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ON ANTIADJACENCY MATRIX OF A DIGRAPH WITH DIRECTED DIGON(S)

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Abstract. *The antiadjacency matrix is one representation matrix of a digraph. In this paper, we find the determinant and the characteristic polynomial of the antiadjacency matrix of a digraph with directed digon(s). The digraph that we will discuss is a digraph obtained by adding arc(s) in an arborescence path digraph such that it contained directed digon(s), and a digraph obtained by deleting arc(s) in a complete star digraph. We found that the determinant and the coefficient of the characteristic polynomial of the antiadjacency matrix of a digraph obtained by adding arc(s) in an arborescence path digraph such that it contained directed digon(s) is different depending on the location of the directed digon. Meanwhile, the determinant of the antiadjacency matrix of a digraph obtained by deleting arc(s) in the complete star digraph is zero.*

Keywords: *antiadjacency matrix, arc, characteristic polynomial, determinant, digraph, directed digon.*

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1. INTRODUCTION

Nowadays, graph theory is one of mathematics' branches of study developing expeditiously. Graph theory has many applications in other fields of study, such as data science, chemistry, biology, and economics. A graph is divided into an undirected graph and a directed graph (digraph).

There are numerous applications of the directed path and directed arborescence graph in several studies. In [1], it is shown that the directed path can be used to find the shortest route from one location to another. Meanwhile, finding minimum travel time can be used from clustering strategies using an arborescence graph [2]. Thus, they are motivated us to investigate the directed arborescence path. Nonetheless, since [3], [4], and [5] have already discussed the antiadjacency matrix of the directed arborescence path, then in this paper, we discuss the determinant of the antiadjacency matrix of the directed arborescence path that added arc(s) such that it will have directed digon(s).

A complete star digraph has some critical applications in biology [6]. Using a complete star digraph, people can find Birth-Death Models of Information Spread in Structured Populations [7]. Meanwhile, in [8], a complete star digraph is used to find information spreads in a population. Furthermore, in this paper, we are also motivated to find other complete star digraph properties.

The coefficients of the characteristic polynomial of a matrix can be found by calculating the sum of the determinant of its submatrices [9]. Bapat [3] has found the value of the determinant of the antiadjaceny matrix of a directed graph. It is interesting to see what is happened if the directed graph has digons. Thus, in this paper, we are also motivated to find the determinant of the antiadjacency matrix of a digraph with directed digon(s). By using the determinant of the submatrix, we also discuss the characteristic polynomial of the digraph that has directed digon(s). The digraphs that will be addressed in this paper are arborescence paths and complete star digraph, that we add arc(s) such that there exists a directed digon(s).

A directed graph (digraph) D is an ordered pair $(V(D), A(D))$ that consists of a set of vertices $V \coloneqq V(D)$ and a set of arcs (directed edges) $A(D)$ that is disjoint from $V(D)$, together with an incidence function ψ_D associated with each arc of D as an ordered pair of vertices in D [10].

Let u, v be the distinct vertices of a digraph D. If D has either an arc (u, v) or an arc (v, u), then D is called an oriented digraph. In a digraph D, we can make an underlying graph of D by replacing each arc (u, v) with an edge uv, or both arcs (u, v) and (v, u) by an edge uv. If the underlying graph of D is connected, then D is called weakly connected. Meanwhile, a digraph D is called strongly connected if, for every pair of vertices u and v, we have a $u - v$ directed path and $v - u$ directed path [11].

Let the set of vertices of the digraph D and H, respectively, be $V(D)$ and $V(H)$ where $V(H) \subseteq V(D)$, and let the set of arcs of the digraph D and H, respectively, be $A(D)$ and $A(H)$ where $A(H) \subseteq A(D)$. Then, the digraph H is a subdigraph of a digraph D. A subdigraph H of a digraph D is called induced subdigraph of D if u and v are the vertices of H and (u, v) is the arc of D then (u, v) is also the arc of H as well. [12].

Let $W = (u = u_1, u_2, \dots, u_k = v)$ be a sequence of vertices of D such that the vertex u_i is adjacent to u_{i+1}, where $i \in \{0,1, \dots, k-1\}$. Then W is called an $u - v$ directed walk in D. The length of the directed walk W is the number of visited arcs on D. If in the $u - v$ directed walk we have $u = v$, then the directed walk is called closed. On the other hand, if in an $u - v$ directed walk we have $u \neq v$, then the directed walk is called open. If W is not passing through the vertex more than once, then W is a directed path. A directed cycle is a closed directed walk with a length at least two, where no vertex is repeated except for the initial and terminal vertices. If a digraph D does not have a directed cycle subgraph, then D is called acyclic digraph. Meanwhile, if a digraph D is had a directed cycle, then D is a cyclic digraph [13]. A Hamiltonian directed path in a digraph D is a directed path that includes all vertices in D [3].

In this paper, we define a directed digon as a directed cycle with length two. Furthermore, we denote a directed digon as $\{(v_a, v_b); (v_b, v_a) | v_a, v_b \in V(D)\}$. For simplicity, we use the term digon for directed digon in the rest of the paper.

A rooted tree is a tree in that one vertex, defined as a root, is distinguished from others [14]. Therefore, if a digraph D has a subdigraph which is a rooted directed tree, then D is called rooted digraph. Let D be a rooted digraph with a set of vertices $V \coloneqq V(D)$ and a set of arcs $A(D)$. A rooted digraph D is called an arborescence digraph if u is a root vertex then there exists a unique directed path from u to v for every vertex v in D [15].

If a digraph D has a single central vertex v, such that an arc exists from v to the neighbour vertices and vice versa, then D is called a complete star digraph. Formally, a complete star digraph (CSD), that has a size n, is denoted by S_n, and has a set of vertices $V \coloneqq V(S_n) = \{v_1, v_2, \dots, v_n\}$ and a set of arcs $A \coloneqq A(S_n) = \{(v_1, v_j); (v_j, v_1) \mid j = 2, 3, \dots, n\}$ [6]. Note that we have digons $\{(v_1, v_j); (v_j, v_1) \mid j \in \{2, 3, \dots, n\}\}$ in the CSD.

In this paper, we define a digraph $S_{n,k}$ as a digraph obtained by deleting k-arcs (v_j, v_1) from a CSD, where $j \in \{2, 3, \dots, n\}$ and $0 \leq k \leq n-1$. If $k = n-1$, then $S_{n,k}$ is an arborescence star digraph. Moreover, if $0 \leq k \leq n-1$, then $S_{n,k}$ will contain digon(s).

A digraph can be represented with a representation matrix. Examples of the representation matrices are adjacency and antiadjacency matrices. An adjacency matrix of a digraph D (with a set of vertices $\{v_1, v_2, \dots, v_n\}$) is an $n \times n$ matrix defined by $A(D) = [a_{ij}]$, where a_{ij} is equal to 1 if there exists an arc from v_i to v_j, and equal to 0 elsewhere. On the other hand, the antiadjacency matrix of a digraph D (with a set of vertices $\{v_1, v_2, \dots, v_n\}$) is an $n \times n$ matrix defined by $B(D) = J - A(D)$, where J is an $n \times n$ matrix in which every entry is equal to 1 and $A(D)$ is an adjacency matrix of D [3].

2. RESEARCH RESULTS

This section discusses some known results that are related to the results in Section 3. The first theorem, Theorem 2.1 shows the results on the characteristic equation and the eigenvalues of an $n \times n$ adjacency matrix.

Theorem 2.1 [9]. *If $\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_{n-1}\lambda + c_n = 0$ is the characteristic equation for $A_{n\times n}$ and if s_k is the k-th symmetric function of the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A_{n\times n}$, then $det(A) = \lambda_1\lambda_2 \dots \lambda_n$*

Theorem 2.1 shows the determinant of the antiadjacency matrix of a simple digraph with a hamiltonian path.

Theorem 2.2 [3]. *Let D be an acyclic digraph with the set of verrtices $V(D) = \{v_1, v_2, \dots, v_n\}$ and $B(D)$ be the antiadjacency matrix of D. Then $det(B(D)) = 1$ if D has a Hamiltonian path, and $det(B(D)) = 0$ otherwise.*

Theorem 2.2 shows the determinant of the antiadjacency matrix of a simple digraph with a hamiltonian path. In the next section, we show the determinant of the antiadjacency matrix of a digraph obtained by adding arc(s) in the arborescence path digraph such that it has digon(s).

3. RESULTS AND DISCUSSION

3.1. Determinant of the antiadjacency matrix

We present our results concerning the determinant of the antiadjacency matrix of an arborescence path digraph with digon(s). We also determine the determinant of the antiadjacency matrix of a digraph that is obtained by deleting arcs in a complete star digraph. Proposition 3.1 shows the determinant of the antiadjacency matrix of an arborescence path digraph with digon(s).

Proposition 3.1. *Let D be an arborescence path digraph with the set of vertices $V(D) = \{v_1, v_2, \dots, v_n\}$ where $n \geq 2$, and $B(D)$ be its antiadjacency matrix.*

(i) If D_1 is a digraph obtained by adding arc(s) (v_{i+1}, v_i) in D, where $i \in \{2, 3, \dots, n-2\}$ and $n \geq 4$, then $det(B(D_1)) = 1$.

(ii) If D_2 is a digraph obtained by adding either an arc (v_2, v_1) or an arc (v_n, v_{n-1}) in D, where $n \geq 3$, then $det(B(D_2)) = 0$. Moreover, if D_3 is a digraph obtained by adding arcs (v_2, v_1) and (v_n, v_{n-1}) in D, where $n \in \{3, 5, 6, \dots\}$, then $det(B(D_3)) = 0$. On the other hand, if D_3 is a

digraph that has order $four$ and is obtained by adding arcs (v_2, v_1) and (v_4, v_3), then $det\big(B(D_3)\big) = -2$.

(iii) Let $n \geq 4$. If D_4 is a digraph obtainedby adding an arc (v_2, v_1) and arc(s) (v_{i+1}, v_i) where $i \in \{2, 3, 5, \ldots, n-1\}$, then $det(B(D_4)) = 0$. If D_5 is added an arc (v_2, v_1) and an arc (v_4, v_3), then $det(B(D_5)) = 1$.

(iv) Let $n \geq 4$. If D_6 is a digraph obtained by adding an arc (v_n, v_{n-1}) and arc(s) (v_{i+1}, v_i) where $i \in \{1, 2, \ldots, n-5, n-4, n-2\}$, then $det(B(D_6)) = 0$. If D_7 is a digraph obtained by adding arcs (v_n, v_{n-1}) and (v_{n-2}, v_{n-3}), then $det(B(D_7)) = 1$.

Proof:

Let D be an arborescence directed path graph with the set vertices $V(D) = \{v_1, v_2, \ldots, v_n\}$ where $n \geq 2$; and $B(D) = [b_{i,j}]$ where $i, j \in \{1, 2, \ldots, n\}$; be its antiadjacency matrix.

(i) Let D_1 be a digraph obtained by adding arc(s) (v_{i+1}, v_i) in D, where $i \in \{2, 3, \ldots, n-2\}$ and $n \geq 4$. Moreover, D_1 has digon(s) $\{(v_i, v_{i+1}); (v_{i+1}, v_i)\}$. Then, in its antiadjacency matrix $B(D_1)$, we can subtract the k-th row from the first row where $k \in \{2, 3, \ldots, n\}$, use the cofactor expansion along the n-th row, and subtract the i-th row from $(i-2)$-th row where $i \in \{3, 4, \ldots, n-1\}$. The result show that $\det(B(D_1)) = 1$.

(ii) To prove this, we need to dissect this into three cases: D_2 is a digraph obtained by adding an arc (v_2, v_1); D_3 is a digraph obtained by adding an arc (v_n, v_{n-1}); and D_4 is a digraph obtained by adding arcs (v_2, v_1) and (v_n, v_{n-1}).

a) Let D_2 be a digraph obtained by adding an arc (v_2, v_1) where $n \geq 3$. Moreover, D_2 has a digon $\{(v_1, v_2);\ (v_2, v_1)\}$. Then, in its antiadjacency matrix $B(D_2)$, we can subtract the k-th row from the first row where $k \in \{3, 4, \ldots, n\}$ and use the cofactor expansion along the n-th row to obtain $\det\big(B(D_2)\big) = 0$.

b) Let D_2 be a digraph obtained by adding an arc (v_n, v_{n-1}) where $n \in \{3, 5, 6, \ldots\}$. Moreover, D_3 has a digon $\{(v_{n-1}, v_n);\ (v_n, v_{n-1})\}$. Since in its antiadjacency matrix $B(D_3)$, the n-th row is the same as the $(n-2)$-th row, then we have $det(B(D_3)) = 0$.

c) For this case, we must differ the value of n into $n = 3, n = 4$, and $n \geq 5$.

- Let D_3 be a digraph that has an order $n = 3$, which obtained by adding arcs (v_2, v_1) and (v_3, v_2) in D.Moreover, D_3 has digons $\{(v_1, v_2);\ (v_2, v_1)\}$ and $\{(v_2, v_3);\ (v_3, v_2)\}$. Since in its antiadjacency matrix $B(D_3)$, the third row is the same as the first row, then we have $\det\big(B(D_3)\big) = 0$.

- Let D_3 be a digraph that has an order $n = 4$, which obtained by adding arcs (v_2, v_1) and (v_4, v_3) in D. Moreover, D_3 has digons $\{(v_1, v_2);\ (v_2, v_1)\}$ and $\{(v_4, v_3);\ (v_3, v_4)\}$. Then, in its antiadjacency matrix $B(D_5)$, we can subtract the third row from the first and the second row and use cofactor expansion along the third row to obtain $\det(B(D_3)) = -2$.

- Let D_3 be a digraph that has an order $n \geq 5$, which obtained by adding arcs (v_2, v_1) and (v_n, v_{n-1}) in D. Moreover, D_3 has digons $\{(v_1, v_2);\ (v_2, v_1)\}$ and $\{(v_n, v_{n-1});\ (v_{n-1}, v_n)\}$. Since in its antiadjacency matrix $B(D_3)$, the n-th row is the same as the $(n-2)$-th row, then we have $\det(B(D_6)) = 0$.

(iii) To prove this, we need to dissect it into two cases: the following.

a) Let D_4 be a digraph that has an order $n \geq 4$, which obtained by adding an arc (v_2, v_1) and arc(s) (v_{i+1}, v_i) in D, where $i \in \{2, 3, 5, \ldots, n-1\}$. Moreover, D_4 has digons $\{(v_1, v_2); (v_2, v_1)\}$ and $\{(v_i, v_{i+1}); (v_{i+1}, v_i)\}$ where $i \in \{3, 5, \ldots, n-2\}$. Then, the transpose of its antiadjacency matrix $B(D_7)^T$ will have at least two same rows: first and third rows. Consequently, we have $\det\big(B(D_4)\big) = 0$.

b) Let D_5 be a digraph that has order $n \geq 4$, which obtained by adding an arc (v_2, v_1) and an arc (v_4, v_3) in D. Moreover, D_5 has digons $\{(v_1, v_2); (v_2, v_1)\}$ and $\{(v_3, v_4); (v_4, v_3)\}$. Then, in its antiadjacency matrix $B(D_5)$, we can subtract the k-th row from the first row where $k \in \{3,4,5,\dots,n\}$, use cofactor expansion along the n-th row, sum the fourth row from the second row, sum the second row from the third row, interchange the second row with the fourth row, to obtain $\det\big(B(D_5)\big) = 1$.

(iv) To prove this, we need to dissect it into two cases: the following.

a) Let D_6 be a digraph that has an order $n \geq 4$, which obtained by adding an arc (v_n, v_{n-1}) and arc(s) (v_{i+1}, v_i) in D, where $i \in \{1,2,\dots,n-5,n-4,n-2\}$. Moreover, D_6 has digons $\{(v_{n-1},v_n);(v_n,v_{n-1})\}$ and $\{(v_i,v_{i+1});(v_{i+1},v_i)\}$ where $i \in \{1,2,\dots,n-5,n-4,n-2\}$. Then, either its antiadjacency matrix $B(D_6)$ will have two same rows, that is, the n-th row and the $(n-2)$-th row. Therefore, we have $\det\big(B(D_6)\big) = 0$.

b) Let D_7 be a digraph that has an order $n \geq 4$, which obtained by adding arcs (v_n, v_{n-1}) and (v_{n-2}, v_{n-3}) in D. Moreover, D_7 has digons $\{(v_{n-1},v_n);(v_n,v_{n-1})\}$ and $\{(v_{n-3},v_{n-2});(v_{n-2},v_{n-3})\}$. Then, we can use the similar method as in (iii) to obtain $\det(B(D_7)) = 1$. ■

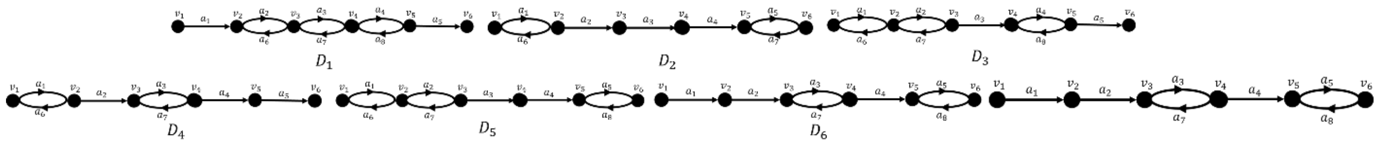

Figure 1. The digraphs $\boldsymbol{D_1}$, $\boldsymbol{D_2}$, $\boldsymbol{D_3}$, $\boldsymbol{D_4}$, $\boldsymbol{D_5}$, $\boldsymbol{D_6}$, $\boldsymbol{D_7}$

From Theorem 2.1 and Proposition 3.1, we have the following Corollary 3.2. Corollary 3.2 shows that the multiplication of all the eigenvalues of the antiadjacency matrix of an arborescence path digraph with digon(s).

Corollary 3.2. *Let D be an arborescence path digraph with the set of vertices $V(D) = \{v_1, v_2, \dots, v_n\}$ where $n \geq 2$, and $B(D)$ be its antiadjacency matrix. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $B(D)$.*

(i) If D_1 is a digraph obtained by adding arc(s) (v_{i+1}, v_i) in D, where $i \in \{2,3,\dots,n-2\}$ and $n \geq 4$, then $\lambda_1\lambda_2 \dots \lambda_n = 1$.

(ii) If D_2 is a digraph obtained by adding either an arc (v_2, v_1) or an arc (v_n, v_{n-1}) in D, where $n \geq 3$, then $\lambda_1\lambda_2 \dots \lambda_n = 0$. Moreover, if D_3 is a digraph obtained by adding arcs (v_2, v_1) and (v_n, v_{n-1}) in D, where $n \in \{3,5,6,\dots\}$, then $\lambda_1\lambda_2 \dots \lambda_n = 0$. On the other hand, if D_3 is a digraph that has an order $four$ and is obtained by adding arcs (v_2, v_1) and (v_4, v_3), then $\lambda_1\lambda_2 \dots \lambda_n = -2$.

(iii) Let $n \geq 4$. If D_4 is a digraph obtained by adding an arc (v_2, v_1) and arc(s) (v_{i+1}, v_i) where $i \in \{2,3,5,\dots,n-1\}$, then $\lambda_1\lambda_2 \dots \lambda_n = 0$. If D_5 is added an arc (v_2, v_1) and an arc (v_4, v_3), then $\lambda_1\lambda_2 \dots \lambda_n = 1$.

(iv) Let $n \geq 4$. If D_6 is a digraph obtained by adding an arc (v_n, v_{n-1}) and arc(s) (v_{i+1}, v_i) where $i \in \{1,2,\dots,n-5,n-4,n-2\}$, then $\lambda_1\lambda_2 \dots \lambda_n = 0$. If D_7 is a digraph obtained by adding arcs (v_n, v_{n-1}) and (v_{n-2}, v_{n-3}), then $\lambda_1\lambda_2 \dots \lambda_n = 1$.

Proof:

Let D be an arborescence directed path graph with the set vertices $V(D) = \{v_1, v_2, \dots, v_n\}$ where $n \geq 2$; and $B(D) = [b_{i,j}]$ where $i,j \in \{1,2,\dots,n\}$; be its antiadjacency matrix. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $B(D)$.

(i) Let D_1 be a digraph obtained by adding arc(s) (v_{i+1}, v_i) in D, where $i \in \{2,3,\dots,n-2\}$ and $n \geq 4$. Moreover, D_1 has digon(s) $\{(v_i,v_{i+1});(v_{i+1},v_i)\}$. Then, from Theorem 2.1 and Proposition 3.1, we have $\lambda_1\lambda_2 \dots \lambda_n = 1$.

(ii) To prove this, we need to dissect this into three cases: D is added an arc (v_2, v_1); D is added an arc (v_n, v_{n-1}); and D is added arcs (v_2, v_1) and (v_n, v_{n-1}).

a) Let D_2 be a digraph obtained by adding an arc (v_2, v_1) where $n \geq 3$. Moreover, D_2 has a digon

$\{(v_1, v_2);\ (v_2, v_1)\}$. Then, from Theorem 2.1 and Proposition 3.1, we have $\lambda_1\lambda_2 \dots \lambda_n = 0$.

b) Let D_2 be a digraph obtained by adding an arc (v_n, v_{n-1}) where $n \in \{3, 5, 6, \dots\}$. Moreover, D_3 has a digon $\{(v_{n-1}, v_n);\ (v_n, v_{n-1})\}$. Then from Theorem 2.1 and Proposition 3.1, we have $\lambda_1\lambda_2 \dots \lambda_n = 0$.

c) For this case, we must differ the value of n into $n = 3, n = 4$, and $n \geq 5$.

- Let D_3 be a digraph that has an order $n = 3$, which obtained by adding arcs (v_2, v_1) and (v_3, v_2) in D.Moreover, D_3 has digons $\{(v_1, v_2);\ (v_2, v_1)\}$ and $\{(v_2, v_3);\ (v_3, v_2)\}$. Then, from Theorem 2.1 and Proposition 3.1, we have $\lambda_1\lambda_2 \dots \lambda_n = 0$.

- Let D_3 be a digraph that has an order $n = 4$, which obtained by adding arcs (v_2, v_1) and (v_4, v_3) in D. Moreover, D_3 has digons $\{(v_1, v_2);\ (v_2, v_1)\}$ and $\{(v_4, v_3);\ (v_3, v_4)\}$. Then, from Theorem 2.1 and Proposition 3.1, we have $\lambda_1\lambda_2 \dots \lambda_n = -2$.

- Let D_3 be a digraph that has an order $n \geq 5$, which obtained by adding arcs (v_2, v_1) and (v_n, v_{n-1}) in D. Moreover, D_3 has digons $\{(v_1, v_2);\ (v_2, v_1)\}$ and $\{(v_n, v_{n-1});\ (v_{n-1}, v_n)\}$. Then, from Theorem 2.1 and Proposition 3.1, we have $\lambda_1\lambda_2 \dots \lambda_n = 0$.

(iii) To prove this, we need to dissect it into two cases, the following.

a) Let D_4 be a digraph that has an order $n \geq 4$, which obtained by adding an arc (v_2, v_1) and arc(s) (v_{i+1}, v_i) in D, where $i \in \{2, 3, 5, \dots, n-1\}$. Moreover, D_4 has digons $\{(v_1, v_2); (v_2, v_1)\}$ and $\{(v_i, v_{i+1}); (v_{i+1}, v_i)\}$ where $i \in \{3, 5, \dots, n-2\}$. Then, from Theorem 2.1 and Proposition 3.1, we have $\lambda_1\lambda_2 \dots \lambda_n = 0$.

b) Let D_5 be a digraph that has order $n \geq 4$, which obtained by adding an arc (v_2, v_1) and an arc (v_4, v_3) in D. Moreover, D_5 has digons $\{(v_1, v_2); (v_2, v_1)\}$ and $\{(v_3, v_4); (v_4, v_3)\}$. Then, from Theorem 2.1 and Proposition 3.1, we have $\lambda_1\lambda_2 \dots \lambda_n = 1$.

(iv) To prove this, we need to dissect it into two cases, the following.

a) Let D_6 be a digraph that has an order $n \geq 4$, which obtained by adding an arc (v_n, v_{n-1}) and arc(s) (v_{i+1}, v_i) in D, where $i \in \{1, 2, \dots, n-5, n-4, n-2\}$. Moreover, D_6 has digons $\{(v_{n-1}, v_n); (v_n, v_{n-1})\}$ and $\{(v_i, v_{i+1}); (v_{i+1}, v_i)\}$ where $i \in \{1, 2, \dots, n-5, n-4, n-2\}$. Then, from Theorem 2.1 and Proposition 3.1, we have $\lambda_1\lambda_2 \dots \lambda_n = 0$.

b) Let D_{10} be a digraph that has an order $n \geq 4$, which obtained by adding arcs (v_n, v_{n-1}) and (v_{n-2}, v_{n-3}) in D. Moreover, D_{10} has digons $\{(v_{n-1}, v_n); (v_n, v_{n-1})\}$ and $\{(v_{n-3}, v_{n-2}); (v_{n-2}, v_{n-3})\}$. Then, from Theorem 2.1 and Proposition 3.1, we have $\lambda_1\lambda_2 \dots \lambda_n = 1$. ■

We will show the result on the determinant of the antiadjacency matrix of a complete star digraph that deleted several arcs such that at least contained one digon.

Proposition 3.3. *Let $S_{n,k}$ be a digraph which obtained by deleting k-arcs (v_j, v_1) in a complete star digraph where $j \in \{2, 3, \dots, n\}$ and $0 \leq k \leq n-1$. Moreover, let $B(S_{n,k})$ be the antiadjacency matrix of $S_{n,k}$. Then, $det(B(S_{n,k})) = 0$.*

Proof:

Let $S_{n,k}$ be a digraph which obtained by deleting k-arcs (v_j, v_1) in a complete star digraph where $j \in \{2, 3, \dots, n\}$ and $0 \leq k \leq n-1$. Moreover, let $B(S_{n,k})$ be the antiadjacency matrix of $S_{n,k}$. To prove this, we need to dissect it into two cases. Those are the following.

i. If $k = 0$, then there will be $(n-1)$-vertices that have the same out-neighborhood vertices. Therefore, $B(S_{n,k})$ will have the $(n-1)$ same rows, which will imply $\det\left(B(S_{n,k})\right) = 0$.

ii. If $1 \leq k \leq n-1$, then there will be at least one vertex that has an empty out-neighborhood vertices. Consequently, $B(S_{n,k})$ will have a zero-row, which will imply $\det(B(S_{n,k})) = 0$.■

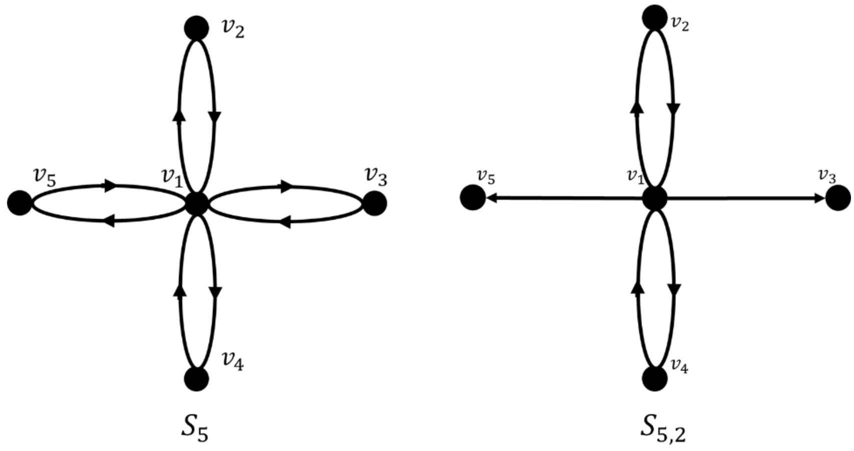

Figure 2. The Digraph $\boldsymbol{S_5}$ and $\boldsymbol{S_{5,2}}$

3.2. The characteristic polynomial of the antiadjacency matrix

We present our results concerning the characteristic polynomial of the antiadjacency matrix of the arborescence path digraph with digon(s), in Proposition 3.4.

Proposition 3.4. *Let D be an arborescence path digraph with the set of vertices $V(D) = \{v_1, v_2, \dots, v_n\}$ where $n \geq 3$, and $B(D)$ be its antiadjacency matrix. Let D_1, D_2, and D_3 respectively be a digraph which obtained by adding arc(s) (v_2, v_1), (v_{i+1}, v_i), and (v_n, v_{n-1}) in D, where $i \in \{2, 3, \dots, n-2\}$; and $n \geq 3$. Let $P(\lambda; B(D_1)) = \lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j}$, $P(\lambda; B(D_2)) = \lambda^n + \sum_{j=1}^{n} b_j \lambda^{n-j}$, and $P(\lambda; B(D_3)) = \lambda^n + \sum_{j=1}^{n} c_j \lambda^{n-j}$, respectively, be the characteristic polynomial of $B(D_1)$, $B(D_2)$, and $B(D_3)$. Then, we have the following properties.*

(i) *Let $j \in \{1, 2, \dots, n\}$. Then $|a_j|$ is equal to the number of directed arborescence paths involving j-vertices.*

(ii) *Let $j \in \{1, 2, \dots, n\}$. Then $|b_j|$ is equal to the number of directed arborescence paths involving j-vertices and digraphs that are obtained by adding arcs such that they contain digons involving j-vertices.*

(iii) *Let $j \in \{1, 2, \dots, n\}$. Then $|c_j|$ is equal to the number of directed arborescence paths involving j-vertices.*

Proof:

Let D be an arborescence path digraph with the set of vertices $V(D) = \{v_1, v_2, \dots, v_n\}$ where $n \geq 2$.

(i) Let D_1 be a digraph obtained by adding arc(s) (v_2, v_1) in D, $B(D_1)$ be its antiadjacency matrix. and $P(\lambda; B(D_1)) = \lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j}$ be the characteristic polynomial of $B(D_1)$. Using elementary methods of finding the principal minor, we have $|a_j| = \sum(\text{ all } j \times j \text{ principal minors of } B(D_1))$, where $j \in \{1, 2, \dots, n\}$. We know that all the $i \times i$ principal minors of $B(D_1)$ are the determinant of the antiadjacency matrix of the induced subdigraphs of D_1. Since D_1 is a digraph obtained by adding arc(s) (v_2, v_1) in D, then the induced subdigraphs of D_1 can be oriented and unoriented. The oriented subdigraphs of D_1 can be weakly connected or unconnected. The unconnected oriented or unoriented induced-subdigraphs of D_1 may have at least one isolated vertex, which implies its antiadjacency matrix has zero rows, and the determinant of its antiadjacency matrix is equal to zero. The weakly connected induced-subdigraphs of D_1 are arborescence path digraphs. Then, according to Theorem 2.2, we have the determinant of the For oriented induced-subdigraphs of D_1. According to Theorem 2.2, we have all determinants of the antiadjacency matrices of those induced-subdigraphs are equal to 1. Meanwhile, according to Proposition 3.1, we have all determinants of the weakly connected and unoriented induced-subdigraphs of D_1 are equal to zero. Therefore, we have $|a_j|$ is equal to the number of directed arborescence paths involving j-vertices, where $j \in \{1, 2, \dots, n\}$.

(ii) Let D_2 be a digraph obtained by adding arc(s) (v_{i+1}, v_i) where $i \in \{2, 3, \dots, n-2\}$ in D, $B(D_2)$ be its antiadjacency matrix. and $P(\lambda; B(D_2)) = \lambda^n + \sum_{j=1}^{n} b_j \lambda^{n-j}$ be the characteristic polynomial of $B(D_2)$. Using elementary methods of finding the principal minor, we have $|b_j| = \sum(\text{ all } j \times j \text{ principal minors of } B(D_1))$ where $j \in \{1, 2, \dots, n\}$. We know that all the $i \times i$ principal minors of $B(D_2)$ are the determinant of the antiadjacency matrix of the induced subdigraphs of D_2. Since D_2 is a digraph obtained by adding arc(s) (v_{i+1}, v_i) where $i \in \{2, 3, \dots, n-2\}$ in D, then the induced subdigraphs of D_1 can be oriented and unoriented. The oriented subdigraphs of D_1 can be weakly connected or unconnected. The unconnected oriented or unoriented induced-subdigraphs of D_1 may have at least one isolated vertex, which implies its antiadjacency matrix has zero rows, and the determinant of its antiadjacency matrix is equal to zero. The weakly connected induced-subdigraphs of D_1 are arborescence path digraphs. Then, according to Theorem 2.2, we have the determinant of the For oriented induced-subdigraphs of D_1. According to Theorem 2.2, we have all determinants of antiadjacency matrices of those induced-subdigraphs are equal to 1. Meanwhile, according to Proposition 3.1, we have all determinants of the weakly connected and unoriented induced-subdigraphs of D_1 are equal to 1. Therefore, we have Then $|b_j|$ is equal to the number of directed arborescence paths involving j-vertices and digraphs obtained by adding arcs such that they contain digons involving j-vertices, where $j \in \{1, 2, \dots, n\}$.

(iii) Let D_3 be a digraph obtained by adding arc(s) (v_n, v_{n-1}) in D, $B(D_3)$ be its antiadjacency matrix. and $P(\lambda; B(D_3)) = \lambda^n + \sum_{j=1}^{n} c_j \lambda^{n-j}$ be the characteristic polynomial of $B(D_3)$. Using elementary methods of finding the principal minor, we have $|c_j| = \sum(\text{ all } j \times j \text{ principal minors of } B(D_3))$ where $j \in \{1, 2, \dots, n\}$. We know that all the $i \times i$ principal minors of $B(D_3)$ are the determinant of the antiadjacency matrix of the induced subdigraphs of D_3. Since D_3 is a digraph obtained by adding arc(s) (v_n, v_{n-1}) in D, then the induced subdigraphs of D_3 can be oriented and unoriented. The oriented subdigraphs of D_3 can be weakly connected or unconnected. The unconnected oriented or unoriented induced-subdigraphs of D_3 may have at least one isolated vertex, which implies its antiadjacency matrix has zero rows, and the determinant of its antiadjacency matrix is equal to zero. The weakly connected induced-subdigraphs of D_3 are arborescence path digraphs. Then, according to Theorem 2.2, we have the determinant of the For oriented induced-subdigraphs of D_3According to Theorem 2.2, we have all determinants of antiadjacency matrices of those induced-subdigraphs are equal to 1. Meanwhile, according to Proposition 3.1, we have all determinants of the weakly connected and unoriented induced-subdigraphs of D_3 are equal to zero. Therefore, we have $|c_j|$ is equal to the number of directed arborescence paths involving j-vertices, where $j \in \{1, 2, \dots, n\}$. ■

Let $S_{n,k}$ be a digraph which obtained by deleting k-arcs (v_j, v_1) in a complete star digraph where $j \in \{2, 3, \dots, n\}$ and $0 \le k \le n-1$. Let $B(S_{n,k})$ be the antiadjacency matrix of $S_{n,k}$. We put the characteristic polynomial of $B(S_{n,k})$ in Problem 3.5.

Problem 3.5. Let $S_{n,k}$ be a digraph that is obtained by deleting k-arcs $(v_j, v_1)$of a complete star digraph where $j \in \{2, 3, \dots, n\}$ and $0 \le k \le n-1$. Moreover, let $B(S_{n,k})$ be the antiadjacency matrix of $S_{n,k}$. Is there any specific properties of coefficients of the characteristic polynomial of $B(S_{n,k})$?

4. CONCLUSIONS

In this paper, we have found several results on the determinant and the characteristic polynomial of the antiadjacency matrix of a digraph obtained by adding arc(s) in the arborescence path digraph. Table 1 shows the determinant matrix of the antiadjacency of each digraph that added arc(s) in arborescence path digraph such that it has digon(s). We also have found the determinant of the antiadjacency matrix of a digraph obtained by deleting k-arcs (v_j, v_1) in a complete star digraph where $j \in \{2, 3, \dots, n\}$ and $0 \le k \le n-1$. is equal to zero. Meanwhile, the coefficient of that digraph's characteristic polynomial, we leave it as Problem 3.5.

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