BAREKENG: Jurnal Ilmu Matematika dan Terapan
June 2022 Volume 16 Issue 2 Page 507–514
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol16iss2pp507–514

SIMPLIFIED FORMULAS FOR SOME BESSEL FUNCTIONS AND THEIR APPLICATIONS IN EXTENDED SURFACE HEAT TRANSFER

Irvan[1], Zahedi[2*], Anjar Agus[3], Sarmin Suparni[4], Harahap Amin[5]

[1]*Mathematics Education, Graduate program, University of Muhammadiyah Sumatera Utara*
Denai St., No.217, Tegal Sari Mandala II, Kec. Medan Denai, Kota Medan, 20371, Indonesia
[2] *Mathematics Graduate Program, Faculty of Mathematics and Natural Science, Universitas Sumatera Utara,*
Dr. T. Mansyur St., No. 9, Medan, Indonesia.
[3]*Mathematics Education, Under graduate program, FKIP Universitas Labuhan Batu,*
Jln. SM. Raja No. 126, Aek Tapa, Rantau Prapat, Kab. Labuhan Batu, Sumatera Utara, Indonesia
[4] *Mathematics Education, Under graduate program, IAIN Padangsidimpuan,*
T. Rizal Nurdin St., Km 4.5 Sihitang, Padangsidimpuan, Indonesia
[5] *Mathematics Education, Under graduate program, FKIP Universitas Labuhan Batu,*
SM. Raja St., No. 126, Aek Tapa, Rantau Prapat, Kab. Labuhan Batu, Sumatera Utara, Indonesia

Corresponding author's e-mail: [2]*zahedi@usu.ac.id

Abstract. *Bessel functions find many applications in Physics and Engineering fields. Some of these applications are in the analysis of extended surface heat transfer where the cross-sections vary. Tables of various kinds of Bessel functions are available in most handbooks of mathematics. However, the use of tables is not always convenient, particularly for applications where many values must be computed. In the applications of Bessel functions in extended surface heat transfer, graphs are also available to provide quick evaluations of the values needed. However, reading these graphs always needs interpolation; this will be cumbersome and time-consuming if there are many readings to be taken. Mathematical formulas for Bessel functions are available but they are usually complicated. Software to calculate values of Bessel functions is also available. Excel, Maple, and Mathematica can also be used to compute the values of Bessel functions. A user can write a program for an application that involves Bessel functions. However, the use of Bessel functions in Excel is limited while Maple and Mathematica are expensive commercial software. In this paper, formulas for Bessel functions of $I_0(x)$ and $I_1(x)$ are simplified with adequate accuracy that can be used to easily compute values needed in the extended surface heat transfer analysis. It is found that errors for $I_0(x)$ and $I_1(x)$ are relatively small (maximum errors are 0.004% and 0.003%, respectively) in the range of 0.05 to 3.75 while the maximum error for $I_2(x)$ is 3.678% for the same range. However, the maximum error for $I_2(x)$ is reduced to 0.166 if the range is from 0.25 to 3.75.*

Keywords: *approximation functions, Bessel functions, extended surfaces, simplified formula*

Article info:

Submitted: 9th February 2022 *Accepted: 20th April 2022*

How to cite this article:
Irvan, Zahedi, A. Agus, S. Suparni and H. Amin, "SIMPLIFIED FORMULAS FOR SOME BESSEL FUNCTIONS AND THEIR APPLICATIONS IN EXTENDED SURFACE HEAT TRANSFER", *BAREKENG: J. Il. Mat. & Ter.*, vol. 16, iss. 2, pp. 507-514, June, 2022.

1. INTRODUCTION

Bessel functions are special functions which have many applications in engineering and science. The applications include problem solving in stationary problems in quantum mechanics to those of spherical and cylindrical wave propagation [1], transversal motion of a circular membrane [4], thermal stress and elasticity and plasticity [8], [14] and electromagnetics [11]. In heat transfer, applications of Bessel functions include conduction problems and extended surface heat transfer with variable cross-sections such as cones and hyperbolic; see [6], [9], [12], [13],[20], [21], for example. A rather short but useful exploration of the use of Bessel functions in the fin analysis is given by [16]. Various authors have analyzed heat conduction using Bessel functions; see [12], for example. Analysis of some fluid problems using which involves Bessel functions is given by [21].

A body or a structure at a high temperature must have its heat removed continuously due to different purposes such as for reducing thermal stress of the material at high temperature and heating a cooler fluid in a heat exchanger. Extended surfaces (or fins) are commonly utilized to increase the heat removal between a structure and a surrounding ambient fluid. Here, they are attached to the primary surface [12], [13], [16]. The primary surface is one at high temperature where the heat will be removed to the surrounding fluid through the fins by convection. There are many various shapes of fins and different arrangements employed in engineering applications such as longitudinal fins, traverse fins, spine- or stud-type fins. The type of fin chosen and its arrangement will affect the heat transfer performance, resistance to the flow of the surrounding fluid, cost of materials and the ease of fabrication [13]. Most fins have a uniform cross-section such as rectangular. The cross-section can also vary such as trapezoidal with rounded edges. Analysis for variable cross-sections is more difficult compared to that of uniformed cross-sections.

2. RESEARCH METHODS

2.1 Conduction−Convection Systems

An example of extended surfaces can be seen in a motor cycle; see Figure 1. The hot engine is cooled by air using an array of fins. If the heat is not removed, the engine temperature will be very high and the pistons will be stuck in the cylinder, making it damaged.

Figure 1. an air-cooled engine
(Source: https://faculty.virginia.edu/ribando/modules/ExtendedSurface/).

A simple extended surface with a uniform cross section area is shown in Figure 2. The extended surface which protrudes from a wall (called the base) will release heat to the surrounding fluid (which can be a gas or a liquid). Here, the heat is conducted by some convection process. Since the heat is conducted through the body and finally released or dissipated to the surrounding by convection, the combination is called conduction-convection systems. Heat transfer in this system is assumed to be one dimension (1-D) where exact solution can be sought. If system is 2-D or 3-D, we must resort to numerical solution. Moreover, if the surrounding is a gas or air, radiation may involve; this makes the analysis quite very complicated. In this paper, only conduction-convection system is treated.

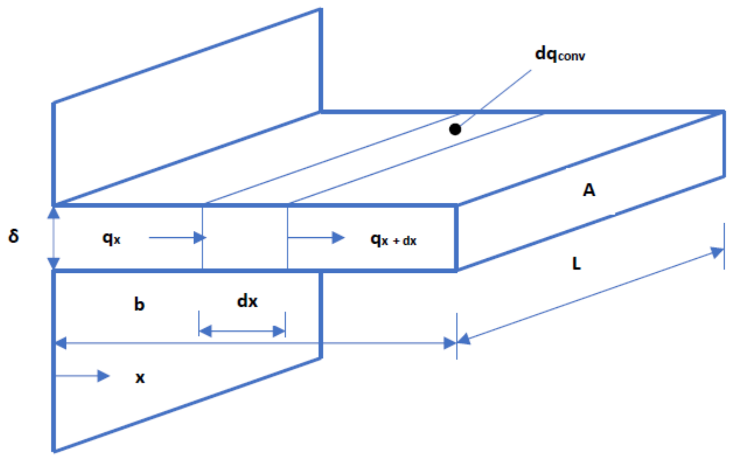

Figure 2. 1-D conduction and convection through a rectangular fin

The differential equation for the energy balance is given by

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_s) = 0 \tag{1}$$

Here, A is cross-sectional area and P is perimeter of the fin. Other variables are described in Section 3.1.

Let $\theta = T - T_s$. Then Equation (1) becomes

$$\frac{d^2\theta}{dx^2} - \frac{hP}{kA}\theta = 0 \tag{2}$$

Let $m^2 = hP/kA$. Then Equation (2) becomes

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{3}$$

The general solution for Equation (3) is given by

$$\theta(x) = C_1e^{-mx} + C_2e^{mx} \tag{4}$$

Constants C_1 and C_2 can be found from the boundary conditions at the base and at the fin tip. At the base or $x = 0$, $T = T_b$. Here, $\theta(0) = T_b - T_s = \theta_b$. At the fin tip, there are four possible cases. Interested readers are referred to textbooks in heat transfer such as [9], [13] and [16]. However, we will not discuss them because we are interested in the extended heat transfer in which the cross-sectional areas are not uniform.

There are two situations where fin cross-section areas are not uniform. First when we need to save material costs by reducing cross section area in the direction of the conduction. Second when the fin is attached to a circular tube, forming an annulus fin. Here, the heat conduction through the fin is still one dimension but with variable cross-sectional area. See [9], [12], [16], [20], [21] for various shapes of fins with variable cross-sectional area. When the cross-sectional area varies along the thickness of the fin, it is found that the solution for the temperature distribution along the fin (or θ(x)) involves Bessel functions, which are the topic of this paper.

2.2 A Short Intoduction to Bessel Functions

Bessel functions come from the solution of second order differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{5}$$

Equation (5) is called Bessel equation of order n. If n is not an integer, the solution is given by

$$y = C_1J_n(x) + C_2J_{-n}(x) \tag{6}$$

and when n is an integer,

$$y = C_1J_n(x) + C_2Y_n(x) \tag{7}$$

C_1 and C_2 are constants while $J_n(x)$ is the Bessel function of the first kind, of order n and argument x, and $Y_n(x)$ is the Bessel function of the second kind, of order n and argument x.

A modified Bessel equation which resembles Equation (5) is given by

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+n^2)y=0 \tag{8}$$

If n is not an integer, the solution is given by

$$y=C_1I_n(x)+C_2I_{-n}(x) \tag{9}$$

and when n is an integer,

$$y=C_1I_n(x)+C_2K_n(x) \tag{10}$$

Here, $I_n(x)$ the modified Bessel function of the first kind, of order n and argument x, and $K_n(x)$ is the modified Bessel function of the second kind, of order n and argument x.

A general formula of Bessel's equation is given by

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda^2x^2-n^2)y=0 \tag{11}$$

which has the general solution

$$y=C_1J_n(\lambda x)+C_2Y_n(\lambda x) \tag{12}$$

Similarly, a general formula of modified Bessel's equation is given by

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(\lambda^2x^2+n^2)y=0 \tag{13}$$

which has the general solution

$$y=C_1I_n(\lambda x)+C_2K_n(\lambda x) \tag{14}$$

Equations (11) and (13) arise in many applications; examples will be given in the next section. Complete treatments of Bessel functions are given by [2] and [18]. Various relationships involving Bessel functions are given in [3], [17]. Readable analysis of Bessel functions as applied in extended surface heat transfer are presented in [13].

Solutions of Bessel functions are given in infinite series, which may not be convenient for applications which need quick solutions. Tables of Bessel functions which list values of arguments are therefore preferable. Short tables of Bessel functions are given by [10], [13].

We can also use software to compute Bessel functions for arbitrary order. Here, we can use Maple, Excel or Scilab. Scilab is very useful for evaluating Bessel functions of arbitrary orders. Moreover, the software can be freely downloaded; see [18] which has also many mathematical functions apart from Bessel functions. Excel can also compute Bessel functions but the orders are limited; they are just integer. Maple and Mathematica have excellent libraries to compute Bessel functions. However, Maple and Mathematica are commercial software which are quite expensive.

3 RESULTS AND DISCUSSION

3.1 Applications of Bessel Functions in Extended Surface Heat Transfer

Since there are almost unlimited configurations of extended surfaces (or often called fins), we will only take two examples which can be analyzed using Bessel functions; other examples will be presented in different paper. All formulas are given without derivation. Interested readers can see the derivations in most textbooks in mathematical physics and higher engineering mathematics [3-4][17][19]. Figure 3 shows a straight fin with triangular cross section or profile. Here,

b = height of the fin
δ_b = thickness of the fin
L = width of the fin

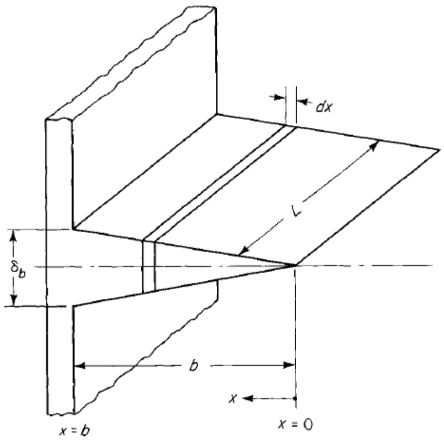

Figure 3. a straight fin with triangular profile

We will seek temperature of the fin at a distance x from the tip, $T(x)$; look at the coordinate of the system. Let T_S be the surrounding temperature. We will then work with the temperature excess defined by $\theta(x) = T(x) - T_S$. Without derivation, the governing differential equation for the temperature excess is given by

$$x\frac{d^2\theta}{dx^2} + \frac{d\theta}{dx} - m^2 b\theta = 0 \quad (15)$$

where

$$m = (2h/k\delta)^{1/2} \quad (16)$$

Here, h: heat-transfer coefficient (W/m^2.°C, k: conductivity of the fin (W/m.°C), b: the height of the fin (m) and δ: the thickness of the fin (m).

The general solution of Equation (15) is given by

$$\theta(x) = C_1 I_0\left(2m\sqrt{bx}\right) + C_2 K_0\left(2m\sqrt{bx}\right) \quad (17)$$

Since temperature excess at the tip ($x = 0$) is finite, C_2 must be zero since $K_0(0)$ is unbounded. So, we have

$$\theta(x) = C_1 I_0\left(2m\sqrt{bx}\right) \quad (18)$$

Initial condition: at $x = b,\ \theta = \theta_b$. So, C_1 is found and substituting C_1 back to Equation (18) yields

$$\theta(x) = \frac{\theta_b I_0\left(2m\sqrt{bx}\right)}{I_0(2mb)} \quad (19)$$

The heat dissipated by the fin is given by

$$q_b = \frac{kAdT}{dx}\bigg|_{x=b} = \frac{2hL\theta_b I_1(2mb)}{mI_0(2mb)} \quad (20)$$

while efficiency of the fin is given by

$$\eta = \frac{I_1(2mb)}{mbI_0(2mb)} \quad (21)$$

As the second example, consider a conical spine as shown in Figure 4.

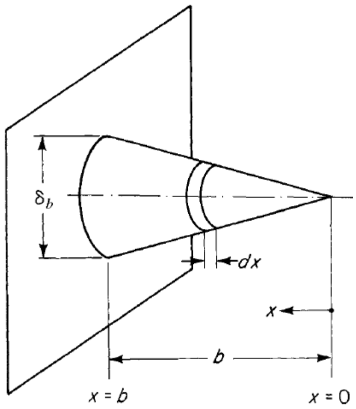

Figure 4. a conical spine

Without derivation, the governing differential equation for the temperature excess, $\theta(x) = T(x) - T_s$, is given by

$$x^2 \frac{d^2\theta}{dx^2} + 2x\frac{d\theta}{dx} - M^2 x\theta = 0 \tag{22}$$

where

$$M = (2m^2 b)^{1/2} \tag{23}$$

and M is given by Equation (16).

$$\theta(x) = x^{-1/2}\left[C_1 I_1\left(2M\sqrt{x}\right) + C_2 K_1\left(2M\sqrt{x}\right)\right] \tag{24}$$

Since temperature excess at the tip i.e. $x = 0$ is finite, C_2 must be zero since $K_1\left(2M\sqrt{x}\right)/\sqrt{x}$ is unbounded at $x = 0$. Initial condition: at $x = 0$, $\theta = \theta b$. So, C_1 is found and substituting C_1 back to Equation (18) yields

$$\theta(x) = \theta_b \left(\frac{b}{x}\right)^{\frac{1}{2}} \frac{I_1\left(2M\sqrt{x}\right)}{I_1\left(2M\sqrt{b}\right)} \tag{25}$$

The heat flow through the base is given by

$$q_b = \frac{\pi k \delta_b^2 \theta_b M}{4\sqrt{b}} \frac{I_2\left(2M\sqrt{b}\right)}{I_1\left(2M\sqrt{b}\right)} \tag{26}$$

while efficiency of the fin is given by

$$\eta = \frac{q_b}{q_{id}} = \frac{\sqrt{2} I_2\left(2\sqrt{2}mb\right)}{(mb) I_1\left(2\sqrt{2}mb\right)} \tag{27}$$

3.2 Approximations of Bessel Functions

Values of Bessel functions $J_n(x), Y_n(x), I_n(x)$ and $K_n(x)$ for $n = \{0,1\}$ can be computed using approximated polynomials given by [2][5][15]. Bessel functions for other orders can be computed using recurrence relations. We will only list approximation functions for I_0(x) and I_1(x)which are relevant to extended surfaces described in the previous section.

Let u be $\frac{x}{3.75}$ and v be $\frac{1}{u}$ (or $\frac{3.75}{x}$). We then have the following approximation functions for I_0(x) and I_1(x)

$$I_0(x) = \sum_{i=0}^{6} a_i u^{2i} + \varepsilon \tag{28}$$

Where $-3.75 \le x \le 3.75$ and $|\varepsilon| < 1.6\times10^{-7}$.

$$x^{½} e^{-x} I_0(x) = \sum_{i=0}^{8} b_i u^i + \varepsilon \tag{29}$$

where $3.75 \le x < \infty$ and $|\varepsilon| < 1.9\times10^{-7}$.

$$x^{-1} I_1(x) = \sum_{i=0}^{6} c_i u^{2i} + \varepsilon \tag{30}$$

where $-3.75 \le x \le 3.75$ and $|\varepsilon| < 8\times10^{-9}$.

$$x^{½} e^{-x} I_1(x) = \sum_{i=0}^{8} d_i u^i + \varepsilon \tag{31}$$

where $3.75 \le x < \infty$ and $|\varepsilon| < 2.2\times10^{-7}$.

3.3 Simplified Formulas for Bessel Functions

We have developed simplified formulas which are less accurate to that used to develop standard tables of Bessel functions such as in [10][13] but still acceptable for engineering purposes particularly for the extended surface heat transfer problems.

Using $I_0(x)$ values from 0 to 3.75, $I_0(x)$ has been regressed into

$$I_0(x) = \sum_{i=0}^{3} a_i\, u^{2i} + \frac{a_4}{2.1061-u} \qquad (32)$$

where $u = \left(\frac{x}{3.75}\right)^2$. Constants a_1 to a_4 are given as follow: a_0 = -4.41575916E+01, a_1 = -2.14395508E+01, a_2 = -6.70459742E+00, a_3 = -4.56248175E+00, and a_4 = 9.51063985E+01.

Relative errors for that equation are as follow: maximum = 0.004 %, average = 0.002%, standard deviation = 0.001%. At first, it was planned to regressed to I_0(x) to a quartic. However, the errors found were bigger.

Using I_1(x) values from 0 to 3.75, I_1(x) has been regressed into

$$I_1(x) = \sum_{i=0}^{4} b_i\, u^{2i} + \frac{b_5}{2.3613-u} \qquad (33)$$

where $u = \left(\frac{x}{3.75}\right)^2$. Constants b_1 to b_5 are given as follow: b_0 = -1.01670E+02, b_1 = -4.11823E+01 , b_2 = -1.82262E+01, b_3 = -4.49396E+00, b_4 = -3.00366E+00 , and b_5 = 2.40074E+02.

Relative errors for that equation are as follow: maximum = 0.003 %, average = 0.000%, standard deviation = 0.001%.

Formulas for a conical spine need the value of I_2(x). However, it is not necessary to develop a separate function for I_2(x) since for $n > 1$, I_n(x)can be computed from the recurrence relation (see [9] or any book on Bessel functions)

$$\frac{2n}{x} I_n(x) = I_{n-1}(x) - I_{n+1}(x) \qquad (34)$$

For $n = 1$, $I_2(x) = I_0(x) - \frac{2I_1(x)}{x}$. Knowing I_0(x) and I_1(x), I_2(x) can then be easily computed. Relative errors for I_2(x) are found to be maximum = 3.678 %, average = 0.132%, standard deviation = 0.535%.

Admittedly, the errors are relatively big. However, big errors are found at low values of x. If we limit the argument from 0.25 to 3.75, the relative errors are much smaller as follow: maximum = 0.166 %, average = 0.024%, standard deviation = 0.040%.

Now, we will compute errors for efficiency of a triangular fin. From Equation (21) we see that the efficiency is proportional to I_1(2mb) / I_0(2mb). Rather than just computing one particular value of the argument *2mb*, we will take it from 0.05 to 3.75 with an interval of 0.05 and we find that relative errors are as follow:

Maximum = 0.262 %, Average = 0.043%, Standard deviation = 0.058%.

These errors are relatively small and acceptable for most engineering applications! Errors for the heat dissipated by the fin are the same for those of the efficiency of the fin because both of them are proportional to I_1(2mb) / I_0(2mb).

For the spine, the efficiency is proportional to $I_2(2mb\sqrt{2}) / I_1(2mb\sqrt{2})$ .So, we will compute the ratio from 0.25 to 3.75 because we limit the lowest argument to be 0.25. we find that relative errors are as follow:

Maximum = 0.167 %, Average = 0.024%, Standard deviation = 0.040%.

These errors are relatively small and acceptable for most engineering applications! Errors for the heat dissipated by the fin are the same for those of the efficiency of the fin because both of them are proportional to $I_2(2mb\sqrt{2}) / I_1(2mb\sqrt{2})$

Approximation functions for I_0(x) and I_1(x) for $x > 3.75$ will be treated in another paper, together with approximation functions for K_0(x) and K_1(x) The latter two Bessel functions will be needed for radial fins.

4 CONCLUSIONS

Bessel functions find numerous applications in various engineering and physics fields. One of practical application is in extended surface heat transfer which needs evaluation of modified Bessel functions. Simplified functions have been developed to compute those Bessel functions which are less accurate than given in [10][13] but still acceptable for most engineering applications. For I_0(x) in the range of 0.05 to 3.75, relative errors are as follow: Maximum = 0.004 %, Average = 0.002% and Standard deviation = 0.001%. For I_1(x) in the same range, relative errors are as follow: Maximum = 0.003 %, Average = 0.000% and Standard deviation = 0.001%. Values for I_2(x) are computed by using recurrence relations between I_0(x) and I_1(x). The errors incurred are much bigger. However, when the range for I_2(x)is from 0.25 to 3.75, the errors become smaller (Maximum = 0.166 %, Average = 0.024% and Standard deviation = 0.040%). When applied to the computation of fin efficiencies, their errors are still relatively small (maximum errors are less than 0.3% for a triangular fin and less than 0.2% for a spine). It is therefore concluded that simplified formulas developed in this paper are acceptable for most engineering applications, at least for extended surface heat transfer discussed here.

REFERENCES

[1] Arfken, G.B., Weber, H.J. and Harris, F.E. Mathematical Methods for Physicists Seventh Edition. Academic Press. Oxford: 2013.

[2] Baricz, A., Masirevic, D.J. and Pogany, T.K. Series of Bessel and Kummer-Type Functions. Springer International Publishing. New York: 2017.

[3] Bayin, S. Mathematical Methods in Science and Engineering. John Wiley & Sons. New York: 2018.

[4] Blennow, M. Mathematical Methods for Physics and Engineering. CRC Press. Boca Raton: 2018.

[5] Caruso, F. and Siviera, F. On new approximations for the modified Bessel function of the second kind K_0(x). Open Journal of Mathematical Sciences Vol. 5 pp. 11-17, 2021.

[6] Chhabra, R.P. (Editor). CRC Handbook of Thermal Engineering Second Edition. CRC Press. Boca Raton: 2018.

[7] Duffy, D.G. Green's Functions with Applications Second Edition. CRC Press. Boca Raton: 2015.

[8] Eslami, M.R., Hetnarski, R.B., Ignaczak, J., Noda, N., Sumi, N. and Tanigawa, Y. Theory of Elasticity and Thermal Stresses: Explanations, Problems and Solutions. Springer Science + Business Media. Dordrecht: 2013.

[9] Han, J.C. Analytical Heat Transfer. CRC Press. Boca Raton: 2012.

[10] Haynes, W.M. (Editor-in-Chief). CRC Handbook of Chemistry and Physics 97th Ed. CRC Press. Boca Raton: 2017.

[11] Ishimaru, A. Electromagnetic Wave Propagation, Radiation, and Scattering: From Fundamentals to Applications. John Wiley & Sons, Inc. Hoboken: 2017.

[12] Kakac, S., Yener, Y. and Naveira-Cotta, C.P. Heat Conduction Fifth Edition. CRC Press. Boca Raton: 2018.

[13] Karwa, R. Heat and Mass Transfer. Springer Science + Business Media. Singapore: 2017.

[14] Kassir, M. Applied Elasticity and Plasticity. CRC Press. Boca Raton: 2018.

[15] Krasikov, I. Approximations for the Bessel and Airy functions with an explicit term. LMS Journal of Computation and Mathematics. Vol. 17(1), pp. 209-225, 2014.

[16] Lienhard IV, J.H. and Lienhard V, J.H. A Heat Transfer Textbook Fifth Edition. Phlogiston. Cambridge: 2019.

[17] Marin, M. and Ochsner, A. Complements of Higher Mathematics. Springer International Publishing. Cham: 2018.

[18] Nagar, S. Introduction to Scilab for Engineers and Scientists. Apress: 2017.

[19] Raine, D. Mathematical Physics: An Introduction. Mercury Learning Information LLC. Dulles: 2019.

[20] Wang, L., Zhou, X. and Wei, X. Heat Conduction: Mathematical Models and Analytical Solutions. Springer-Verlag. Berlin-2018.

[21] Weigand, B. Analytical Methods for Heat Transfer and Fluid Flow Problems Second Edition. Springer-Verlag. Berlin-2015.