THE ENTIRE FACE IRREGULARITY STRENGTH OF A BOOK WITH POLYGONAL PAGES

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Abstract

A face irregular entire labeling is introduced by Baca et al. recently, as a modification of the well-known vertex irregular and edge irregular total labeling of graphs and the idea of the entire colouring of plane graph. A face irregular entire $k$-labeling $\lambda: V \cup E \cup F \rightarrow \{1,2,\ldots,k\}$ of a 2-connected plane graph $G = (V,E,F)$ is a labeling of vertices, edges, and faces of $G$ such that for any two different faces $f$ and $g$, their weights $w_f(f)$ and $w_g(f)$ are distinct. The minimum $k$ for which a plane graph $G$ has a face irregular entire $k$-labeling is called the entire face irregularity strength of $G$, denoted by $efs(G)$.

This paper deals with the entire face irregularity strength of a book with $m$ $n$-polygonal pages, where embedded in a plane as a closed book with $n$ -sided external face.

Keywords and phrases: Book, entire face irregularity strength, face irregular entire $k$-labeling, plane graph, polygonal page.

NILAI KETAKTERATURAN SELURUH MUKA GRAF BUKU SEGI BANYAK

Abstrak

Pelabelan tak teratur seluruh muka diperkenalkan oleh Baca et al. baru-baru ini, sebagai suatu modifikasi atas pelabelan total tak teratur titik dan tak teratur sisi suatu graf serta ide tentang pewarnaan lengkap pada graf bidang. Pelabelan $k$- tak teratur seluruh muka $\lambda: V \cup E \cup F \rightarrow \{1,2,\ldots,k\}$ dari suatu graf bidang 2-connected $G = (V,E,F)$ adalah suatu pelabelan seluruh titik, sisi, dan muka internal dari $G$ sedemikian sehingga untuk sebarang dua muka $f$ dan $g$ berbeda, bobot muka $w_f(f)$ dan $w_g(f)$ juga berbeda. Bilangan bulat terkecil $k$ sedemikian sehingga suatu graf bidang $G$ memiliki suatu pelabelan $k$-tak teratur seluruh muka disebut nilai ketakteraturan seluruh muka dari $G$, dinotasikan oleh $efs(G)$.

Kami menentukan nilai eksak dari nilai ketakteraturan seluruh muka graf buku segi-$n$, dimana pada bidang datar dapat digambarkan seperti suatu buku tertutup.

Kata Kunci: Graf bidang, graf buku segi-$n$, nilai ketakteraturan seluruh muka, pelabelan lengkap $k$-tak teratur muka.

1. Introduction

Let $G$ be a finite, simple, undirected graph with vertex set $V(G)$ and edge set $E(G)$. A total labeling of $G$ is a mapping that sends $V \cup E$ to a set of numbers (usually positive or nonnegative integers). According to the condition defined in a total labeling, there are many types of total labeling have been investigated.

Baca, Jendrol, Miller, and Ryan in [1] introduced a vertex irregular and edge irregular total labeling of graphs. For any total labeling $f: V \cup E \rightarrow \{1,2,\ldots,k\}$, the weight of a vertex $v$ and the weight of an edge $e = xy$ are defined by $w(v) = f(v) + \sum_{uv \in E} f(uv)$ and $w(xy) = f(x) + f(y) + f(xy)$, respectively. If all the vertex weights are distinct, then $f$ is called a vertex irregular total $k$-labeling, and if all the edge weights are distinct, then $f$ is called an edge irregular total $k$-labeling. The minimum value of $k$ for which there exist a vertex (an edge) irregular total labeling $f: V \cup E \rightarrow \{1,2,\ldots,k\}$ is called the total vertex (edge) irregularity...
strength of $G$ and is denoted by $tvs(G)$ ($tes(G)$), respectively. There are several bounds and exact values of $tvs$ and $tes$ were determined for different types of graphs given in [1] and listed in [2].

Furthermore, Ivanco and Jendrol in [3] posed a conjecture that for arbitrary graph $G$ different from $K_5$ and maximum degree $\Delta(G)$,

$$tes(G) = \max \left\{ \frac{|E(G)| + 2}{3}, \frac{\Delta(G) + 1}{2} \right\}.$$ 

Combining previous conditions on irregular total labeling, Marzuki et al. [4] defined a totally irregular total labeling. A total $k$-labeling $f : V \cup E \rightarrow \{1, 2, \ldots, k\}$ of $G$ is called a totally irregular total $k$-labeling if for any pair of vertices $x$ and $y$, their weights $w(x)$ and $w(y)$ are distinct and for any pair of edges $x_1x_2$ and $y_1y_2$, their weights $w(x_1x_2)$ and $w(y_1y_2)$ are distinct. The minimum $k$ for which a graph $G$ has totally irregular total labeling, is called total irregularity strength of $G$, denoted by $ts(G)$. They have proved that for every graph $G$,

$$ts(G) \geq \max\{tes(G), tvs(G)\} \tag{6}$$

Several upper bounds and exact values of $ts$ were determined for different types of graphs given in [4], [5], [6], and [7].

Motivated by this graphs invariants, Baca et al. in [8] studied irregular labeling of a plane graph by labeling vertices, edges, and faces while considering the weights of faces. They defined a face irregular entire labeling.

A 2-connected plane graph $G = (V, E, F)$ is a particular drawing of planar graph on the Euclidean plane where every face is bound by a cycle. Let $G = (V, E, F)$ be a plane graph. A labeling $\lambda : V \cup E \cup F \rightarrow \{1, 2, \ldots, k\}$ is called a face irregular entire $k$-labeling of the plane graph $G$ if for any two distinct faces $f$ and $g$ of $G$, their weights $w_3(f)$ and $w_3(g)$ are distinct. The minimum $k$ for which a plane graph $G$ has a face irregular entire $k$-labeling is called the entire face irregularity strength of $G$, denoted by $efs(G)$. The weight of a face $f$ under the labeling $\lambda$ is the sum of labels carried by that face and the edges and vertices of its boundary. They also provided the boundaries of $efs(G)$.

**Teorema A.** Let $G = (V, E, F)$ be a 2-connected plane graph $G$ with $n_i$ $i$-sided faces, $i \geq 3$. Let $a = \min\{i | n_i \neq 0\}$ and $b = \max\{i | n_i \neq 0\}$. Then

$$\left\lceil \frac{2a + n_3 + n_4 + \cdots + n_b}{2b + 1} \right\rceil \leq efs(G) \leq \max\{n_i | 3 \leq i \leq b\}.$$ 

For $n_b = 1$, they gave the lower bound as follow

**Teorema B.** Let $G = (V, E, F)$ be a 2-connected plane graph $G$ with $n_i$ $i$-sided faces, $i \geq 3$. Let $a = \min\{i | n_i \neq 0\}$, $b = \max\{i | n_i \neq 0\}$, $n_b = 1$ and $c = \max\{i | n_i \neq 0, i < b\}$. Then

$$efs(G) \geq \left\lceil \frac{2a + |F| - 1}{2c + 1} \right\rceil.$$ 

Moreover, by considering the maximum degree of a 2-connected plane graph $G$, they obtained the following theorem.

**Theorem C.** Let $G = (V, E, F)$ be a 2-connected plane graph $G$ with maximum degree $\Delta$. Let $x$ be a vertex of degree $\Delta$ and let the smallest (and biggest) face incident with $x$ be an $a$-sided (and $b$-sided) face, respectively. Then

$$efs(G) \geq \left\lceil \frac{2a + \Delta - 1}{2b} \right\rceil.$$ 

They proved that Theorem B is tight for Ladder graph $L_n$, $n \geq 3$, and its variation and Theorem C is tight for wheel graph $W_n$, $n \geq 3$. In this paper, we determine the exact value of $efs$ of a book with $m$ $n$-polygonal pages which is greater than the lower bound given in Theorem A - C.
2. Main Results

Considering Theorem C, $efs(W_n)$, and a condition where every face of a plane graph shares common vertices or edges, our first result provide a lower bound of the entire face irregularity strength of a graph with this condition. This can be considered as generalization of Theorem A, B, and C.

**Lemma 2.1.** Let $G = (V, E, F)$ be a 2-connected plane graph with $n_i$ $i$-sided faces, $i \geq 3$. Let $a = \min\{i | n_i \neq 0\}$, $b = \max\{i | n_i \neq 0\}$, $c = \max\{i | n_i \neq 0, i < b\}$, and $d$ be the number of common labels of vertices and edges which have bounded every face of $G$. Then

$$efs(G) \geq \begin{cases} \frac{2a + |F| - d - 1}{2c - d + 1}, & \text{for } n_b = 1, \\ \frac{2a + |F| - d}{2b - d + 1}, & \text{otherwise.} \end{cases}$$

**Proof.** Let $\lambda : V \cup E \cup F \rightarrow \{1, 2, \ldots, k\}$ be a face irregular entire $k$-labeling of 2-connected plane graph $G = (V, E, F)$ with $efs(G) = k$. Our first proof is for $n_b \neq 1$. By Theorem A, the minimum face-weight is at least $2a + 1$ and the maximum face-weight is at least $2a + |F|$. Since $G$ is 2-connected, each face of $G$ is a cycle. It implies that every face might be bounded by common vertices and edges.

Let $d$ be the number of common labels of vertices and edges which have bounded every face of $G$ and $D$ be the sum of all common labels. Then the face-weights $w_\lambda(f_1), w_\lambda(f_2), \ldots, w_\lambda(f_{|F|})$ are all distinct and each of them contains $D$, implies the variation of face-weights is depend on $2a - d + 2 \leq i \leq 2b - d + 1$ labels. Without adding $D$, the maximum sum of a face label and all vertices and edges-labels surrounding it is at least $2a + |F| - d$. This is the sum of at most $2b - d + 1$ labels. Thus, we have $efs(G) \geq \frac{2a + |F| - d}{2b - d + 1}$. 

For $n_b = 1$, it is a direct consequence from Theorem B with the same reason as in the result above. ■

This lower bound is tight for ladder graphs and its variation and wheels given in [8].

A book with $m$ $n$-polygonal pages $B_m^n$, $m \geq 1, n \geq 3$, is a plane graph obtained from $m$-copies of cycle $C_n$ that share a common edge. There are many ways drawing $B_m^n$ for which the external face of $B_m^n$ can be an $n$-sided face or a $(2n - 2)$-sided face.

By considering that topologically, $B_m^n$ can be drawn on a plane as a closed book such that $B_m^n$ has an $n$-sided external face, an $n$-sided internal face, and $m - 1$ number of $(2n - 2)$-sided internal faces, the entire face irregularity strength of $B_m^n$ is provided in the next theorem.

**Theorem 2.2.** For $B_m^n$, $m \geq 1, n \geq 3$, be a book with $m$ $n$-polygonal pages whose an $n$-sided external face, an $n$-sided internal face, and $m - 1$ $(2n - 2)$-sided internal faces, we have

$$efs(B_m^n) = \begin{cases} 2, & \text{for } m \in \{1, 2\}; \\ \left\lfloor \frac{4n + m - 7}{4n - 5} \right\rfloor, & \text{otherwise.} \end{cases}$$

**Proof.** Let $B_m^n, m \geq 1, n \geq 3$, be a 2-connected plane graph. For $m \in \{1, 2\}$, by Lemma 2.1, we have $efs(B_m^n) \geq 2$. Labeling the $n$-sided external face by label 2 and all the rests by label 1, then all face-weights are distinct. Thus, $efs(B_m^n) = 2$.

Now for $m > 2$, let $z = efs(B_m^n)$. Since every internal face of $B_m^n$ shares 2 common vertices, $a = n$, $b = 2n - 2$, and $n_b > 1$, by Lemma 2.1, we have $z \geq \frac{2a + |F| - 2}{2b - 1} = \frac{2n + m - 1}{4n - 5}$. Consider that $z = \frac{2n + m - 1}{4n - 5}$ is not valid, since for $m \leq 2n - 4$, the maximum label is 1.

Moreover, since $B_m^n$ has at least 2 face-weights which are contributed by the same number of labels, there must be 2 faces of the same weight. Then the divisor must be at least $4n - 4$. Thus we have $z \geq \frac{4n + m - 7}{4n - 5}$. 


Next, to show that $z$ is an upper bound for entire face irregularity strength of $B^n_m$, let $B^n_m$, $m \geq 1, n \geq 3$, be the 2-connected plane graph with an $n$-sided internal face $f^n_{int}$, $m - 1 (2n - 2)$-sided internal faces and an external $n$-sided face $f^n_{ext}$.

Let $m_1 = \left\lceil \frac{m}{2} \right\rceil$ and $m_2 = m - m_1$. Our goal is to have $m_1$ distinct even face-weights and $m_2$ distinct odd face-weights such that $m (2n - 2)$-sided face-weights are distinct and form an arithmetic progression.

Let $z = \left\lceil \frac{4n+m-7}{4n-5} \right\rceil$. It can be seen that $B^n_m$ has $m$ different paths of length $(n - 1)$. Next, we divide $m_1$ paths into $S = \left\lceil \frac{m_1}{4n-5} \right\rceil$ parts, where part $s$-th consists of $(4n - 5)$ paths, for $1 \leq s \leq S - 1$, and part $S$-th consists of $r_1 = m_1 - (S - 1)(4n - 5)$ paths. Also, we divide $m_2$ paths into $T = \left\lceil \frac{m_2+1}{4n-5} \right\rceil$ parts, where the first part consists of $(4n - 6)$ paths, part $t$-th consists of $(4n - 5)$ paths, for $2 \leq t \leq T - 1$, and part $T$-th consists of $r_2 = m_2 - (T - 1)(4n - 5)$ paths.

Let

$$V(B^n_m) = \{x, y, u(s)_{i,j}, u(S)_{i,j}, v(t)_{i,j}, v(T)_{i,j} | 1 \leq s \leq S - 1, 1 \leq t \leq T - 1, 1 \leq i \leq 4n - 5, 1 \leq j \leq 2n - 2, 1 \leq k \leq r_1, 1 \leq l \leq r_2\};$$

$$E(B^n_m) = \{xy\} \cup$$

$$\{u(s)_i^1 = u(s)_i^{2j}, u(s)_i^{2j-1} = u(s)_i^{2j-2} u(s)_i^{2j}, u(s)_i^{2n-3} = u(s)_i^{2n-4} y | 1 \leq s \leq S - 1, 1 \leq i \leq 4n - 5, 2 \leq j \leq n - 2 \} \cup$$

$$\{u(S)_{i,j}^1 = u(x), u(S)_{i,j}^{2j-1} = u(S)_{i,j}^{2j-2} u(S)_{i,j}^{2j}, u(S)_{i,j}^{2n-3} = u(S)_{i,j}^{2n-4} y | 1 \leq i \leq r_1, 2 \leq j \leq n - 2 \} \cup$$

$$\{v(t)_i^1 = x v(t)_i^{2j}, v(t)_i^{2j-1} = v(t)_i^{2j-2} v(t)_i^{2j}, v(t)_i^{2n-3} = v(t)_i^{2n-4} y | 1 \leq t \leq T, 1 \leq i \leq 4n - 5, 2 \leq j \leq n - 2 \} \cup$$

$$\{v(T)_i^1 = x v(T)_i^{2j}, v(T)_i^{2j-1} = v(T)_i^{2j-2} v(T)_i^{2j}, v(T)_i^{2n-3} = v(T)_i^{2n-4} y | 1 \leq i \leq r_2, 2 \leq j \leq n - 2 \};$$

$$F(B^n_m) = \{f^n_{ext}, f^n_{int}, u(s)_{2n-2}, u(S)_{2n-2}^k, v(t)^{2n-2}, v(T)^{2n-2} | 1 \leq s \leq S - 1, 1 \leq t \leq T - 1, 1 \leq i \leq 4n - 5, 1 \leq k \leq r_1, 1 \leq l \leq r_2 \};$$

Where $f^n_{ext}$ is bounded by cycle $x v(1)^2 v(1)^2 \cdots v(1)^2 \cdots y x$;

$f^n_{int}$ is bounded by cycle $x u(1)^2 u(1)^2 \cdots u(1)^2 \cdots y x$;

$u(s)_{i}^{2n-2}$ is bounded by cycle $x u(s)_{i}^{2} u(s)_{i}^{4} \cdots u(s)_{i}^{2n-4} y u(s)_{i+1}^{2n-4} u(s)_{i+1}^{2n-6} \cdots u(s)_{i+1}^{2} x$, for $1 \leq s \leq S, i \neq r_1$;

$u(S)_{r_1}^{2n-2}$ is bounded by cycle $x u(S)_{r_1}^{2} u(S)_{r_1}^{4} \cdots u(S)_{r_1}^{2n-4} y v(T)_{r_2}^{2n-4} v(T)_{r_2}^{2n-6} \cdots v(T)_{r_2}^{2} x$; and

$v(t)_i^{2n-2}$ is bounded by cycle $x v(t)_i^{2} v(t)_i^{4} \cdots v(t)_i^{2} v(t)_i^{2n-4} y v(t)_i^{2n-4} v(t)_i^{2n-6} \cdots v(t)_i^{2} x$, for $1 \leq t \leq T, i \neq r_2$;

Our notations above imply that, without losing generality, for $v(t)_i^1$, we let $2 \leq i \leq 4n - 5$ for $t = 1$. It means that there is no vertex or edge or face $v(1)^1$.

Now, we divide our labeling of $B^n_m$ into 2 cases as follows:

**Case 1. For odd $m$ with $2 \leq r_2 \leq 2n - 1$ or even $m$;**

Define an entire $k$-labeling $\lambda : V \cup E \cup F \rightarrow \{1, 2, \cdots, k\}$ of $B^n_m$ as follows.

$\lambda(x) = \lambda(y) = \lambda(xy) = \lambda(f^n_{ext}) = 1$;

$\lambda(f^n_{int}) = 2$.
Hence, we propose the following open problem.

**Case 2. For odd \( m \) with \( r_2 = 1 \) or \( 2n \leq r_2 \leq 4n - 5 \):**

Define an entire \( k \)-labeling \( \lambda^* : V \cup U \cup F \rightarrow \{1, 2, \ldots, k\} \) of \( B^n_\text{m} \) as follows.

\[
\begin{align*}
\lambda^*(x) &= \lambda^*(y) = \lambda^*(xy) = \lambda^*(f^n_{\text{int}}) = 1; \\
\lambda^*(f^n_{\text{int}}) &= 2; \\
\lambda^*(u(s)_1) &= \lambda(u(s)_1); \\
\lambda^*(v(t)_1) &= \lambda(v(t)_1).
\end{align*}
\]

\[
\begin{align*}
\lambda^*(u(s)_1) &= \lambda(u(s)_1) \\
&= \begin{cases} 
2T - 2, & \text{for } r_2 = 1, \ t = T, \ i = 1, \ j = 1; \\
2T - 1, & \text{for } r_2 = 1, \ t = T - 1, \ i = 4n - 5, \ j = 2n - 2; \\
\lambda(v(t)_1) + 1, & \text{for } r_2 \text{ odd, } 2n \leq r_2 \leq 4n - 5, \ t = T, \ i = r_2, \ j = 1; \\
\lambda(v(t)_1) - 1, & \text{for } r_2 \text{ odd, } 2n \leq r_2 \leq 4n - 5, \ t = T, \ i = r_2 - 1, \ j = 2n - 2; \\
\lambda(v(t)_1) - 1, & \text{for } r_2 \text{ even, } 2n \leq r_2 \leq 4n - 5, \ t = T, \ i = r_2 - 1, \ j = 2n - 3; \\
\lambda(v(t)_1), & \text{for otherwize},
\end{cases}
\end{align*}
\]

It is easy to check that the labeling \( \lambda \) is an entire \( z \)-labeling. Then we have evaluate the face –weights set \( \{w(f^n_{\text{ext}}), w(f^n_{\text{int}}), w(u(s)^{2n-2}), w(v(t)^{2n-2}) \mid 1 \leq s \leq S, \ 1 \leq t \leq T, \ 1 \leq i \leq 4n - 5 \} \) as follows.

\[
\begin{align*}
w(f^n_{\text{ext}}) &= 2n + 1; \\
w(f^n_{\text{int}}) &= 2n + 2; \\
w(u(s)^{2n-2}) &= \begin{cases} 
(2s - 1)(4n - 5) + 2i, & \text{for } s = S - 1, \ 1 \leq i \leq 4n - 5; \\
(2s - 1)(4n - 5) + 2i, & \text{for } s = S - 1, \ 1 \leq i \leq r_1; \\
(2s - 1)(4n - 5) + 2r_1, & \text{for even } m, \ s = S - 1, i = r_1; \\
(2s - 1)(4n - 5) + 2r_1 - 1, & \text{for odd } m, \ s = S - 1, i = r_1,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
w(v(t)^{2n-2}) &= \begin{cases} 
(2t - 1)(4n - 5) + 2i + 1, & \text{for } 1 \leq t \leq T - 1, \ 1 \leq i \leq 4n - 5; \\
(2t - 1)(4n - 5) + 2i + 1, & \text{for } t = T, \ 1 \leq i \leq r_2 - 1.
\end{cases}
\end{align*}
\]

Since all face-weights are distinct, then \( \lambda \) is a face irregular entire \( z \)-labeling of \( B^n_\text{m} \) where \( m \) is odd with \( 2 \leq r_2 \leq 2n - 1 \) or \( m \) is even; and \( \lambda^* \) is a face irregular entire \( z \)-labeling of \( B^n_\text{m} \) where \( m \) is odd with \( r_2 = 1 \) or \( 2n \leq r_2 \leq 4n - 5 \). Thus, \( z = \frac{4n + m - 7}{4n - 5} \) is the entire face irregularity strength of \( B^n_\text{m} \).

Note that our result in Theorem 2.2 show that the ef \( s(B^n_\text{m}) \) is greater than the lower bound in Lemma 2.1.

Hence, we propose the following open problem.
Open Problems
1. Find a class of graph which satisfy a condition where the lower bound in Lemma 2.1 is sharp;
2. Generalize the lower bound for any condition.

References


