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ODD HARMONIOUS LABELING ON SOME STRING GRAPH CLASSES

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Abstract. *A graph with the labeling properties of odd harmonious is called an odd harmonious graph. The purpose of this research was to get labeling properties of odd harmonious on the class of string graphs. The research used was a qualitative research method. The result of the research was that the definition and construction of a string graph, the union of a string graph, and the multiple string graph are obtained. Furthermore, it has been proved that a string graph, the union of a string graph, and the multiple string graph is an odd harmonious graph.*

Keywords: *odd harmonious graph, odd harmonious labeling, multiple graph, string graph, union operation*

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1. INTRODUCTION

Graph labeling is one of the research topics on graph theory that has developed very rapidly in recent years. In 2019, Liang and Bai construct the definition of an odd harmonious graph as follows. For example, $G(p,q)$ is a graph with the set of vertices $V(G)$ and the set of edges $E(G)$ with order $p = |V(G)|$ and size $q = |E(G)|$. If the graph $G(p,q)$ fulfills the injective $f: V(G) \to \{0,1,2,3,4,5,\dots,2q-1\}$ that induces $f^*: E(G) \to \{1,3,5,7,9,\dots,2q-1\}$ which is bijective with the definition of $f^*(x,y) = f(x) + f(y)$ so the graph $G(p,q)$ is an odd harmonious graph [1].

Gallian in 2019 collected all the research on graph labeling theory and its applications [2]. There are several classes of odd harmonious graphs that have been found, including: Srividya and Govindarajan in 2020 found a circular graph with parallel chord [3]. Ferbriana and Sugeng in 2020 found a squid graph [4], Jeyanthi and Philo in 2020 found a ladder graph [5]. Mumtaz et al in 2021 found a matting graph [6], Mumtaz and Silaban in 2021 found that the snake hair graph [7]. Pujiwati et al in 2021 found two star graphs [8]. Other relevant research results can be seen in [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], and [20].

In 2017, Firmansah and Yuwono found a snake net graph [21], and Firmansah, in 2020, found a multiple net snake graph which is an odd harmonious graph [22]. The two results in this paper are the basis for research to obtain new class constructions in the form of string graphs, the union of string graphs, and multiple string graphs. This research aimed to find the labeling properties of odd harmonious in several classes of string graphs. The research method used was qualitative research. In this study, the construction and definition of the string graph $R(s)$ with $s \geq 1$, the union of a string graph $R(s) \cup R(s)$ with $s \geq 1$, and the multiple string graph $R(s,t)$ with $s \geq 1$ and $t \geq 1$ will be given. Next, we will prove that the string graph $R(s)$ with $s \geq 1$, the union of a string graph $R(s) \cup R(s)$ with $s \geq 1$, and the multiple string graph $R(s,t)$ with $s \geq 1$ and $t \geq 1$ fulfill the labeling properties of odd harmonious such that it is an odd harmonious graph. Furthermore, the results of this study also add new properties of odd harmonious graphs from previous studies.

2. RESEARCH METHOD

This research is a type of qualitative research that focuses on developing new properties of odd harmonious graphs. The research stages were as follows: Data collection stage, looking for relevant reference sources to obtain new graph class constructions in the form of graph class definitions and graph class images to obtain vertex notation and edge notation from the graph class. The data analysis stage will be given the construction of a set of vertices, a set of edges, labeling vertices that are injective, and labeling edges that are bijective. Stages of theorem construction and proof, at this stage, the construction of the theorem and its proof will be given. Then, the theorem was proven using the direct proof method, namely by showing that the labeling function of the vertices that have been constructed fulfills the injective properties to induce a bijective edge labeling function.

3. RESULTS AND DISCUSSION

In this chapter, odd harmonious labelling will be given to several classes of string graphs, namely string graphs, the union of a string graphs and multiple string graphs.

3.1 Odd Harmonious Labeling on String Graphs

Definition 1. *String graph $R(s)$ with $s \geq 1$ is a graph with*
$V(R(s)) = \{u_i | 0 \leq i \leq s+1\} \cup \{v_i{}^j | 1 \leq i \leq s+1,\ j = 1,2\} \cup \{w_i{}^j | 1 \leq i \leq s,\ j = 1,2\}$ *and*
$E(R(s)) = \{u_i v_{(i+1)}{}^j | 0 \leq i \leq s,\ j = 1,2\} \cup \{v_i{}^j u_i | 1 \leq i \leq s+1, j = 1,2\} \cup$
$\{v_i{}^j w_i^j | 1 \leq i \leq s, j = 1,2\} \cup \{w_i{}^j v_{(i+1)}^j | 1 \leq i \leq s, j = 1,2\}$.

In Figure 1 given the construction of a string graph $R(s)$ with $s \geq 1$.

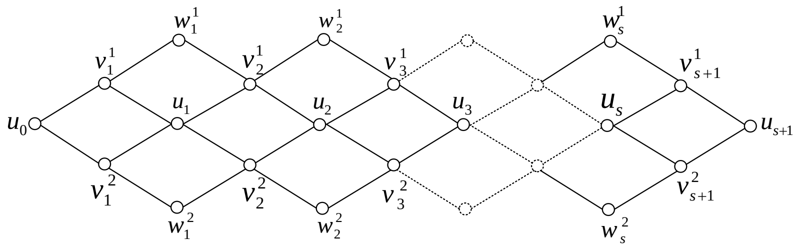


Figure 1. String graph $R(s)$

Next, we will prove that the string graphs $R(s)$ with $s \geq 1$ is an odd harmonious graph expressed in Theorem 2.

Theorem 2. *The string graph $R(s)$ with $s \geq 1$ is odd harmonious graph*

Proof. Based on Definition 1 obtained $p = |V(R(s))| = 5s + 4$ and $q = |E(R(s))| = 8s + 4$. given $f: V(R(s)) \to \{0,1,2,3,4,5,\dots,16s+7\}$

$$f(u_i) = 4i, 0 \leq i \leq s+1 \quad (1)$$

$$f(v_i{}^j) = 4i + 2j - 5, j = 1,2 \text{ and } 0 \leq i \leq s+1 \quad (2)$$

$$f(w_i{}^j) = 16k - 12i - 4j + 18, j = 1,2 \text{ and } 1 \leq i \leq s \quad (3)$$

Based on (1), (2), and (3) it is obtained that each vertices has a different table and $V(R(s)) \subseteq \{0,1,2,3,4,5,\dots,16s+7\}$ so f is injective. Given $f^*: E(R(s)) \to \{1,3,5,7,9,\dots,16s+7\}$.

$$f^*(u_i v_{(i+1)}{}^j) = 2j + 8i - 1, j = 1,2 \text{ and } 0 \leq i \leq s \quad (4)$$

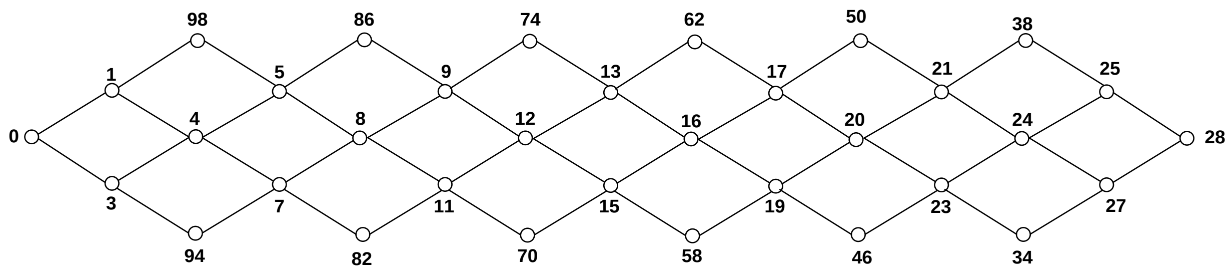
$$f^*(v_i{}^j u_i) = 2j + 8i - 5, j = 1,2 \text{ and } 1 \leq i \leq s+1 \quad (5)$$

$$f^*(v_i{}^j w_i^j) = 16k - 2j - 8i + 13, j = 1,2 \text{ and } 1 \leq i \leq s \quad (6)$$

$$f^*\left(w_i{}^j v_{(i+1)}^j\right) = 16k - 2j - 8i + 17, j = 1,2 \text{ and } 1 \leq i \leq s \quad (7)$$

According to (4), (5), (6) and (7) It is obtained that each edges has a different label and $E(R(s)) = \{1,3,5,7,9,\dots,16s+7\}$ so f^* is bijective. Therefore, the string graph $R(s)$ with $s \geq 1$ is an odd harmonious graph □

Here is an example of a string graph: $R(6)$ in Figure 2 which is an odd harmonious graph.



Gambar 2. String Graph $R(6)$

3.2 Odd Harmonious Labeling on The Union of a String Graphs

Definition 3. *The union of a string graphs $R(s) \cup R(s)$ with $s \geq 1$ is a graph with* $V(R(s) \cup R(s)) = \{u_i | 0 \leq i \leq s+1\} \cup \{v_i{}^j | 1 \leq i \leq s+1,\ j = 1,2\} \cup \{w_i{}^j | 1 \leq i \leq s,\ j = 1,2\} \cup \{x_i | 0 \leq i \leq s+1\} \cup \{y_i{}^j | 1 \leq i \leq s+1,\ j = 1,2\} \cup \{z_i{}^j | 1 \leq i \leq s,\ j = 1,2\}$ *and*

$E\big(R(s) \cup R(s)\big) = \{u_i v_{(i+1)}{}^j \,|\, 0 \le i \le s,\ j = 1,2\} \cup \{v_i{}^j u_i \,|\, 1 \le i \le s+1, j = 1,2\} \cup$ $\{v_i{}^j w_i^j \,|\, 1 \le i \le s, j = 1,2\} \cup \left\{w_i{}^j v_{(i+1)}^j \,\middle|\, 1 \le i \le s, j = 1,2\right\} \cup \{x_i y_{(i+1)}{}^j \,|\, 0 \le i \le s,\ j = 1,2\} \cup$ $\{y_i{}^j x_i \,|\, 1 \le i \le s+1, j = 1,2\} \cup \{y_i{}^j z_i^j \,|\, 1 \le i \le s, j = 1,2\} \cup \left\{z_i{}^j y_{(i+1)}^j \,\middle|\, 1 \le i \le s, j = 1,2\right\}$.
In Figure 3, construction the union of a string graph is given $R(s) \cup R(s)$ with $s \ge 1$.

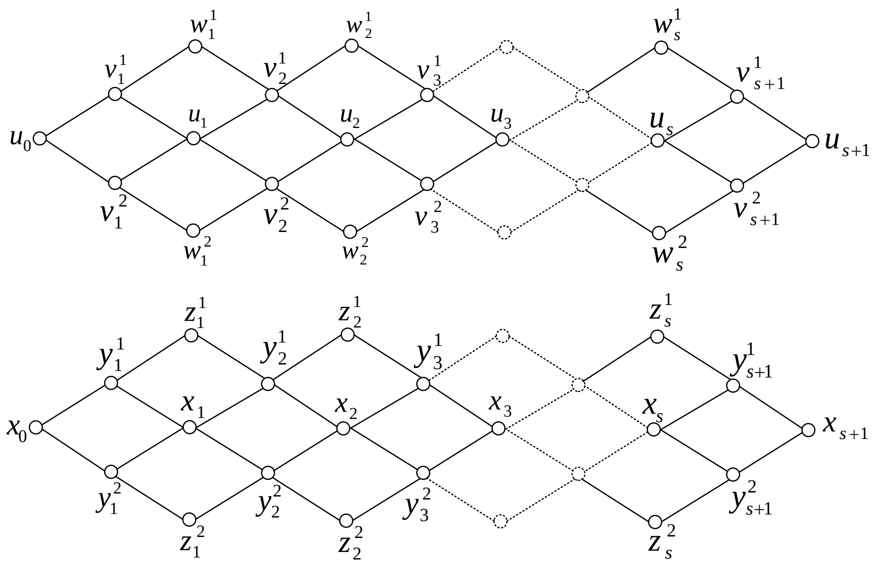


Figure 3. The union of a string graphs $\boldsymbol{R(s) \cup R(s)}$

Furthermore, it will be proved that the union of a string R(s)∪R(s) is an odd harmonious graph which is stated in Theorem 4.

Theorem 4. *The union of a string graphs $R(s) \cup R(s)$ with $s \ge 1$ is odd harmonious graph*

Proof. Based on Definition 3 obtained $p = \left|V\big(R(s) \cup R(s)\big)\right| = 10s + 8$ and $q = \left|E\big(R(s) \cup R(s)\big)\right| = 16s + 8$. Given $f: V\big(R(s) \cup R(s)\big) \to \{0,1,2,3,4,5,\dots,32s+15\}$.

$$f(u_i) = 4i, 0 \le i \le s+1 \tag{8}$$

$$f(v_i{}^j) = 4i + 2j - 5, j = 1,2 \text{ and } 0 \le i \le s+1 \tag{9}$$

$$f(w_i{}^j) = 16k - 12i - 4j + 18, j = 1,2 \text{ and } 1 \le i \le s \tag{10}$$

$$f(x_i) = 4i + 2, 0 \le i \le s+1 \tag{11}$$

$$f(y_i{}^j) = 16k + 4i + 2j + 1, j = 1,2 \text{ and } 0 \le i \le s+1 \tag{12}$$

$$f(z_i{}^j) = 16k - 12i - 4j + 20, j = 1,2 \text{ and } 1 \le i \le s \tag{13}$$

According to (8), (9), (10), (11), (12), and (13) it is obtained that each vertices has a different label and $V\big(R(s) \cup R(s)\big) \subseteq \{0,1,2,3,4,5,\dots,32s+15\}$ so f is injective. Defined edges labeling function $f^*: E\big(R(s) \cup R(s)\big) \to \{1,3,5,7,9,\dots,32s+15\}$.

$$f^*(u_i v_{(i+1)}{}^j) = 8i + 2j - 1, j = 1,2 \text{ and } 0 \le i \le s \tag{14}$$

$$f^*(v_i{}^j u_i) = 8i + 2j - 5, j = 1,2 \text{ and } 1 \le i \le s+1 \tag{15}$$

$$f^*(v_i{}^j w_i^j) = 16k - 8i - 2j + 13, j = 1,2 \text{ and } 1 \le i \le s \tag{16}$$

$$f^*\left(w_i{}^j v_{(i+1)}^j\right) = 16k - 8i - 2j + 17, j = 1,2 \text{ and } 1 \le i \le s \tag{17}$$

$$f^*(x_i y_{(i+1)}{}^j) = 16k + 8i + 2j + 7, j = 1,2 \text{ and } 0 \le i \le s \tag{18}$$

$$f^*(y_i{}^j x_i) = 16k + 8i + 2j + 3, j = 1,2 \text{ and } 1 \le i \le s+1 \tag{19}$$

$$f^*(y_i{}^j z_i^j) = 32k - 8i - 2j + 21, j = 1,2 \text{ and } 1 \le i \le s \tag{20}$$

$$f^*\left(z_i{}^j y_{(i+1)}^j\right) = 32k - 8i - 2j + 25, j = 1,2 \text{ and } 1 \le i \le s \tag{21}$$

Based on (14), (15), (16), (17), (18), (19), (20), and (21) it is obtained that each edges has a different label and $E\big(R(s) \cup R(s)\big) = \{1,3,5,7,9, \dots ,32s + 15\}$ so f^* is bijective. Therefore, the union of a graph of $R(s) \cup R(s)$ with $s \geq 1$ is an odd harmonious graph. □

The following is an example of the union of a string graph $R(4) \cup R(4)$ in Figure 4 which is an odd harmonious graph.

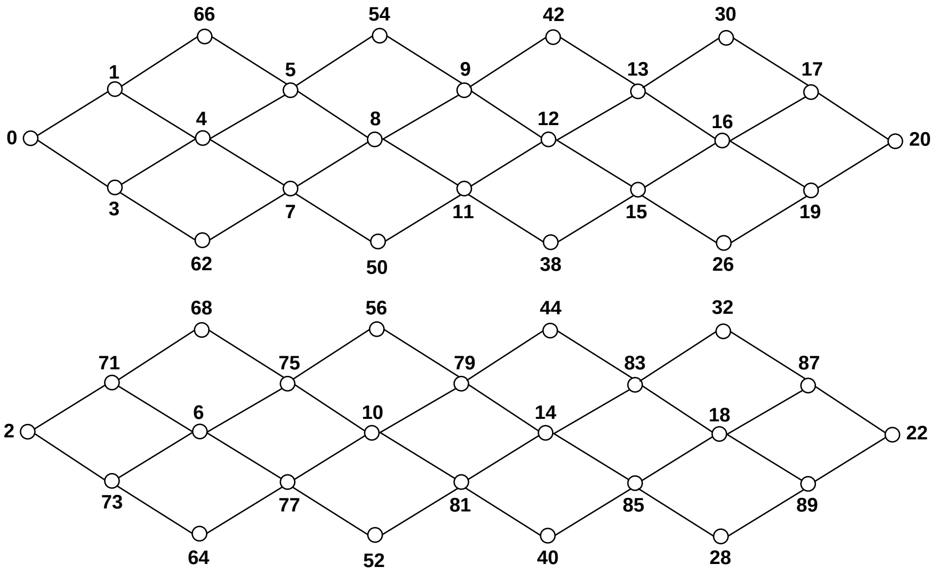


Figure 4. The union of a string graphs $R(4) \cup R(4)$

3.3 Odd Harmonious Labeling on a Multiple String Graph

Definition 5. *Multiple string graph* $R(s,t)$ *with* $s \geq 1$ *and* $t \geq 1$ *is a graph with* $V\big(R(s,t)\big) = \{u_i | 0 \leq i \leq s+1\} \cup \{{v_i}^j | 1 \leq i \leq s+1,\ j = 1,2\} \cup \{{w_l}^{i,j} | 1 \leq i \leq s, 1 \leq l \leq t, j = 1,2\}$ *and* $E\big(R(s,t)\big) = \{u_i {v_{(i+1)}}^j | 0 \leq i \leq s,\ j = 1,2\} \cup \{{v_i}^j u_i | 1 \leq i \leq s+1, j = 1,2\} \cup \{{v_i}^j w_l^{i,j} | 1 \leq i \leq s, 1 \leq l \leq t, j = 1,2\} \cup \left\{{w_l}^{i,j} v_{(i+1)}^j \middle| 1 \leq i \leq s, 1 \leq l \leq t, j = 1,2\right\}$.

In Figure 5, the construction of a multiplied string graph is given $R(s,t)$ with $s \geq 1$ and $t \geq 1$

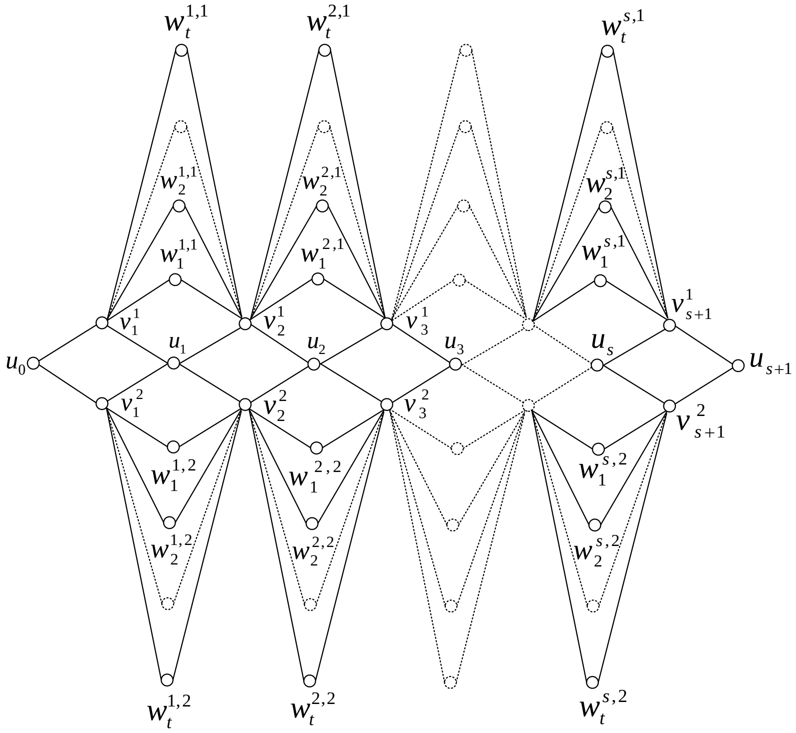


Figure 5. The multiple string graph $R(s,t)$

Next, it will be proved that the multiple string graph $R(s,t)$ with $s \geq 1$ and $t \geq 1$ is an odd harmonious graph expressed in Theorem 6.

Theorem 6. *The multiple string graph $R(s,t)$ with $s \geq 1$ and $t \geq 1$ is odd harmonious graph.*

Proof. Based on Definition 5, it is obtained $p = \left|V\big(R(s,t)\big)\right| = 2st + 5s + 4$ and $q = \left|E\big(R(s)\big)\right| = 4st + 8s + 4$. Given $f: V\big(R(s,t)\big) \rightarrow \{0,1,2,3,4,5,\dots,8st + 16s + 7\}$.

$$f(u_i) = 4i, 0 \leq i \leq s + 1 \quad (23)$$

$$f\big({v_i}^j\big) = 4i + 2j - 5, 0 \leq i \leq s + 1, j = 1,2 \quad (24)$$

$$f\big({w_l}^{i,j}\big) = (8t + 16)s + 8l - (8t + 12)i - 4j + 10, 1 \leq i \leq s, 1 \leq l \leq t + 1, j = 1,2 \quad (25)$$

Based on (23), (24), (25) it is obtained that each vertices has a different label and $V\big(R(s,t)\big) \subseteq \{0,1,2,3,4,5,\dots,8st + 16s + 7\}$ so f is injective. Given $f^*: E\big(R(s)\big) \rightarrow \{1,3,5,7,9,\dots,8st + 16s + 7\}$.

$$f^*\big(u_i {v_{(i+1)}}^j\big) = 8i + 2j - 1,\ 0 \leq i \leq s, j = 1,2 \quad (26)$$

$$f^*\big({v_i}^j u_i\big) = 8i + 2j - 5,\ 1 \leq i \leq s + 1, j = 1,2 \quad (27)$$

$$f^*\big({v_i}^j w_l^{i,j}\big) = (8t + 16)s + 8l - (8t + 8)i - 2j + 5, 1 \leq i \leq s, 1 \leq l \leq t, j = 1,2 \quad (28)$$

$$f^*\left({w_l}^{i,j} v_{(i+1)}^j\right) = (8t + 16)s + 8l - (8t + 8)i - 2j + 9, 1 \leq i \leq s, 1 \leq l \leq t + 1, j = 1,2 \quad (29)$$

Based on (26), (27), (28) and (29) it is obtained that each edges has a different label and $E\big(R(s)\big) = \{1,3,5,7,9,\dots,8st + 16s + 7\}$ so f^* is bijective. Therefore, multiple string graph $R(s,t)$ with $s \geq 1$ and $t \geq 1$ is an odd harmonious graph. □

The following is an example of the multiple string graph $R(4,3)$ in Figure 6 which is an odd harmonious graph.

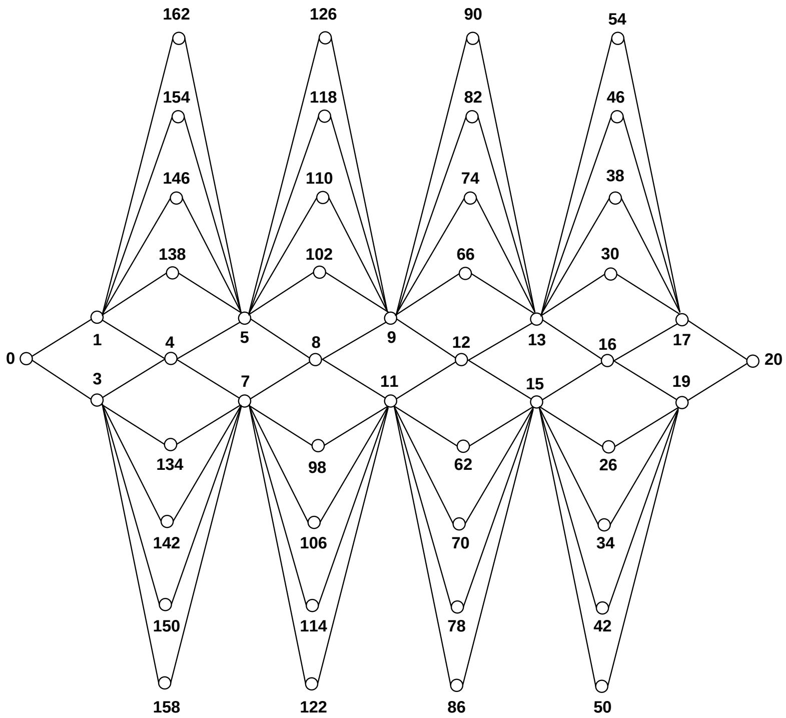


Figure 6. The multiple string graph $R(4,3)$

Based on the results in Theorem 2, Theorem 4 and Theorem 6, it has been proved that the string graph $R(s)$ with $s \geq 1$, the union of a string graph $R(s) \cup R(s)$ with $s \geq 1$, and the multiple string graph $R(s,t)$ with $s \geq 1$ and $t \geq 1$ are odd harmonious graphs. The results and discussion show that this research is a development of previous studies [21] and [22], namely the addition of family graph classes from odd harmonious graphs.

4. CONCLUSION

Based on the results and discussion, it is obtained that the construction of the definition of the string graph $R(s)$ with $s \geq 1$, the union of a string graph $R(s) \cup R(s)$ with $s \geq 1$, and the multiple string graph $R(s,t)$ with $s \geq 1$ and $t \geq 1$. Furthermore, it has been proved that the string graph $R(s)$ with $s \geq 1$, the union of a string graph $R(s) \cup R(s)$ with $s \geq 1$, and the multiple string graph $R(s,t)$ with $s \geq 1$ and $t \geq 1$ fulfils the labelling properties of odd harmonious such that it is an odd harmonious graph.

This research can be continued by looking for odd harmonious labelling on the union of multiple string graphs $R(s,t) \cup R(s,t)$ with $s \geq 1$ and $t \geq 1$.

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