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THE NON-DEGENERACY OF THE SKEW-SYMMETRIC BILINEAR FORM OF THE FINITE DIMENSIONAL REAL FROBENIUS LIE ALGEBRA

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Abstract. *A Frobenius Lie algebra is recognized as the Lie algebra whose stabilizer at a Frobenius functional is trivial. This condition is equivalent to the existence of a skew-symmetric bilinear form which is non-degenerate. On the other hand, the Lie algebra is Frobenius as well if its orbit on the dual vector space is open. In this paper, we study the skew-symmetric bilinear form of finite dimensional Frobenius Lie algebra corresponding to its Frobenius functional. The work aims to prove that a Lie algebra of dimension $2n$ is Frobenius if and only if the n-th derivation of the Frobenius functional is not equal to zero. Indeed, this condition implies that the skew-symmetric bilinear form is non-degenerate and vice versa. In addition, some properties of Frobenius functionals are obtained. Furthermore, the computations are given using the coadjoint orbits and the structure matrix. As a discussion, we can investigate these results in the j −algebra case whether giving rise to a left-invariant Kähler structure of a Frobenius Lie group or not.*

Keywords: *Bilinear form, Frobenius lie algebras, Frobenius functionals, stabilizer.*

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1. INTRODUCTION

The notion of a Lie algebra arises naturally from the notion of a vector space over a field with certain properties. One of the most important properties in the Lie algebra is a bilinear form. Roughly speaking, let $\mathfrak{g}$ be a real vector space. The bilinear form in $\mathfrak{g}$ is map given by

$$\beta : \mathfrak{g} \times \mathfrak{g} \ni (a,b) \mapsto \beta(a,b) \in \mathbb{R}. \tag{1}$$

which is a linear map on each component. This notion corresponds to value of a linear functional $\psi \in \mathfrak{g}^*$ at a point $X \in \mathfrak{g}$ which is denoted by $\langle X, \psi \rangle$. In this case, $\mathfrak{g}^*$ is a dual vector space of $\mathfrak{g}$. On the other hand, it is well known the classification of Lie algebras based on their dimension consisting of odd and even dimension [1]. The Frobenius Lie algebra has always even dimension while the contact Lie algebra is odd dimension. Our research will concern to the notion of real Frobenius Lie algebra, particularly in providing the necessary and sufficient conditions for non-degeneracy the skew-symmetric bilinear form.

The research of Frobenius Lie algebras can be found in [2] focusing on $\mathfrak{g}$-quasi Frobenius Lie algebra. In this paper, we can see the necessary and sufficient conditions for a quasi Frobenius Lie algebra to be a Frobenius Lie algebra. We note that a quasi Frobenius Lie algebra is a Frobenius Lie algebra if and only if there exists a linear functional or 1-form $\alpha \in \mathfrak{g}^*$ such that β is exact. Namely, we have

$$\beta(a,b) = (\mathrm{d}\alpha)(a,b) = \langle \alpha, [a,b] \rangle \text{ for all } a,b \in \mathfrak{g}. \tag{2}$$

where $\mathrm{d}\alpha$ is 2-form in $\mathfrak{g}$. Another research of Frobenius Lie algebras can be found in ([3], [4], [5]) as well.

The research aims to describe the non-degeneracy of the skew-symmetric bilinear form corresponding to its Frobenius functional in order a Lie algebra $\mathfrak{g}$ is Frobenius. The problem arise because the skew-symmetric bilinear form can be defined in the notion of a linear functional. It means that we have

$$\beta_\alpha : \mathfrak{g} \times \mathfrak{g} \ni (a,b) \mapsto \beta(a,b) \coloneqq \langle \alpha, [a,b] \rangle \in \mathbb{R}. \tag{3}$$

We shall complete the prove in [1] that the skew-symetric bilinear form β_α of the $2n$-dimensional Frobenius Lie algebra is non-degenerate if and only if $(\mathrm{d}\alpha)^n \neq 0$. In the previous results (see [2], [4],[6]), we found that the non-degeneracy of the skew-symmetric bilinear form in a Frobenius Lie algebra is described using a Frobenius functional such that the determinant of a brackets matrix under the Frobenius functional is not equal to zero. Different from the previous results, in this work, we observe that that the non-degeneracy corresponds to a derivative of a certain Frobenius functional. In other words, we show that the skew-symmetric bilinear form in a Frobenius Lie algebra is non-degenerate if only if $(\mathrm{d}\alpha)^n \neq 0$ with α is the Frobenius functional in its dual space.

This paper is organized as follows. Section 1 consists of introduction which provides the state of art of research, the gap of research, and the aim of reseach. In Section 2, we provide reseach methods, the basic definitions, some reviews of Frobenius Lie algebras. In section 3, we provide result and discussion. In this section as well, we prove the non-degeneracy of the skew-symmetric bilinear form β_α corresponding to the 1-form α. Finally, in section 4, we conclude our results and some discussions for the future research.

2. RESEARCH METHODS

The method in this reseach is a literature review. In particular, we review the important notion of the bilinear forms and their properties, Frobenius Lie algebra, the necessary and sufficient conditions of a Lie algebra $\boldsymbol{\mathfrak{g}}$ to be a Frobenius. We have a result that a Lie algebra $\boldsymbol{\mathfrak{g}}$ if Frobenius if the determinant of a representation matrix of the skew-symmetric bilinear form $\boldsymbol{B_{f_0}}$ for some linear functionals $\boldsymbol{f_0}$ is not equal to zero. We consider the latter statement to the derivation of $\boldsymbol{\mathfrak{g}}$ and we shall prove these conditions.

2.1 Bilinear Form

We start from the notion of a bilinear form.

Definition 1 [7]. Let $\mathfrak{g}$ be a real vector space. A map $\beta: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is called a bilinear map if the map β is linear in each component when the other one is fixed i.e.

$$\beta(a+c,b) = \beta(a,b) + \beta(c,b), \qquad \beta(ka,b) = k\beta(a,b) \tag{4}$$

and

$$\beta(a, b+c) = \beta(a,b) + \beta(a,c), \qquad \beta(a, lb) = l\beta(a,b), \tag{5}$$

for all $a, b, c \in \mathfrak{g}, k, l \in \mathbb{R}$. Moreover, the bilinear form β is called skew-symmetric if

$$\beta(a,b) = -\beta(b,a), \forall\, a, b \in \mathfrak{g}. \tag{6}$$

The vector space $\mathfrak{g}$ which is equipped with a choice bilinear form is called a bilinear space and it is denoted by a pair $(\mathfrak{g}, \beta)$.

Let us see the example of the skew-symmetric bilinear form on $\mathbb{R}^2$. We define the bilinear form as follows

$$\det: \mathbb{R}^2 \times \mathbb{R}^2 \ni \big((a,b),(a',b')\big) \mapsto ab' - a'b = \det\begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \in \mathbb{R}. \tag{7}$$

Using the determinant properties, it can be observed that the map $\det$ is skew-symmetric bilinear form.

Theorem 2 [7]. *Let $(\mathfrak{g}, \beta)$ be a bilinear space. The following statements are equivalent :*
1. *The matrix realization of β, denoted by $[\beta(e_i, e_j)]$, is invertible for some basis $S = \{e_1, e_2. e_3, \dots, e_n\}$ of $\mathfrak{g}$.*
2. *If $\beta(a,b) = 0$, for all $b \in \mathfrak{g}$ then $a = 0$.*

Definition 3 [7]. Let $(\mathfrak{g}, \beta)$ be a non-zero bilinear space. The bilinear form β is called non-degenerate if the conditions in Theorem 2 hold.

In the real Lie algebra case, it is well known that that the symmetric bilinear form can be defined as what we called the Killing form. Namely, for all $a, b \in \mathfrak{g}$, we have

$$\beta(a,b) := \mathrm{Tr}\big((\mathrm{ad}\ a)(\mathrm{ad}\ b)\big), \tag{8}$$

where Tr is trace of adjoint representation ad of $\mathfrak{g}$. It is useful to determine that the Lie algebra is semi-simple if and only if the Killing form $\mathfrak{g}$ is non-degenerate.

2.2 Frobenius Lie Algebra

The notion of Frobenius Lie algebra $\mathfrak{g}$ corresponds to the characterization of the Lie algebra of finite dimension having a universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ which is primitive [8]. We recall that the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ is a pair $(\mathfrak{U}(\mathfrak{g}), \sigma)$ where $\sigma: \mathfrak{U}(\mathfrak{g}) \to \mathfrak{g}$ is a Lie algebra homomorphism satifying the universal property [9]. In view of geometric aspect, the Frobenius Lie algebra can be related to the coadjoint orbit Ω_{f_0} for certain $f_0 \in \mathfrak{g}^*$. In this view, let G be the Lie group of the Frobenius Lie algebra $\mathfrak{g}$. The openness of the orbit

$$\Omega_{f_0} := \{\mathrm{Ad}^*(g) f_0 \ \ ; \ \ g \in G\} \subseteq \mathfrak{g}^* \tag{9}$$

is necessary and sufficient condition for $\mathfrak{g}$ to be the Frobenius Lie algebra [6]. The openness of Ω_{f_0} can be examined by using a stabilizer $\mathfrak{g}^{f_0} = \{x \in \mathfrak{g}\ \ ; \mathrm{ad}^*(x) f_0 = 0\} \subseteq \mathfrak{g}$. Namely, Ω_{f_0} is open if and only if $\mathfrak{g}^{f_0} = \{0\}$. The linear functional f_0 satisfying Ω_{f_0} open is called a Frobenius functional. As example, let $\mathfrak{g}$ be a Lie algebra whose basis is $\{y_1, y_2, y_3, y_4\}$ and its bracket are $[y_4, y_1] = [y_3, y_2] = y_1, [y_4, y_2] = \frac{1}{2} y_2, [y_4, y_3] = \frac{1}{2} y_3$. Let $\mathfrak{g}^*$ be a dual space of $\mathfrak{g}$ whoses basis is $\{y_1^*, y_2^*, y_3^*, y_4^*\}$. The coadjoint orbit Ω_{f_0} can be written in the following form

$$\Omega_{\pm y_1^*} = (\pm\mathbb{R}_{>0}) y_1^* \oplus \mathbb{R} y_2^* \oplus \mathbb{R} y_3^* \oplus \mathbb{R} y_4^*.$$

From some $\pm y_1^* \in \mathfrak{g}^*$, the stabilizer $\mathfrak{g}^{\pm y_1^*} = \{0\}$. Therefore, $\Omega_{\pm y_1^*}$ is open.
We resume some results of Frobenius Lie algebras as follows:

Theorem 4 [2]. *Let $\mathfrak{g}$ be a real finite dimensional Lie algebra of dimension n whose basis is $B = \{e_1, e_2, e_3, \dots, e_n\}$. Let $M^{\mathfrak{g}}([e_i, e_j])$ be a structure matrix consisting of all brackets $[e_i, e_j]$. The Lie algebra $\mathfrak{g}$ is Frobenius if th following conditions are equivalent:*
1. *The stabilizer of $\mathfrak{g}$ at some point in $\mathfrak{g}^*$ is trivial.*
2. *The determinant of $M^{\mathfrak{g}}([e_i, e_j])$ is not equal to zero.*
3. *There exists a linear functional $f_0 \in \mathfrak{g}^*$ such that the determinant $M^{\mathfrak{g}}(\langle f_0, [e_i, e_j]\rangle)$ is not equal to zero.*

4. There exists a linear functional $f_0 \in \mathfrak{g}^*$ such that the coadjoint orbit $\Omega_{f_0} \subseteq \mathfrak{g}^*$ is open.

Let $\mathfrak{g}$ be a Lie algebra whose basis is $\{y_1, y_2, y_3, y_4\}$ and its bracket are $[y_4, y_1] = [y_3, y_2] = y_1, [y_4, y_2] = \frac{1}{2}y_2, [y_4, y_3] = \frac{1}{2}y_3$ as mentioned above.We have already choosen the linear functional $f_0 = y_1^*$ above. The matrix $M^{\mathfrak{g}}([e_i, e_j])$ can be written in the following form

$$M^{\mathfrak{g}}([e_i, e_j]) = \begin{bmatrix} 0 & 0 & 0 & -y_1 \\ 0 & 0 & -y_1 & -\frac{1}{2}y_2 \\ 0 & y_1 & 0 & -\frac{1}{2}y_3 \\ y_1 & \frac{1}{2}y_2 & \frac{1}{2}y_3 & 0 \end{bmatrix}$$

The direct computations we have $\det\left(M^{\mathfrak{g}}([e_i, e_j])\right) = y_1^4 \neq 0$. Thus, $\mathfrak{g}$ is Frobenius Lie algebra. In the other hand, the matrix $M^{\mathfrak{g}}(\langle f_0, [e_i, e_j]\rangle)$ for choice $f_0 = y_1^*$ can be written in the following form

$$M^{\mathfrak{g}}(\langle y_1^*, [e_i, e_j]\rangle) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

whose determinant is equal to 1. We show again that $\mathfrak{g}$ is Frobenius Lie algebra.

The more example of Frobenius Lie algebras can be found in [10]. We can see there are some classifications the low dimensional Frobenius Lie algebras of dimension ≤ 6. Other examples can be observed in [11]. We remark here that the Frobenius Lie algebra is not nilpotent ([12], [13], and [14]) and we can observe that some Frobenius Lie algebras are solvable while others are not [11].

3. RESULT AND DISCUSSION

The notion of Frobenius Lie algebra can be viewed by some criteria. The first, the Lie algebra $\mathfrak{g}$ is Frobenius if its stabilizer $\mathfrak{g}^{f_0} = \{x \in \mathfrak{g} \ ; \mathrm{ad}^*(x)f_0 = 0\}$ is trivial for some $f_0 \in \mathfrak{g}^*$. The second, we can consider that $\mathfrak{g}$ is Frobenius if the determinant of matrix structure of $\mathfrak{g}$ is not equal to zero. The third, the Lie algebra $\mathfrak{g}$ is Frobenius if and only if the skew-symmetric bilinear form of $\mathfrak{g}$ is non degenerate. In this result, we relate the notion of a skew-symmetric bilinear form of $\mathfrak{g}$ to the notion of $(n+1)$-form. Namely, we shall complete the proof of the following Proposition which was obtained in previous result [1].

Proposition 1. *Let $\mathfrak{g}$ be a Lie algebra of dimension $2n$. For a linear functional $f_0 \in \mathfrak{g}^*$, the skew-symmetric bilinear form B_{f_0} is non degenerate if and only if $(\mathrm{d}f_0)^n \neq 0$. Moreover, the Lie algebra $\mathfrak{g}$ is Frobenius if and only if $(\mathrm{d}f_0)^n \neq 0$.*

Proof.

($\Rightarrow$) Let $\mathfrak{g}$ be a Frobenius Lie algebra of dimension $2n$ whose basis is $B = \{x_1, x_2, x_3, \dots, x_{2n}\}$ and let $\mathfrak{g}^*$ be a dual vector space of $\mathfrak{g}$ whose basis is $B^* = \{x_1^*, x_2^*, x_3^*, \dots, x_{2n}^*\}$. The value of a linear functional $x_i^* \in \mathfrak{g}^*$ at a point $x_j \in \mathfrak{g}$ is given by $\langle x_i^*, x_j\rangle = \delta_{ij}$ which are $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Let us assume that there exists a linear functional $f_0 \in \mathfrak{g}^*$. Then we assume that a skew-symmetric bilinear form B_{f_0} is non-degenerate for some Frobenius functionals $f_0 \in \mathfrak{g}^*$. In this case, we define B_{f_0} as follows:

$$B_{f_0} : \mathfrak{g} \times \mathfrak{g} \ni (a, b) \mapsto \langle f_0, [a, b]\rangle \in \mathbb{R}.$$

Let $M^{\mathfrak{g}}([a, b])$ be a matrix structure of $\mathfrak{g}$ whose entries of brackets $[a, b]$, $\forall a, b \in \mathfrak{g}$. The matrix realization of B_{f_0} can be given as matrix $M^{\mathfrak{g}}(f_0, [a, b])$ containing all entries of the form $\langle f_0, [a, b]\rangle$.

In the other hand, B_{f_0} can be defined as $B_{f_0}(a, b) = \mathrm{d}f_0(a, b)$, $\forall a, b \in \mathfrak{g}$. Since f_0 is 1-form then $\mathrm{d}f_0$ is 2-form.

Furthermore, $(\mathrm{d}f)^n$ is $n+1$-form of $\mathfrak{g}$. Since B_{f_0} is nondegenerate then the determinant of $M^{\mathfrak{g}}([a,b])$ is non zero. Moreover, the determinant of the matrix $M^{\mathfrak{g}}(f_0,[a,b])$ is non-zero as well. This implies that all rows in $M^{\mathfrak{g}}(f_0,[a,b])$ are non zero. But since all entris of $M^{\mathfrak{g}}(f_0,[a,b])$ are of the form $\langle f_0, [a,b]\rangle$ then we can find element $a_0, b_0 \in \mathfrak{g}$ such that $\mathrm{d}f_0(a_0,b_0) \neq 0$. In other words, $\mathrm{d}f_0 \neq 0$. Since, $\mathfrak{g}$ is Frobenius then $\mathfrak{g}$ is never nilpotent. Thus, the $n+1$-form $(\mathrm{d}f_0)^n \neq 0$.

($\Leftarrow$) We assume that $(\mathrm{d}f_0)^n \neq 0$. There exists a point $(x_1, x_2, x_3, \dots, x_{n+1}) \in \underbrace{\mathfrak{g} \times \mathfrak{g} \times \dots \times \mathfrak{g}}_{(n+1)\text{ times}}$ such that $(\mathrm{d}f_0)^n(x_1, x_2, x_3, \dots, x_{n+1}) \neq 0$. We observe that all rows of matrix realization of $(\mathrm{d}f_0)^n$ are nonzero and linearly independent. In other words, the determinant of $(n+1)\times(n+1)$ is non zero. Furthermore, since $B_{f_0}(a,b) = \mathrm{d}f_0(a,b),\ \forall a,b \in \mathfrak{g}$ then rows of $2n \times 2n$ matrix are non zero and linear independent. Therefore, determinant of matrix $M^{\mathfrak{g}}(f_0,[a,b])$ is nonzero. Thus, B_{f_0} is nondegenerate.

■

As example, let $\mathfrak{g}$ be a Lie algebra whose basis is $B = \{x, y\}$ and its bracket is $[x,y] = y$. Indeed, the dimension of $\mathfrak{g}$ is equal to 2. Let $\mathfrak{g}^*$ be its dual whose basis is $B^* = \{x_1^*, x_2^*\}$. The relation of elements in $\mathfrak{g}^*$ and $\mathfrak{g}$ is given by kronecker function δ_{ij} that is δ_{ij} equals 1 if $i = j$ and 0 otherwise. A straightforward computations, we choose $f_0 = -y^* \in \mathfrak{g}^*$. If we consider the skew-symmetric bilinear form

$$\beta_{f_0}(a,b) = \mathrm{d}f_0(a,b) = \langle -y^*, [a,b]\rangle, \tag{10}$$

For $a = b = x$, then we have $\mathrm{d}f_0(x,x) = 0$. The same case if $a = b = y$ then $\mathrm{d}f_0(y,y) = 0$. Moreover, if we consider $a = x$ and $b = y$ then we obtain

$$\mathrm{d}f_0(x,y) = \langle -y^*, [a,b]\rangle = \langle -y^*, [x,y]\rangle = \langle -y^*, y\rangle = -1. \tag{11}$$

If we consider $a = y$ and $b = x$ then we obtain

$$\mathrm{d}f_0(y,x) = \langle -y^*, [a,b]\rangle = \langle -y^*, [y,x]\rangle = \langle -y^*, -y\rangle = 1. \tag{12}$$

A straightforward computation, we have $\mathrm{d}f_0 = -x^* \wedge y^*$ and by our computations above in equations (11) and (12), the 2-form $\mathrm{d}f_0 \neq 0$. This shows that β_{y^*} is non-degenerate if and only if $\mathrm{d}f_0 \neq 0$. In other words, the Lie algebra $\mathfrak{g}$ is Frobenius if and only if $\mathrm{d}f_0 \neq 0$.

As a discussion, it will be interesting to study the notion of contact Lie algebra with odd dimension. Eventhough, some results have been obtained in [1], but there are some open problems to be investigated more. Furthermore, we can investigate these results to the j −algebra [15] case whether give rise to a left-invariant Kähler structure of a Frobenius Lie group or not. Moreover, the principal elements of the Frobenius Lie algebras are interesting as well to invesigate more [16].

4. CONCLUSION

We proved the necessary and sufficient condition for the nondegeneracy of the skew-symmetric bilinear form B_{f_0}. We proved the skew-symmetric bilinear form B_{f_0} of $2n$-dimensional Frobenius Lie algebra is non-degenerate if and only if $(\mathrm{d}f_0)^n \neq 0$. For the future research, the result can be observed for case a contact Lie algebra of dimension $2n+1$. The study of non-generacy of the skew-symmetric bilinear form also encourage to the notion of the j −algebra. The open problem is how to consider a left-invariant Kähler structure of a Frobenius Lie group. We guess that our result motivate to this case.

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