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DETERMINATION OF THE RESTRAINED DOMINATION NUMBER ON VERTEX AMALGAMATION AND EDGE AMALGAMATION OF THE PATH GRAPH WITH THE SAME ORDER

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Abstract. *Graph theory is a mathematics section that studies discrete objects. One of the concepts studied in graph theory is the restrained dominating set which aims to find the restrained dominating number. This research was conducted by examining the graph operation result of the vertex and edges amalgamation of the path graph in the same order. The method used in this research is the deductive method by using existing theorems to produce new theorems that will be proven deductively true. This research aimed to determine the restrained dominating number in vertex and edges amalgamation of the path graph in the same order. The results obtained from this study are in the form of the theorem about the restrained dominating number of vertex and edges amalgamation of the path graph in the same order, namely: for* $n \geq 1$, $\gamma_r(Amal\ (P_n, P_n, x_1, y_1)) = (2n-1) - 2\lfloor \frac{(2n-2)}{3} \rfloor$, *and for* $n \geq 2$, $\gamma_r(Amal\ (P_n, P_n, x_1x_2,\ y_1y_2)) = (2n-2) - 2\lfloor \frac{(2n-3)}{3} \rfloor$.

Keywords: *edges amalgamation, path graph, restrained dominating number, vertex amalgamation.*

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1. INTRODUCTION

Mathematics is one of the sciences that provides many alternatives to solving problems in various fields of life. One of the branches of mathematics used to solve problems is graph theory. Mathematically, graph G is a pair of sets (V,E) written with the notation $G = (V, E)$ where V is a non-empty set of points (vertices or nodes) and E (edges or arcs) that connect a pair of points. [1].

One of the concepts studied in graph theory is the domination set which has historical roots since the 1850s when European chess enthusiasts studied the problem of "queen domination." This fan is thinking of determining the minimum number of queens needed so that each square on a standard 8 x 8 chessboard can be occupied by a queen or can be directly attacked by a queen. In other words, the square is dominated by a queen. This situation can be modeled by graph theory. A square is a dot (V) on a chessboard, and the two dots are connected in G if each square can reach the queen on the other square with one move. The minimum number of queens that is possible not to collide with other queens with one move is the domination number of a domination set in G [2].

A set $D \subseteq V(G)$ is a dominating set if every vertex $v \in V(G) - D$ corresponds to at least one vertex in the set D, where $V(G)$ is the set of vertices of a graph. G. The minimum cardinality of the dominant set in graph G is called the domination number of graph G and is denoted by $\gamma(G)$. There are several kinds of domination sets, for example signed dominating sets, restrained dominating sets, roman dominating sets, and so on. The type of domination set discussed in this study was the restrained dominating set. A domination set D is said to be a restrained dominating set if $\forall v \in V(G) - D$ is neighbors with at least one member of $V(G) - D$. The restrained dominating set is denoted by D_r. Meanwhile, the restrained domination number is the minimum cardinality of a restrained dominating set denoted by $\gamma_r(G)$ [3]

Research on restrained domination numbers has been carried out for path graphs (P_n), cycle graphs (C_n), and complete graphs (K_n), entitled *"Restrained domination in graphs"*[4]. In addition, other studies have also been carried out in the development of restrained domination number theory, including research entitled "Dominating Set on the Results of Special Graph Operations and Its Applications"[5], "Restrained dominating set on Coronal Path Graph with Path Graph, Graph Cycle with Cycle Graph, and Complete Graph with Complete Graph"[3], *"Secure Restrained Domination in the Join and Corona of Graphs"* [6], *"On Disjoint Restrained Domination in Graphs"*[7], "*Critical Graphs in Restrained Domination"*[8], *"Restrained domination in claw-free graphs with minimum degree at least two"*[9], *"Restrained domination in cubic graphs"*[10], *"Inverse and Disjoint Restrained Domination in Graphs"*[11], *"Total restrained domination in unicyclic graphs"*[12], dan *"Restrained domination in graphs under some binary operations"*[13].

The study of the restrained dominating set of special graphs and their operating results has been carried out. Studies have been carried out using addition, multiplication, combination and corona operations. However, the restrained dominating set with amalgamation operations has not been carried out so that researchers are interested in studying the restrained dominating set in vertex amalgamation and edge amalgamation to determine the restrained domination number. The special graph used in this study is a path graph with the same order.

The vertex amalgamation operation is denoted by $\mathrm{Amal}\,(\mathrm{G}, \mathrm{v}, \mathrm{r})$ where G is a finite family of graphs. Each G has a point v which is called a terminal vertex and r represents the number of graphs G to be amalgamated. If the terminal is an edge, then the amalgamation is called edge amalgamation which is denoted by $\mathrm{Amal}\,(\mathrm{G}, \mathrm{e}, \mathrm{r})$. If the terminal is a nontrivial connected subgraph, then the amalgamation is called an amalgamation subgraph denoted by Amal $\mathrm{Amal}\,(\mathrm{G}, \mathrm{G}_1, \mathrm{r})$ where G_1 is a nontrivially connected subgraph of G [14]. Meanwhile, the path graph is a non-empty graph $\mathrm{P} = (\mathrm{V}, \mathrm{E})$ with $\mathrm{V} = \{\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_\mathrm{n}\}$ and $\mathrm{E} = \{\mathrm{x}_0\mathrm{x}_1, \mathrm{x}_1\mathrm{x}_2, \mathrm{x}_2\mathrm{x}_3, \dots, \mathrm{x}_{\mathrm{n}-1}\mathrm{x}_\mathrm{n}\}$ with $\mathrm{x}_\mathrm{i} \neq \mathrm{x}_\mathrm{j}, \forall \mathrm{i}, \mathrm{j} \in \mathbb{N}$ [15]

2. RESEARCH METHOD

The method used to prove the results of this research was the deductive method, which is a research method that uses the principles of deductive proof that apply in mathematical logic by using existing theorems to produce new theorems that will be proven deductively true. The steps taken in this research were to apply the path graph to the vertex amalgamation operation and edge amalgamation with terminal vertex v_1 and terminal edge v_1v_2 so that $Amal(P_n, P_n, v_1, v_1)$ and $Amal(P_n, P_n, v_1v_2, v_1v_2)$ were gotten. Then, the restrained dominating set from $D_r(Amal(P_n, P_n, v_1, v_1))$ and $D_r(Amal(P_n, P_n, v_1v_2, v_1\ v_2\ ))$ was determined, and check the restrained dominating set is minimum. Then, the restrained domination number from the amalgamation of the path graph $\gamma_r(Amal(P_n, P_n, v_1, v_1))$ and $\gamma_r(Amal(P_n, P_n, v_1v_2, v_1v_2))$ were analyzed so that the theorems for vertex amalgamation and edge amalgamation were obtained with their own proofs

3. RESULT AND DISCUSSION

The results of this research are in the form of a new theorem and proof related to the restrained dominating set in vertex amalgamation and edge amalgamation. The graph of the results of the amalgamation operation is carried out only on a path graph with the same order. The terminal vertex and edge are v_1 and v_1v_2, respectively, while the number of graphs amalgamated is r = 2. The Proving of the theorem was obtained by using the previous theorems. The theorems include:

$$\gamma_r\,(P_n) = n - 2\left\lfloor \frac{(n-1)}{3} \right\rfloor [4] \tag{1}$$

$$\gamma_r\,(G)\ \geq n - \frac{2m}{3} [16] \tag{2}$$

However, before presenting the theorem and its proof, it is first shown some definitions and observations from the development of the concept of the restrained dominating set and the amalgamation of the vertices and edges of the path graph that will be used in proving the theorem.

3.1. Restrained Domination Number in Vertex Amalgamation of Path Graph With Same Order Path Graph

Definition 1. It is given the first path graph with $V(P_n) = \{x_i; 1 \leq i \leq n\},\ E(P_n) = \{x_ix_{i+1}; 1 \leq i \leq n - 1\}$ and the second path graph with $V(P_n) = \{y_i; 1 \leq i \leq n\}$, $E(P_n) = \{y_iy_{i+1}; 1 \leq i \leq n-1\}$, and x_1, y_1 is the terminal vertex of the amalgamated graph to be x_1, then graph $Amal(P_n, P_n, x_1, y_1)$ is a graph that has $V\big(Amal(P_n, P_n, x_1, y_1)\big) = \{\,x_i; 1 \leq i \leq n\} \cup \big\{\,y_j; 2 \leq j \leq n\big\}$ and $E\big(Amal(P_n, P_n, x_1, y_1)\big)\ = \{x_ix_{i+1}\,; 1 \leq i \leq n-1\} \cup \big\{y_jy_{j+1}\,; 2 \leq i \leq n-1\big\} \cup \{\,x_1y_2\}$.

Based on Definition 1, the following observations are obtained:

Observation 1.

$$p = \left|\,V\big(Amal(P_n, P_n, x_1, y_1)\big)\right| = 2n - 1 \tag{3}$$

$$q = \left|\,E\big(Amal(P_n, P_n, x_1, y_1)\big)\right| = 2n - 2 \tag{4}$$

$$Amal(P_n, P_n, x_1, y_1) =\ P_{2n-1} \tag{5}$$

Theorem 1. *For* $n \geq 1$ *restrained domination number of Amal* $(P_n, P_n\,, x_1, y_1) = (2n-1) - 2\left\lfloor \frac{(2n-2)}{3} \right\rfloor$ *or* $\gamma_r(Amal\ (P_n, P_n\,, x_1, y_1)) = (2n-1) - 2\left\lfloor \frac{(2n-2)}{3} \right\rfloor$

Proof : Based on the equation (1) that $\gamma_r(P_n) = n - 2\left\lfloor \frac{(n-1)}{3} \right\rfloor$ then for $n = 2n - 1$, it is obtained $\gamma_r(P_{2n-1}) = (2n-1) - 2\left\lfloor \frac{(2n-2)}{3} \right\rfloor$. Because of the equation (5), namely $Amal(P_n, P_n, x_1, y_1) =\ P_{2n-1}$, then it can be concluded that $\gamma_r\ (Amal(P_n, P_n, x_1, y_1)) =\ \gamma_r\ (P_{2n-1}) = (2n-1) - 2\left\lfloor \frac{(2n-2)}{3} \right\rfloor$.

Furthermore, according to the equation (2), namely $\gamma_r\,(G)\ \geq n - \frac{2m}{3}$, then it will be proved that $\gamma_r\,\big(Amal(P_n, P_n, x_1, y_1)\big) = (2n-1) - 2\left\lfloor \frac{(2n-2)}{3} \right\rfloor$ always above or equal to the lower limit of the restrained domination number of a graph.

Based on the equation (2), $\gamma_r\,(G)\ \geq n - \frac{2m}{3}$ or $\gamma_r\,(G)\ \geq |\,V(G)| - \frac{2(|\,E(G)|)}{3}$ it is obtained :

$$\Rightarrow \quad \gamma_r\,(Amal(P_n, P_n, x_1, y_1))\ \geq \left|\,V\big(Amal(P_n, P_n, x_1, y_1)\big)\right| - \frac{2(\left|\,E\big(Amal(P_n, P_n, x_1, y_1)\big)\right|)}{3}$$

$\Rightarrow \ (2n-1) - 2\lfloor \frac{(2n-2)}{3} \rfloor \geq (2n-1) - 2(\frac{(2n-2)}{3})$

because $\lfloor \frac{(2n-2)}{3} \rfloor$ always less than or equal to $(\frac{2n-2}{3})$, so $2\lfloor \frac{(2n-2)}{3} \rfloor$ will also always be less than or equal to $2(\frac{2n-2}{3})$, and result in $(2n-1) - 2\lfloor \frac{(2n-2)}{3} \rfloor$ always be greater than or equal to $(2n-1) - 2(\frac{(2n-2)}{3})$. then it is proved that $(2n-1) - 2\lfloor \frac{(2n-2)}{3} \rfloor \geq (2n-1) - 2(\frac{(2n-2)}{3})$ or $\gamma_r\left(Amal(P_n, P_n, x_1, y_1)\right) = (2n-1) - 2\lfloor \frac{(2n-2)}{3} \rfloor$ always above or equal to the lower limit of the restrained domination number of a graph ∎

To better understand the theorem, consider Figure 1, which is the restrained dominating set of graph $Amal(P_4, P_4, x_1, y_1)$ with the red dot is the restrained dominating set.

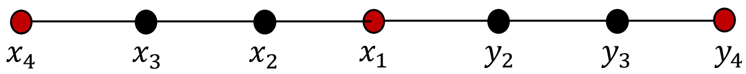

Figure 1. The restrained dominating set $D_r\ (Amal(P_4, P_4, x_1, y_1)) = \{x_1, x_4, y_4$ and $\gamma_r(Amal(P_4, P_4, x_1, y_1)) = 3$

3.2. The Restrained Domination Numbers in Edge Amalgamation of Path Graphs with Equal Order Path Graphs

Definition 2. It is given the first path graph with $V(P_n) = \{x_i; 1 \leq i \leq n\}$, $E(P_n) = \{x_i x_{i+1}; 1 \leq i \leq n-1\}$ and the second path graph with $V(P_n) = \{y_i; 1 \leq i \leq n\}$, $E(P_n) = \{y_i y_{i+1}; 1 \leq i \leq n-1\}$, and x_1x_2, y_1y_2 are the terminal edge of the graph which is amalgamated to x_1x_2 , with $x_1 = y_2, x_2 = y_1$, then graph $Amal(P_n, P_n, x_1x_2, y_1y_2)$ adalah graf yang memiliki $V\left(Amal(P_n, P_n, x_1x_2, y_1y_2)\right) = \{x_i; 1 \leq i \leq n\} \cup \left\{y_j; 3 \leq j \leq n\right\}$ and $E\left(Amal(P_n, P_n, x_1x_2, y_1y_2)\right) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \left\{y_j y_{j+1}; 3 \leq j \leq n-1\right\} \cup \{x_1 y_3\}$ for $n \geq 2$.

Based on Definition 1, the following observations are obtained:

Observation 2.

$$p = \left|V\left(Amal(P_n, P_n, x_1x_2, y_1y_2)\right)\right| = 2n-2 \quad (6)$$
$$q = \left|E\left(Amal(P_n, P_n, x_1x_2, y_1y_2)\right)\right| = 2n-3 \quad (7)$$
$$Amal(P_n, P_n, x_1x_2, y_1y_2) = P_{2n-2} \quad (8)$$

Theorem 2. *For* $n \geq 2$ *restrained domination number of* $Amal\ (P_n, P_n, x_1x_2, y_1y_2) = (2n-2) - 2\lfloor \frac{(2n-3)}{3} \rfloor$ *or* $\gamma_r(Amal\ (P_n, P_n, x_1x_2, y_1y_2)) = (2n-2) - 2\lfloor \frac{(2n-3)}{3} \rfloor$

Proof : Based on the equation (1) that $\gamma_r(P_n) = n - 2\lfloor \frac{(n-1)}{3} \rfloor$, then for $n = 2n-2$ it is obtained $\gamma_r(P_{2n-2}) = (2n-2) - 2\lfloor \frac{(2n-3)}{3} \rfloor$. Because of the equation (8), namely $Amal(P_n, P_n, x_1x_2, y_1y_2) = P_{2n-2}$, then it can be concluded that $\gamma_r\ (Amal(P_n, P_n, x_1x_2, y_1y_2)) = \gamma_r\ (P_{2n-2}) = (2n-2) - 2\lfloor \frac{(2n-3)}{3} \rfloor$.

Furthermore, according to the equation (2), namely $\gamma_r\ (G) \geq n - \frac{2m}{3}$, then it will be proved that $\gamma_r\left(Amal(P_n, P_n, x_1x_2, y_1y_2)\right) = (2n-2) - 2\lfloor \frac{(2n-3)}{3} \rfloor$ always above or equal to the lower limit of the restrained domination number of a graph.

Based on the equation (2), $\gamma_r\ (G) \geq n - \frac{2m}{3}$ or $\gamma_r\ (G) \geq |V(G)| - \frac{2(|E(G)|)}{3}$ it is obtained :

$\Rightarrow \ \gamma_r\ (Amal(P_n, P_n, x_1x_2, y_1y_2)) \geq \left|V\left(Amal(P_n, P_n, x_1x_2, y_1y_2)\right)\right| - \frac{2(\left|E\left(Amal(P_n, P_n, x_1x_2, y_1y_2)\right)\right|)}{3}$

$\Rightarrow \ (2n-2) - 2\lfloor \frac{(2n-3)}{3} \rfloor \geq (2n-2) - 2(\frac{(2n-3)}{3})$

because $\lfloor \frac{(2n-3)}{3} \rfloor$ always less than or equal to$(\frac{2n-3}{3})$, so that $2 \lfloor \frac{(2n-3)}{3} \rfloor$ will also always be less than or equal to $2(\frac{2n-3}{3})$, and resulted in$(2n-2) - 2 \lfloor \frac{(2n-3)}{3} \rfloor$ always be greater than or equal to $(2n-2) - 2(\frac{(2n-3)}{3})$. Then, it is proved that $(2n-2) - 2 \lfloor \frac{(2n-3)}{3} \rfloor \geq (2n-2) - 2(\frac{(2n-3)}{3})$ or $\gamma_r\left(Amal(P_n, P_n, x_1x_2, y_1y_2)\right) = (2n-2) - 2 \lfloor \frac{(2n-3)}{3} \rfloor$ always above or equal to the lower limit of the restrained domination number of a graph ■

To better understand the theorem, consider Figure 2, which is the restrained dominating set of graph $Amal(P_4, P_4, x_1x_2, y_1y_2)$ with the red dot is the restrained dominating set.

Figure 2. The restrained dominating set $D_r\ (Amal(P_4, P_4, x_1x_2, y_1y_2)) = \{x_1, x_4, y_3, y_4\}$ and $\gamma_r(Amal(P_4, P_4, x_1x_2, y_1y_2)) = 4$

4. CONCLUSION

The conclusions that can be drawn from the results of this study are in the form of two theorems, namely:

Theorem 1. *For* $n \geq 1$ *restrained domination number from* $Amal\ (P_n, P_n, x_1, y_1) = (2n-1) - 2\lfloor \frac{(2n-2)}{3} \rfloor$ *or* $\gamma_r(Amal\ (P_n, P_n, x_1, y_1)) = (2n-1) - 2\lfloor \frac{(2n-2)}{3} \rfloor$

Theorem 2. *For* $n \geq 2$ *restrained domination number from* $Amal\ (P_n, P_n, x_1x_2, y_1y_2) = (2n-2) - 2\lfloor \frac{(2n-3)}{3} \rfloor$ *or* $\gamma_r(Amal\ (P_n, P_n, x_1x_2, y_1y_2)) = (2n-2) - 2\lfloor \frac{(2n-3)}{3} \rfloor$

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