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CHARACTERISTIC ANTIADJACENCY MATRIX OF GRAPH JOIN

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Abstract. *Let $G = (V, E)$ be a simple, connected, and undirected graph. The graph $G = (V, E)$ can be represented as a matrix such as antiadjacency matrix. An antiadjacency matrix for an undirected graph with order n is a matrix that has an order $n \times n$ and symmetric so that the antiadjacency matrix has a determinant and characteristic polynomial. In this paper, we discuss the properties of antiadjacency matrix of a graph join, such as its determinant and characteristic polynomial. A graph join $G = (V, E)$ is obtained of a graph join operation obtained from joining two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.*
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Keywords: *antiadjacency matrix, graph join, characteristic polynomial of antiadjacency matrix*

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1. INTRODUCTION

Let $G = (V, E)$ be a simple, connected and undirected graph with n vertices. A graph G can be represented by antiadjacency matrix. Antiadjacency matrix $B = J - A$, where A is the $n \times n$ adjacency matrix of graph G, and J the matrix whose entries are all one. Therefore, B is a symmetric matrix so that the antiadjacency matrix has a determinant and a characteristic polynomial for each graph. The characteristic of matrix adjacency can be seen in [1][2]. Diwyacitta et. al. [3] has determined determinant of antiadjacency matrix for directed cycle graph $\vec{C}_n$. Edwina and Sugeng [4] determined determinant of antiadjacency matrix of some undirected graphs, such as $K_n \cup K_m$, wheels W_n, bipartite $K_{n,m}$ and star S_n. In this paper, we discussed the determinant and characteristic polynomials of antiadjacency matrix of undirected graph G obtained from join operation graph.

2. BASIC THEORY

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be finite graphs. A join operation of graphs G_1 and G_2 is denoted by $G = G_1 + G_2$, where $V_1 \cap V_2 = \emptyset$ and $V = V_1 \cup V_2$ is a set of vertices of graph G and $E = E_1 \cup E_2 \cup \{\{x, y\}; x \in V_1, y \in V_2\}$ is a set of edges of graph G [5]. An example of the join operation of graph G_1 and G_2 is given in Figure 1.

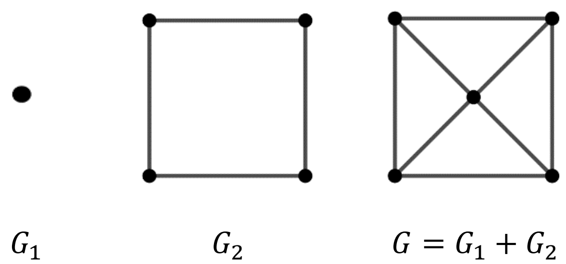

Figure 1. Graph join G_1 and G_2

Let G be a graph with $V(G) = \{1, \dots, n\}$ and $E(G) = \{e_1, \dots, e_m\}$. The adjacency matrix of the graph G, denoted by A, is the $n \times n$ matrix. The rows and the columns of A are indexed by $V(G)$. If $i \neq j$ then the (i, j)-entry of A is 0 for vertices i and j nonadjacent, and the (i, j)-entry is 1 for i and j adjacent. The (i, i)-entry of A is 0 for $i = 1, \dots, n$. The matrix $B = J - A$ will be called the antiadjacency of graph G [1].
The adjacency matrix of the graph $G = G_1 + G_2$ is written in a block matrix form as follows:

$$A = \begin{bmatrix} A_1 & J \\ J & A_2 \end{bmatrix},$$

where A_1 is an adjacency matrix of the graph G_1 and A_2 is an adjacency matrix of the graph G_2.
Therefore, the antiadjacency matrix of the graph G is as follows:

$$B = J - A = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where B_1 is the antiadjacency matrix of the graph G_1 and B_2 is the antiadjacency matrix of the graph G_2.
Let M be a square matrix in a block matrix form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (1)$$

where A and D are $n \times n$ and $m \times m$ matrices, respectively. Thus, the determinant of M can be obtained as stated in Theorem 1.

Theorem 1. [7] *Let M be a square matrix partitioned as (1). Then*

$$\det M = \det A \det(D - CA^{-1}B), \text{ if } A \text{ is invertibel, and}$$
$$\det M = \det(AD - CB), \text{ if } AC = CA.$$

Theorem 2. [4] *Let W_n be a wheel graph with $n, n > 3$ vertices. If C_n be a cycle graph with m vertices, $n > 2$ then*

$$\det\big(B(W_n)\big) = \det(B(C_{n-1})).$$

Furthermore, the relationship between symmetric functions, principal minors, and the coefficient of the characteristic polynomial is given in the following Theorem 3.

Theorem 3. [6] *if* $\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \cdots + c_n = 0$ *is the characteristic polynomial for* $A_{n\times n}$ *and if* s_i *is the* i^{th} *symmetric function of the eigenvalue* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *of* A. *Then*

- $c_i = (-1)^i \sum(all\ i \times i\ principal\ minors)$,
- $s_i = \sum(all\ i \times i\ principal\ minors)$,
- $trace(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = -c_1$,
- $det(A) = \lambda_1\lambda_2 \ldots \lambda_n = (-1)^n c_n$.

The i^{th} symmetric function of $\lambda_1, \lambda_2, \ldots, \lambda_n$ is defined to be the sum of the product of the eigenvalues taken i at a time. That is,

$$s_k = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1} \ldots \lambda_{i_k}.$$

For example, when $n = 3$,

$$s_1 = \lambda_1 + \lambda_2 + \lambda_3,$$
$$s_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$$
$$s_3 = \lambda_1\lambda_2\lambda_3.$$

3. RESULTS AND DISCUSSION

3.1. Graph join

Let $G_i = (V_i, E_i)$ for $i = 1,2$ be a finite graph with $V_1 \cap V_2 = \emptyset$. The graph $G = (V, E)$ is a graph join of G_1 and G_2, denoted by $G = G_1 + G_2$ where $V = V_1 \cup V_2$ is a set of vertices and $E = E_1 \cup E_2 \cup \{\{x, y\}; x \in V_1, y \in V_2\}$ is a set of edges. The adjacency matrix of graph G is written in a block matrix form

$$A = \begin{bmatrix} A_1 & J \\ J & A_2 \end{bmatrix}.$$

Let $G = G_1 + G_2$. As mentioned before, the antiadjacency matrix of graph G is as follows:

$$B = J - A = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where $B_i = J - A_i$ is an antiadjacency matrix of graph G_i for $i = 1,2$. Theorem 4 stated the value of det $B(G)$.

Theorem 4. *Let* $G = (V, E)$ *is a graph join of* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *then* $det(B(G)) = det\,B(G_1).det\,B(G_2)$.

Proof. Let $G = (V, E)$ is a graph join that denoted by $G = G_1 + G_2$ so that the antiadjacency matrix of graph G is written in the form of a block matrix as follows

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

We obtain,

$$\det B(G) = \det \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} = \det B_1 . \det B_2 = \det B(G_1). \det B(G_2).$$ ◻

In Theorem 5 and 6, we give the determinant from the example of graph join.

Theorem 5. *Let* K_n *be a complete graph with* $n \ge 2$ *and* $B(K_n)$ *be an antiadjacency matrix of* K_n, *then* $det\,B(K_n) = 1$.

Proof. Given a graph K_n with $B(K_n)$ is an antiadjacency matrix of graph K_n. Then the principal diagonal matrix is 1. Clearly, the determinant $B(K_n) = 1$. ◻

Theorem 6. *Let fan graph* $F_{n,1}$ *be a graph join of path* $P_n, n \ge 2$ *and complete graph* K_1. *Then*

$$det\, B(F_{n,1}) = det\, B(P_n).$$

Proof. Let $F_{n,1} = P_n + K_1$ be a fan graph. Then $|V| = n + 1$. Thus,

$$\begin{aligned}\det B(F_{n,1}) &= \det B(P_n) . \det B(K_1)\\ &= \det B(P_n) . (1)\\ &= \det B(P_n).\end{aligned}$$

□

3.2. Characteristic Polynomial

Theorem 7. The coefficients of the antiadjacency matrix graph G satisfy

1) *$-c_1$ is the number of vertices of graph G;*
2) *c_2 is the number of edges of graph G;*
3) *$-c_3$ is the number of $C_3 \subset G$ – number of $\{v_i v_j, v_k | i, j, k = 1, \dots, n\}$ and v_k nonadjacent with v_i and v_j.*

Proof. For $i \in \{1,2, \dots, n\}$, the number $(-1)^i c_i$ is the sum of those principal minors of B which have i rows and i columns. Thus, it is clear that for $i = 1$ then $-c_1$ is the sum of the diagonal elements of matrix B, because $b_{ii} = 1$ for $i = 1, \dots, n$ so that $-c_1$ represents the number of vertices of graph G. For $i = 2$, a principal minor with two rows and columns, and which has non-zero entry, must be of the form

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

This represents every edge of the graph G and is 1, So, $(-1)^2 c_2 = |E(G)|$. This means that, $c_2 = |E(G)|$. for $i = 3$ there are essentially four possibilities for non-trivial principal minors with three rows and columns

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}.$$

The first form is worth 1 and the other is worth -1. The first principal minor denotes a triangle in graph G and the number of $\{v_i v_j, v_k | i, j, k = 1, \dots, n\}$ and v_k not adjacent with v_i and v_j. So, $-c_3$ is the number of $C_3 \subset G -$ number of $\{v_i v_j, v_k | i, j, k = 1, \dots, n\}$ and v_k not adjacent with v_i and v_j. □

Theorem 8. *For graph $\overline{K}_n$ and $B(\overline{K}_n)$ antiadjacency matrix of graph $\overline{K}_n$ then characteristic polynomial for $n \geq 1$ that is*

$$P(\lambda) = \lambda^{n-1}(\lambda - n).$$

Proof. Let $B(\overline{K}_n)$ antiadjacency matrix with all entries are equal to one. Thus, matrix $B(\overline{K}_n)$ equivalent to matrix J. This implies that $P(\lambda) = \det(\lambda I - J) = \lambda^{n-1}(\lambda - n)$. □

Theorem 9. *For $G = (V, E)$ is a graph join of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ then $P(\lambda) = P_1(\lambda). P_2(\lambda)$, where $P(\lambda), P_1(\lambda)$ and $P_2(\lambda)$ are the characteristic polynomial of antiadjacency matrix of G, G_1 and G_2.*

Proof. Let $G = (V, E)$ be a graph join, which is denoted by $G = G_1 + G_2$. Then the antiadjacency matrix of the graph G can be written in a block matrix form as follows

$$B = \begin{bmatrix}B_1 & 0\\ 0 & B_2\end{bmatrix},$$

with B_1 is the antiadjacency matrix of the graph G_1 and B_2 is the antiadjacency matrix of the graph G_2. Thus,

$$\begin{aligned}P(\lambda) = \det(B - \lambda I) &= \det\begin{bmatrix}B_1 - \lambda I & 0\\ 0 & B_2 - \lambda I\end{bmatrix}\\ &= \det(B_1 - \lambda I). \det(B_2 - \lambda I) = P_1(\lambda). P_2(\lambda)\end{aligned}$$

□

A bipartite graph $K_{n,m}$ can be considered as the graph join $K_{n,m} = \overline{K}_n + \overline{K}_m$, where $\overline{K}_n$ and $\overline{K}_m$ are the empty graphs on m and n vertices, respectively.

Corollary 10. *For bipartite graph $K_{n,m} = \overline{K}_n + \overline{K}_m$ with $n, m \geq 1$ and $B(K_{n,m})$ is an antiadjacency matrix of graph $K_{n,m}$ then characteristic polynomial of the bipartite graph $K_{n,m}$,*

$$P(\lambda) = \lambda^{n+m-2}(\lambda - n)(\lambda - m).$$

Proof. Let $K_{n,m} = \overline{K}_n + \overline{K}_m$ be a bipartite graph So, the antiadjacency matrix of the graph $K_{n,m}$ can be written in the form of a block as follows

$$B(K_{n,m}) = \begin{bmatrix} J_{(n\times n)} - A_{1(n\times n)} & 0_{(n\times n)} \\ 0_{(m\times m)} & J_{(m\times m)} - A_{2(m\times m)} \end{bmatrix} = \begin{bmatrix} J_{n\times n} & 0 \\ 0 & J_{m\times m} \end{bmatrix},$$

where A_1 is an adjacency matrix of the graph $\overline{K}_n$, A_2 is an adjacency matrix of the graph $\overline{K}_m$ and J is the matrix whose entries are all one. Thus,

$$\begin{aligned} P(\lambda) &= \det \begin{bmatrix} J_{n\times n} - \lambda I_{n\times n} & 0 \\ 0 & J_{m\times m} - \lambda I_{m\times m} \end{bmatrix}, \\ &= \det(J_{n\times n} - \lambda I_{n\times n}).\det(J_{m\times m} - \lambda I_{m\times m}) \\ &= \lambda^{n-1}(\lambda - n).\lambda^{m-1}(\lambda - m) \\ &= \lambda^{n+m-2}(\lambda - n)(\lambda - m). \end{aligned}$$

□

Corollary 11. *For a complete split graph $K_n + \overline{K}_m$ with $n, m \geq 1$ and $B(K_n + \overline{K}_m\ )$ is an antiadjacency matrix of the graph $K_n + \overline{K}_m$ then characteristic polynomial of a complete split graph is as follows,*

$$P(\lambda) = \lambda^{m-1}(\lambda - 1)^n(\lambda - m).$$

Proof. Let $K_n + \overline{K}_m$ be a complete split graph with $n, m \geq 1$. Thus, the antiadjacency matrix of graph $K_n + \overline{K}_m$ can be written in the form of a block as follows

$$\begin{aligned} B(K_n + \overline{K}_m) &= \begin{bmatrix} J_{(n\times n)} - A_{1(n\times n)} & 0_{(n\times n)} \\ 0_{(m\times m)} & J_{(m\times m)} - A_{2(m\times m)} \end{bmatrix} \\ &= \begin{bmatrix} I_{n\times n} & 0 \\ 0 & J_{m\times m} \end{bmatrix}, \end{aligned}$$

where A_1 is an adjacency matrix of graph K_n, A_2 is an adjacency matrix of graph $\overline{K}_m$ and J is the matrix whose entries are all equal to one. The we have

$$\begin{aligned} P(\lambda) &= \det \begin{bmatrix} I_{n\times n} - \lambda I_{n\times n} & 0 \\ 0 & J_{m\times m} - \lambda I_{m\times m} \end{bmatrix}, \\ &= \det(I_{n\times n} - \lambda I_{n\times n}).\det(J_{m\times m} - \lambda I_{m\times m}) \\ &= (\lambda - 1)^n.\lambda^{m-1}(\lambda - m) \\ &= \lambda^{m-1}(\lambda - 1)^n(\lambda - m). \end{aligned}$$

□

The friendship graph F_n on $2n + 1$ vertices is a graph join $F_n = nK_2 + K_1$, where nK_2 is the disjoint union of n copies of K_2.

Corollary 12. *For friendship graph $F_n = nK_2 + K_1$ with $n \geq 1$ with $B(\ F_n)$ is an antiadjacency matrix of the graph F_n then characteristic polynomial of graph F_n is*

$$P(\lambda) = (\lambda - 2n + 1)(\lambda - 1)^{n+1}(\lambda + 1)^{n-1}.$$

Proof. Let $F_n = nK_2 + K_1$ be a friendship graph with $n \geq 1$. Then the antiadjacency matrix of the graph *friendship* F_n written in the form of a block matrix as follows

$$\begin{aligned} B(\ F_n) &= \begin{bmatrix} J_{(n\times n)} - A_{1(n\times n)} & 0_{(n\times n)} \\ 0_{(m\times m)} & J_{(m\times m)} - A_{2(m\times m)} \end{bmatrix} \\ &= \begin{bmatrix} B_1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

where A_1 is an adjacency matrix of the graph nK_2, A_2 is an adjacency matrix of the graph K_1 and B_1 is an antiadjacency matrix of the graph nK_1. Then we have

$$\begin{aligned} P(\lambda) &= \det \begin{bmatrix} B_1 - \lambda I & 0 \\ 0 & 1 - \lambda I \end{bmatrix}, \\ &= \det(B_1 - \lambda I).\det(1 - \lambda I) \\ &= (\lambda - 2n + 1)(\lambda - 1)^{n+1}(\lambda - 1)^{\mathrm{n}}.(\lambda - 1) \\ &= (\lambda - 2n + 1)(\lambda - 1)^{n+1}(\lambda - 1)^{\mathrm{n+1}}. \end{aligned}$$

□

4. CONCLUSIONS

In this paper, we prove the correlation of the characteristic polynomial coefficients of the antiadjacency matrix of undirected graph and determined determinant of antiadjacency matrix of the graph join F_n and complete graph K_n with $n \geq 2$. Then, we determined the characteristic polynomial of the antiadjacency matrix of some graphs such as bipartite graph, complete split graph, and friendship graph. Further work can be conducted to find the determinant and characteristic polynomial of other graphs.

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