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SOME BASIC PROPERTIES OF THE NOISE REINFORCED BROWNIAN MOTION

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Abstract. *Noise reinforced Brownian motion appears as the universal limit of the step reinforced random walk. This article aims to study some basic properties of the noise reinforced Brownian motion. As main results, we prove integral representation, series expansion, Markov property, and martingale property of the noise reinforced Brownian motion.*

Keywords: *integral representation, Markov property, martingale property, noise reinforced Brownian motion.*

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1. INTRODUCTION

Stochastic processes with reinforcement has been an active research area in the last two decades. A survey on various stochastic processes with reinforcement as well as their applications has been given by Pemantle in [1]. The paper [2] contains some recent results in this area. In [3] Bertoin proved a version of the Donsker invariance principle: every step reinforced random walk with reinforcement parameter p, whose every step has finite second order moment, converges to a stochastic process called noise reinforced Brownian motion. Previously invariance principle for step reinforced random walk has been studied in the case of elephant random walk, that is one with steps follow Rademacher distribution, see [4] and [5]. Elephant random walk is a simple random walk which was introduced by Schütz dan Trimper in [6] and has attracted many researchers to study, see e.g. [5], [7], [8], [9], dan [10]. Some results on random walk with memory were given in [11] dan [12].

The noise reinforced Brownian motion $B^p = \left(B_t^p\right)_{t\geq 0}$ is a real valued centered Gaussian process with covariance function

$$\mathbb{E}\left(B_t^p B_s^p\right) = \frac{1}{1-2p}(t \vee s)^p (t \wedge s)^{1-p}, \quad s,t \geq 0 \qquad (1)$$

with $t \vee s = max\{t,s\}$, $t \wedge s = min\{t,s\}$, and $p \in \left(0, \frac{1}{2}\right)$ is a reinforcement parameter. By letting $p \to 0$ in equation (1), one gets

$$\mathbb{E}\left(B_t^p B_s^p\right) = t \wedge s.$$

In other words, if one drops the reinforcement parameter, then B^p is nothing else the standard Brownian motion. In this paper we will prove some basic properties of the noise reinforced Brownian motion including the Wiener integral representation, series expansion, Markov property, and martingale property. Having the basic properties will provide foundations for further research on the noise reinforced Brownian motion such as the study of the sample paths (continuity, differentiability), stochastic calculus with respect to noise reinforced Brownian motion, local time and self-intersection local time, asset modeling etc.

2. RESEARCH METHODS

The research method used here is a literature study. We study articles related to the noise reinforced Brownian motion and books related to probability theory and stochastic processes. The research begins by determining the topic, objectives to be achieved, the title of the research and is followed by collecting library resources. In particular, basic properties of the noise reinforced Brownian motion were investigated and analyzed in detail together with their proofs. The study ended with drawing conclusions based on literature review and research results.

3. RESULTS AND DISCUSSION

3.1. Noise Reinforced Brownian Motion

A standard Brownian motion can be constructed as a limit of simple random walks, see e.g. [13]. A similar construction of the noise reinforced Brownian motion has been proved in [3]. We summarize it as follows. We fix a reinforcement parameter $p \in \left(0, \frac{1}{2}\right)$. For every discrete time step reinforced random walk repeats one of its previous steps randomly and uniformly with probability p, otherwise it has independent increments with the same distributions with probability $1-p$. More precisely, consider a sequence of random variables $Y_1, Y_2, \ldots$ which consists of independent copies of the real valued random variables Y with identical distribution. Next, we define a sequence of random variables $X_1, X_2, \ldots$ recursively as follows. Let $(\varepsilon_i)_{i\geq 2}$ be an independent sequence of Bernoulli random variables with parameter p. We define $X_1 = Y_1$ and for $i \geq 2$ $X_i = Y_i$ if $\varepsilon_i = 0$ while X_i is a uniform random sampling of $X_1, \ldots, X_{i-1}$ if $\varepsilon_i = 1$. The sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ with

$$S_n = \sum_{i=1}^{n} X_i$$

is called step reinforced random walk. In general, $(S_n)_{n\in\mathbb{N}}$ does not satisfy the Markov property.

Theorem 1. ([3], Theorem 3.3) *Let* $p \in \left(0, \frac{1}{2}\right)$ *and* $Y \in L^2(\mathbb{P})$. *For* $n \to \infty$ *the sequence of random variables*

$$\frac{S_{\lfloor tn \rfloor} - tn\mathbb{E}(Y)}{\sqrt{n\, var(Y)}}, \quad t \geq 0$$

converges in distribution with respect to the Skorokhod topology to the noise reinforced Brownian motion $\left(B_t^p\right)_{t\geq 0}$ *with reinforcement parameter* p.

In [3] the following basic properties of the noise reinforced Brownian motion are also mentioned. The notation $\stackrel{d}{=}$ means equality in the distribution.

1. Scaling property: for every $c > 0$

$$\left(B_t^p\right)_{t\geq 0} \stackrel{d}{=} \left(c^{-1} B_{c^2 t}^p\right)_{t\geq 0}$$

2. Time inversion property:

$$\left(B_t^p\right)_{t\geq 0} \stackrel{d}{=} \left(t B_{\frac{1}{t}}^p\right)_{t> 0}$$

3. The law of iterated logarithm: with probability one

$$\limsup_{t\to\infty} \frac{B_t^p}{\sqrt{2t \ln\ln t}} = \limsup_{t\to 0^+} \frac{B_t^p}{\sqrt{2t \ln\ln \frac{1}{t}}} = \frac{1}{\sqrt{1-2p}}\ .$$

4. For different reinforcement parameter p, the distribution of the noise reinforced Brownian motion, say in the interval [0,1], yields probability distributions on the space of continuous function C[0,1] which are mutually singular and are also singular to the Wiener measure.

For $n \in \mathbb{N}$ the n-th moment of the noise reinforced Brownian motion is given by

$$\mathbb{E}\left(\left(B_t^p\right)^n\right) = \begin{cases} \left(\frac{2}{1-2p}\right)^k \frac{1}{\sqrt{\pi}} \Gamma\left(k + \frac{1}{2}\right) t^k & , n = 2k \\ 0 & , n = 2k-1 \end{cases}$$

where Γ is the Euler gamma function. To prove this, we start with observing that

$$\mathbb{E}\left(\left(B_t^p\right)^n\right) = \frac{\sqrt{1-2p}}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^n e^{-\frac{1-2p}{2t}x^2}\, dx.$$

If $n = 2k-1$ for some $k \in \mathbb{N}$, then the integrand in the above integral is an odd function and hence the value of integral is zero. If $n = 2k$ for some $k \in \mathbb{N}$, then

$$\mathbb{E}\left(\left(B_t^p\right)^{2k}\right) = \frac{2\sqrt{1-2p}}{\sqrt{2\pi t}} \int_0^{\infty} x^{2k} e^{-\frac{1-2p}{2t}x^2} dx.$$

By using substitution $x = \sqrt{\frac{2t}{1-2p}}\, y$, we get

$$\mathbb{E}\left(\left(B_t^p\right)^{2k}\right) = \frac{2}{\sqrt{\pi}}\left(\frac{2}{1-2p}\right)^k t^k \int_0^\infty y^{2k} e^{-y^2} dy = \left(\frac{2}{1-2p}\right)^k \frac{1}{\sqrt{\pi}} \Gamma\left(k+\frac{1}{2}\right) t^k.$$

Some properties of the standard Brownian motion which do not hold for the noise reinforced Brownian motion include stationarity of increments, independence of increments, dan time reversion property.

a. A stochastic process $(X_t)_{t\geq 0}$ has the stationary increment property if for every $0 \leq s < t$ $X_t - X_s \overset{d}{=} X_{t-s}$. Since $\left(B_t^p\right)_{t\geq 0}$ is a Gaussian process, we only need to check its first two moments. It is clear that $\mathbb{E}\left(B_t^p - B_s^p\right) = \mathbb{E}\left(B_{t-s}^p\right) = 0$. For the variance we have

$$\begin{aligned} var\left(B_t^p - B_s^p\right) &= \mathbb{E}\left(\left(B_t^p\right)^2\right) - 2\mathbb{E}\left(B_t^p B_s^p\right) + \mathbb{E}\left(\left(B_s^p\right)^2\right) \\ &= \tfrac{1}{1-2p} t - 2\tfrac{1}{1-2p} t^p s^{1-p} + \tfrac{1}{1-2p} s \\ &= \tfrac{1}{1-2p}\left(t - s - 2s\left(\tfrac{1}{2}\right)^p\right) \\ &\neq \tfrac{1}{1-2p}(t-s) \\ &= var\left(B_{t-s}^p\right). \end{aligned}$$

This shows that the increments of the noise reinforced Brownian motion is not stationary.

b. Recall that Gaussian random vectors $(Z_1, \ldots, Z_n)$ are independent if and only if $cov\left(Z_i, Z_j\right) = 0$ for every $i \neq j$. Let $n \in \mathbb{N}$ be arbitrary and consider the collection of random variables $B_{t_1}^p, B_{t_2}^p - B_{t_1}^p, \ldots, B_{t_n}^p - B_{t_{n-1}}^p$. Next, if $j, k \in \{1,2, \ldots, n\}$ with $0 \leq t_{j-1} < t_j < t_{k-1} < t_k$, then it holds

$$\begin{aligned} &\mathbb{E}\left(\left(B_{t_j}^p - B_{t_{j-1}}^p\right)\left(B_{t_k}^p - B_{t_{k-1}}^p\right)\right) \\ &= \mathbb{E}\left(B_{t_j}^p B_{t_k}^p\right) - \mathbb{E}\left(B_{t_j}^p B_{t_{k-1}}^p\right) - \mathbb{E}\left(B_{t_{j-1}}^p B_{t_k}^p\right) + \mathbb{E}\left(B_{t_{j-1}}^p B_{t_{k-1}}^p\right) \\ &= \tfrac{1}{1-2p} t_k^p t_j^{1-p} - \tfrac{1}{1-2p} t_{k-1}^p t_j^{1-p} - \tfrac{1}{1-2p} t_k^p t_{j-1}^{1-p} + \tfrac{1}{1-2p} t_{k-1}^p t_{j-1}^{1-p} \\ &= \frac{1}{1-2p}\left(t_k^p - t_{k-1}^p\right)\left(t_j^{1-p} - t_{j-1}^{1-p}\right) \\ &\neq 0. \end{aligned}$$

It is proved that the increments of the noise reinforced Brownian motion is not independent.

c. To show that the time reversion property does not hold, we show that the distribution of $B_1^p - B_{1-t}^p$ is not same with the distribution of B_t^p, $0 \leq t \leq 1$. Here it is sufficient to show that the variances are different. It can be checked easily that

$$var\left(B_1^p - B_{1-t}^p\right) = \frac{1}{1-2p}\left(1 - 2(1-t)^{1-p} + (1-t)\right).$$

On the other hand, $var\left(B_1^p - B_{1-t}^p\right) = var\left(B_t^p\right)$ if and only if $t \in \{0,1\}$. We see that noise reinforced Brownian motion does not satisfy the time reversion property.

3.2. Integral Represention and Series Expansion

A representation of the noise reinforced Brownian motion as a Wiener integral was mentioned briefly in [3]. Here we will look into it in more details.

Lemma 2. *The noise reinforced Brownian motion* $\left(B_t^p\right)_{t\geq 0}$ *can be written as a Wiener integral*

$$B_t^p = t^p \int_0^t s^{-p}\, dB_s \ , \ \ t \geq 0$$

with $(B_t)_{t\geq 0}$ is a standard Brownian motion.

Proof. The above stochastic integral is well defined as a Wiener integral since the integrand is a Lebesgue square integrable, that is

$$\int_0^t t^{2p} s^{-2p}\, ds < \infty,$$

since $0 < p < \frac{1}{2}$. According to Theorem 2.3.4 in [14], $\left(B_t^p\right)_{t\geq 0}$ is a Gaussian process with mean 0 and variance

$$\int_0^t t^{2p} s^{-2p}\ ds = \frac{1}{1-2p} t.$$

It is left to show that the covariance function is identical with the expression in the right hand side of (1). From Corollary 2.3.5 in [14]

$$cov\left(B_t^p, B_s^p\right) = \int_0^t t^p u^{-p} s^p u^{-p} du = \frac{1}{1-2p}(t \vee s)^p (t \wedge s)^{1-p}. \qquad \square$$

From the integral representation in Lemma 2 we see that B^p is an integral with respect to time of the noise reinforced white noise. This justifies the name *noise reinforced* Brownian motion. In this context, *reinforcement* means that the noise tends to repeat itself infinitesimally as time passes. In other words, the parameter p represents the probability that an infinitesimal part of the noise is a repetition. The above integral representation allows us to derive the series expansion of the noise reinforced Brownian motion.

Theorem 3. *With probability one the noise reinforced Brownian motion $\left(B_t^p\right)_{t\geq 0}$ can be represented as*

$$B_t^p = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \int_{\mathbb{R}} \phi_n(s)\, dB_s,$$

where $\langle \cdot,\cdot \rangle$ is the standard inner product on $L^2(\mathbb{R})$, $f(s) = \left(\frac{t}{s}\right)^p 1_{[0,t]}(s)$, ϕ_n is the n–th Hermite function which is defined via the Rodrigues formula

$$\phi_n(x) = (-1)^n \frac{1}{\sqrt{n!}} (2\pi)^{-\frac{1}{4}} e^{\frac{x^2}{4}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad n \geq 0$$

and the convergence of the random series is almost surely.

Proof. From Lemma 2 we have

$$B_t^p = \int_{\mathbb{R}} t^p s^{-p} 1_{[0,t]}\, dB_s,$$

where 1_A is the indicator function of the set $A \subset \mathbb{R}$. If we define $f(s) = \left(\frac{t}{s}\right)^p 1_{[0,t]}(s)$, then $f(s) \in L^2(\mathbb{R})$. Recall the fact that the family of Hermite functions $\{\phi_n : n = 0,1,2, \dots\}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Finally, Theorem 2.6.2. in [14] delivers the desired result. □

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which the noise reinforced Brownian motion is defined. From Theorem 3 for every $\omega \in \Omega$ it holds

$$B_t^p(\omega) = \sum_{n=1}^{\infty} \int_0^t t^p s^{-p} \phi_n(s) ds \int_{\mathbb{R}} \phi_n(s)\, dB_s(\omega)\,.$$

From the last formula we obtain the representation of the noise reinforced Brownian motion as the series

$$B_t^p(\omega) = \sum_{n=1}^{\infty} \xi_n(\omega) \int_0^t t^p s^{-p} \phi_n(s) ds,$$

where $(\xi_n)_{n\in\mathbb{N}}$ is an independent and identically distributed sequence of Gaussian random variables with mean 0 and variance 1.

3.3. Markov Property and Martingale Property

In order to show the Markov property we will use the following characterization result.

Proposition 4. ([15], Proposition 14.7) *Real valued Gaussian process $X = (X_t)_{t\geq 0}$ is a Markov process if and only if its covariance function is triangular*:

$$cov(X_s, X_t) = \frac{cov(X_s, X_r) cov(X_r, X_t)}{cov(X_r, X_r)}, \quad 0 < s \leq r \leq t.$$

Proposition 5. *The noise reinforced Brownian motion $\left(B_t^p\right)_{t\geq 0}$ is a Markov process.*

Proof. Since $\left(B_t^p\right)_{t\geq 0}$ is a Gaussian process we employ Proposition 4 to prove the Markov property. For any $s, r, t > 0$ with $0 < s \leq r \leq t$ it holds

$$\frac{cov\left(B_s^p, B_r^p\right) cov\left(B_r^p, B_t^p\right)}{cov\left(B_r^p, B_r^p\right)} = \frac{\frac{1}{1-2p} r^p s^{1-p} \frac{1}{1-2p} t^p r^{1-p}}{\frac{1}{1-2p} r^p r^{1-p}} = \frac{1}{1-2p} t^p s^{1-p} = cov\left(B_s^p, B_t^p\right).$$ □

Specifically, Proposition 5 shows that a non-Markovian stochastic process may converge in distribution to a Markov process.

Proposition 6. *The noise reinforced Brownian motion $B^p = \left(B_t^p\right)_{t\geq 0}$ is a martingale with respect to the natural filtration induced by the standard Brownian motion.*

Proof. The statement in the theorem is a corollary of the Wiener integral representation of B^p (Lemma 2) and Theorem 2.5.4 in [14] by observing that $s^{-p} \in L^2[0, t]$ for $p \in \left(0, \frac{1}{2}\right)$. To be more precise, since $B_t^p = t^p \int_0^t s^{-p}\, dB_s$, $t \geq 0$, then according Theorem 2.5.4. in [14], the stochastic process $\left(B_t^p\right)_{t\geq 0}$ is a martingale with respect to the natural filtration of the standard Brownian motion $\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t)$. □

The importance of Proposition 6 is that the noise reinforced Brownian motion can be used to model a fair game. From stochastic calculus point of view, Proposition 6 also gives the existence of the stochastic integral with respect to the noise reinforced Brownian motion as an Itô integral. Another possible application in this direction is an asset modeling using the noise reinforced Brownian motion.

4. CONCLUSIONS

The noise reinforced Brownian motion $B^p = \left(B_t^p\right)_{t\geq 0}$ is a stochastic process which appears as the universal limit of the step reinforced random walk. Some of its basic properties are as follow:

1. By letting the reinforcement parameter $p \to 0$ in the definition of B^p one reveals the standard Brownian motion.
2. There are some properties of the standard Brownian motion that are not satisfied by noise reinforced Brownian motion including stationary increments property and independent increments property.
3. B^p has a Wiener integral representation since the integral kernel is a Lebesgue square integrable function.
4. B^p can be expanded in the form of random series involving the Hermite functions.
5. B^p is a Markov process although it is a limit of a non-Markov process.
6. B^p is a martingale with respect to the natural filtration generated by the standard Brownian motion.

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