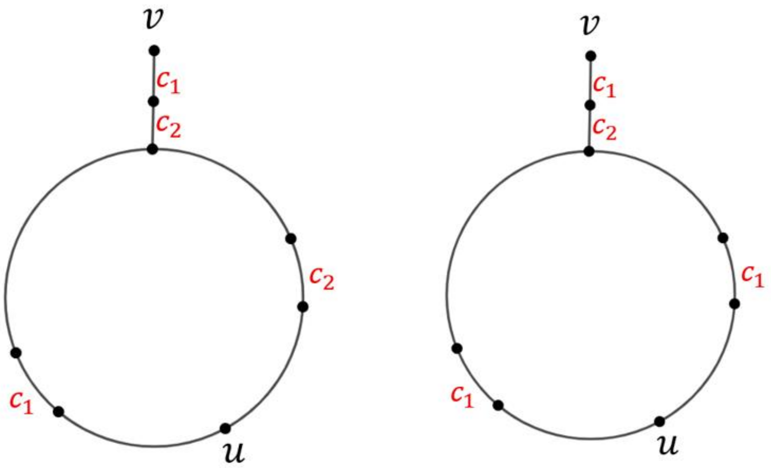


Figure 9. Illustrations for a Contradiction in the Proof of Lemma 3

With the above lemma in hand, we are now ready to establish the value of the locating rainbow edge connection number of leveled generalized sun graphs in the following theorem.

Theorem 2. *Let n and l be two positive integers with $n \geq 3$. Then the locating rainbow edge connection number of the leveled generalized sun graph (n, l) is*

$$recl(Sun_n^l) = \begin{cases} \dfrac{ln^2 - ln + 2n}{2} + 1, & \text{for even } n; \\ \dfrac{ln^2 - ln + 2n}{2}, & \text{for odd } n. \end{cases}$$

Proof. The proof proceeds by considering two cases.

Case 1. For even n

Let $n = 2k$. Remark that Sun_n^l has $\frac{ln^2 - ln + 2n}{2}$ bridges. Suppose there exists an edge coloring c' such an LREC-$\left(\frac{ln^2 - ln + 2n}{2}\right)$ of Sun_n^l. For $1 \leq i \leq n$, let H_i be a maximal subgraph of Sun_n^l that contain no cycle edges and contains the vertex v_i. By Lemma 1, assign distinct colors to all bridge edges of Sun_n^l using colors 1 to $\frac{ln^2 - ln + 2n}{2}$. By Lemma 3, from each H_i we can only reuse at most one color on the cycle and can be used at most once. Let this color be denoted by c_i, taken from H_i. Since there are n uncolored cycle edges and each H_i contributes at most one reuseable color, it follows that each color c_i must be used exactly once on the cycle. Consider subgraphs H_1 and H_{k+1} with their pendant vertices $v_{1,1}$ and $v_{k+1,kl+1}$ respectively.

Since c' is a rainbow coloring, there must exists a rainbow path connecting $v_{1,1}$ and $v_{k+1,kl+1}$. Hence, besides in the bridge edges, the colors c_1 and c_2 must both appear on the $P_1(v_1, v_{k+1})$ or $P_2(v_1, v_{k+1})$. WLOG, let colors c_1 dan c_{k+1} assign on the $P_2(v_1, v_{k+1})$.

Define color set $C = \{c_i | 2 \leq i \leq k\}$. Then $|C| = k - 1$. Since path $P_2(v_1, v_{k+1})$ has k edges and two of them are colored with c_1 and c_{k+1}, there remain $k - 2$ edges on $P_2(v_1, v_{k+1})$ to be colored. Hence, by pigeonhole principle at least one color from C must be assigned to an edge on $P_1(v_1, v_{k+1})$ namely color c_s. Note that vertex v_s lies on $P_1(v_1, v_{k+1})$ Now, consider the following two conditions.

1. If color c_s is assigned to edge e_t with $s > t$, then no rainbow path connecting $v_{s,1+(s-1)l}$ and $v_{1,1}$.
2. If color c_s is assigned to edge e_t with $s \leq t$, then no rainbow path connecting $v_{s,1+(s-1)l}$ and $v_{k+1,kl+1}$.

This contradicts the assumption that c' is an LREC-$\left(\frac{ln^2 - ln + 2n}{2}\right)$. Therefore, we conclude that

$$recl\left(Sun_n^l\right) \geq \frac{ln^2 - ln + 2n}{2} + 1.$$

For the upper bound, we define $c\colon E\left(Sun_n^l\right) \rightarrow \{1,2, \dots, \frac{ln^2 - ln + 2n}{2} + 1\}$ as below.

1. For the bridge edges, we assign colors sequentially from the maximal connected subgraph associated with v_1 to the one that associated with v_n starting with the edge that is closest to the cycle. Let the colors that can be reused in the cycle be denoted by $c_1, c_2, \dots, c_n$, where c_i is the color assigned from the edge $e_{i,1}$ for $1 \leq i \leq n$.

2. For the cycle edges,

$$c(e_l) = \begin{cases} c_{k+l-1}, & \text{for } 1 \leq l \leq k-1; \\ c_{l-k+1}, & \text{for } k \leq l \leq 2k-2; \\ \dfrac{ln^2 - ln + 2n}{2} + 1, & \text{for } l = 2k-1; \\ c_k, & \text{for } l = 2k. \end{cases}$$

For any $x, y \in V\left(Sun_n^l\right)$, we will show that there is a rainbow path from x to y.

1. Let $x = v_i$ and $y = v_j$ with $i \neq j$. For $x, y \in \{v_i \mid 1 \leq i \leq n\}$, the subgraph C_n is a cycle whose edges are colored distinctly. Hence, any path between x and y is a rainbow path.

2. Let $x \in H_i$ and $y = v_j$ with $1 \leq i, j \leq n$. Observe that, c_i appears exactly once on the cycle, either along $P_1(v_i, v_j)$ or $P_2(v_i, v_j)$. If it appears on $P_1(v_i, v_j)$, then the rainbow path between x and y is $x - P_2(v_i, y)$. Otherwise, the rainbow path between x and y is $x - P_1(v_i, v_j)$.

3. Let $x \in H_i$ and $y \in H_j$ with $1 \leq i, j \leq n$. Observe that both c_i and c_j appear on the cycle exactly once, either along $P_1(v_i, v_j)$ or $P_2(v_i, v_j)$. If they appear on $P_1(v_i, v_j)$, then a rainbow path between x and y is $x - P_2(v_i, v_j) - y$. Otherwise, the rainbow path between x and y is $x - P_1(v_i, v_j) - y$.

Hence, c is a rainbow coloring.

Next for the rainbow code, we only consider pairs of edges that receive the same color because for the edges that have distinct colors, their rainbow codes are distinguished by their own color classes. The only pairs of edges assigned to the same color are edges on H_i (say e_{c_i}) and the cycle edges farthest from H_i say e'_{c_i}

1. For $i = 1$, we have
$$d\left(e_{c_i}, R_2\right) = 2 \neq 1 = d\left(e'_{c_i}, R_2\right).$$

2. For $i \geq 2$, there exists a bridge adjacent to edge colored c_i whose color is used exactly once in the graph, say color c'_i. Thus, we have $d\left(e_{c_i}, R_{i'}\right) = 1 \neq k = d\left(e_{c'_i}, R_{i'}\right)$.

Therefore $rc_\Pi\left(e_{c_i}\right) \neq rc_\Pi\left(e'_{c_i}\right)$.

Hence, c is an LREC-$\left(\frac{ln^2 - ln + 2n}{2} + 1\right)$. Thus, $recl\left(Sun_n^l\right) \leq \frac{ln^2 - ln + 2n}{2} + 1$. Combining both upper and lower bounds, we obtain $recl\left(Sun_n^l\right) = \frac{ln^2 - ln + 2n}{2} + 1$.

Case 2. For odd n

Let $n = 2k + 1$. By Lemma 1, we have

$$recl\left(Sun_n^l\right) \geq \frac{ln^2 - ln + 2n}{2}.$$

For the upper bound, define $c\colon E\left(Sun_n^l\right) \rightarrow \{1,2, \dots, \frac{ln^2 - ln + 2n}{2}\}$ as below.

1. For the bridge edges, we assign colors sequentially from the maximal connected subgraph associated with v_1 to that of v_n starting with the edge that is closest to the cycle.

2. For the cycle edges,

$$c(e_{i+k}) = c(e_{i,1}).$$

By using the same method as in previous case, we have c is a rainbow coloring. Next, we show that all rainbow codes are distinct. A pair of edges with distinct colors, their rainbow codes are distinguished by their own color classes. Observe the edges assigned by the same colors namely $e_{i,1}$ and e_{i+k}.

1. For $i = 1$, we have

$$d(e_{1,1}, R_2) = 2 \neq 1 = d(e_{i+k}, R_2).$$

2. For $i \geq 2$, there exists a bridge adjacent to $e_{i,1}$ that has color b. Thus, we have

$$d(e_{i,1}, R_b) = 1 \neq k = d(e_{i+k}, R_b).$$

Therefore, $rc_{\Pi(e_{i,1})} \neq rc_{\Pi(e_{i+k})}$. Hence, c is an LREC-$\left(\frac{ln^2 - ln + 2n}{2}\right)$ and $recl(Sun_n^l) \leq \frac{ln^2 - ln + 2n}{2}$. Combining upper and lower bounds, we obtain $recl(Sun_n^l) = \frac{ln^2 - ln + 2n}{2}$.

■

The following figures illustrate the coloring c for each case of the proof. Fig. 10 corresponds to Case 1 (even n) and Fig. 11 corresponds to Case 2 (odd n).

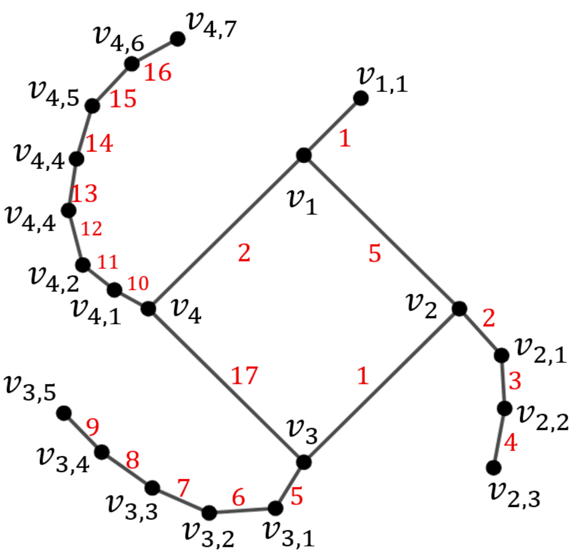


Figure 10. An LREC-17 of Sun_4^2

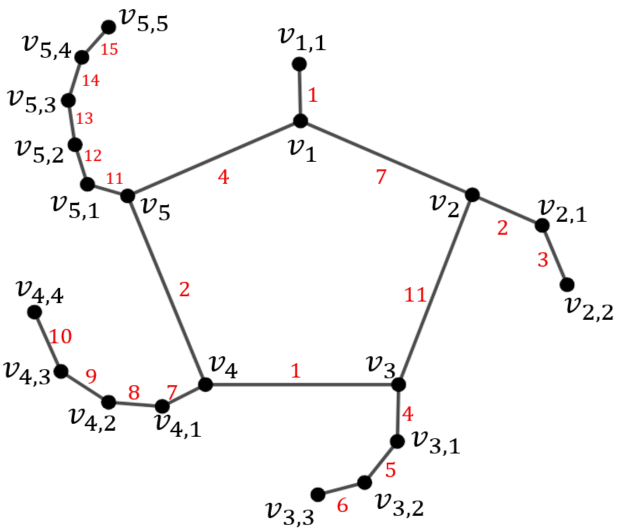


Figure 11. An LREC-15 of Sun_5^1

4. CONCLUSION

In conclusion, the locating rainbow edge connection numbers of the two graph classes considered in this paper are determined by the number of bridges, or the number of bridges plus one. For the standard generalized sun graph, the value of its locating rainbow edge connection number is equal to the number of bridges plus one only when $n = 3$ and $p = 1$ or when n is even and $p = 1$. Meanwhile, for the leveled generalized sun graph, the value of its locating rainbow edge connection number is equal to the number of bridges plus one only when n is even. In all other cases, the locating rainbow edge connection number is exactly equal to the number of bridges.

For future work, it would be interesting to extend this study to other families of unicyclic graphs, such as generalized sun graphs with unequal pendant paths or graphs obtained from cycles by attaching more complex tree graphs. Another possible direction is to investigate the locating rainbow edge connection number of bicyclic graphs to better understand how structural modifications affect the locating rainbow edge connection number.

Author Contributions

Muhammad Ahnaf Yusuf: Formal Analysis, Project Administration, Writing – Original Draft. A.N.M. Salman: Supervision, Validation, Writing-Review and Editing. Mukayis: Writing – Original Draft. All authors discussed the results collaboratively and contributed to the final version of the manuscript.

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Declarations

The authors confirm that there are no conflicts of interest associated with the research, authorship, or publication of this article.

Declaration of Generative AI and AI-assisted technologies

The authors declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, including for writing, editing, data analysis, or the creation of tables and figures.

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