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APPLICATION OF SYSTEM MAX-PLUS LINEAR EQUATIONS ON SERIAL MANUFACTURING MACHINE WITH STORAGE UNIT

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Abstract. *The set $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$ together with the operation maximum (max) denoted as $\oplus$ and addition (+) denoted as $\otimes$ is called max-plus algebra. Max-plus algebra may be used to apply algebraically a few programs of Discrete Event Systems (DES), certainly one of the examples in the production system. In this study, the application of max-plus algebra in a serial manufacturing machine with a storage unit is discussed. The results of this are the generalization system max-plus-linear equations on a production system that is, in addition, noted the max-plus-linear time-invariant system. From the max-plus-linear time-invariant system, it can be obtained the equation $x(k+1) = \bar{A} \otimes x(k)$ which is then used to determine the beginning time of a production system so the manufacturing machine work periodically. The eigenvector and eigenvalue of the matrix $\bar{A}$ are then used to find the beginning time and the period time of the manufacturing machine. Furthermore, the time when the product leaves the manufacturing machine with the time while the raw material enters the manufacturing machine is given and vice versa are obtained from the max-plus-linear time-invariant system that is can be formed in the equation $Y = H \otimes U$.*

Keywords: *discrete event system, max-plus algebra, production system, serial manufacturing machine, time-invariant system.*

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1. INTRODUCTION

In recent years, both the industrial and academic worlds are interested in techniques for modeling, analyzing, and controlling discrete event systems (DES). A DES is the name of the classification of man-made system problems involving limited resources and users [1], [2]. Max-plus Algebra can be used to apply algebraically several of the DES [3]. This article discusses the application of max-plus algebra which focuses on one example of DES, namely the production system. In the production system, there are not only simple production systems but also other, more complex situations. Another more complex situation is the serial manufacturing machine with a container or storage unit. The type of this production system can be worked out the same as in a simple production system.

Max-plus Algebra has been used to model and analyzes algebraically several of the DES, one of which is the production system. The study conducted by [4], [5] describes the max-plus algebra application in DES with the condition "synchronization without concurrency". In this article, the system max-plus-linear equations were explained. In the research results, an example is given to a simple production system. In this case, a maximum time-plus invariant linear system was obtained. Furthermore, also given an explanation of the biggest subsolution of a system's max-plus-linear equations. The research conducted by [6]–[8] found that in the production system, there are not only simple production systems, but also other, more complex situations. Whereas the research conducted [8] describes a simple production system which is further developed into 5 types of manufacturing machines with a storage unit, namely the manufacturing machine type serial, assembly, splitting, parallel, and flexible with certain row activities. The five types of manufacturing machines with these storage units, can all be worked out the same as in a simple production system. The explanation of the five types of manufacturing machines produced by the study written [8] is explained until a nonlinear equation system is obtained in each type of manufacturing machine.

This article is discussed situations that might occur in the production system, and provide examples of illustrations. Therefore we need some underlying concepts in writing this article that refers to some literature. The basic understanding and operation of max-plus algebra and matrix operations for max-plus algebra that used in this article refers to [9]–[11]. This article also describes the eigenvalues and eigenvectors in max-plus algebra referring to [3], [4], [12], [13]. Next, to determine the eigenvalues and eigenvectors of the matrix of max-plus algebra, an algorithm is taken from the article [5], [14]. This article is given an example of using the algorithm to calculate the eigenvalues and eigenvectors of a matrix. Furthermore, in this article the concept of the system max-plus-linear equations refers to [4], [8], [15], [16]. The solution to the system max-plus-linear equations can be found by determining the subsolution of the equation described in the [17]. Furthermore, we also explained the theory of the solution to the max-plus algebra optimization problem taken from the article [4]. Furthermore, this article also provides an explanation for the max-plus time linear system invariant along with the analysis for the time problem output and input referring to [4], [18].

The writing of this article is carried out to determine the system linear equations of the serial manufacturing machine with storage units using max-plus algebra operations. Furthermore, the systems of equations obtained can be formed into a system max-plus-linear equations in general from a production system. In this article, there are also some examples of periodization problems, namely determining the beginning time of a manufacturing machine so that the machine takes place periodically. In addition, in this example with different assumptions, calculations were also carried out to get the time when the product left the manufacturing machine with the time given when the raw material was put into the machine. In further cases, the last time is calculated for the raw material to be entered into the manufacturing machine based on the given time for the product to leave the machine.

2. RESEARCH METHODS

The research method used in the writing of this article is the study of literature by collecting and studying several references in the form of books, thesis, and articles on max-plus algebra and illustrations on production systems. The steps taken in this study are as follows.

a. Specifies the general form of the system max-plus-linear equations in a production system.

b. Calculates the beginning time of a manufacturing machine so that the machine takes place periodically based on eigenvalues and eigenvectors.

c. Understand the biggest subsidence of max-plus-linear equations and solutions to max-plus algebra optimization problems.

d. Learn about the biggest subsystem applications and solutions to optimization problems for production systems.

e. Calculates the time when the product leaves the manufacturing machine with the time given when the raw material is put into the machine based on the Max-plus Linear System Time-Invariant equation form.

f. Calculating the most recent time for raw materials is entered into a manufacturing machine based on the given time for the product to leave the machine using the subsolution theory.

3. RESULTS AND DISCUSSION

This section discusses the production system which is divided into 4 subsections. The first subsection describes the system max-plus-linear equations in serial manufacturing machines with storage units. The second subsection describes the form of the system max-plus-linear equations in general from a production system. The third subsection gives an example of the problem of periodization in serial manufacturing machines. Next, in the fourth subsection, we give an example of the problem of time output-input in the manufacturing machine.

3.1. Manufacturing Machine with Storage Unit

Given a manufacturing machine with a storage unit divided into 5 types of machines that might occur in the manufacturing machine taken from [8], which consists of several processing units (machines) P_i with $i = 1,2,\dots,n$. The processing time required for P_i is d_i unit of time. The travel time for raw materials is assumed to be zero. In this manufacturing machine it is assumed that there is a container unit with a limited capacity of N_i. It is assumed that $x_i(\mathrm{k}) = \varepsilon$ for all k non-positive numbers which means that at the beginning of the process, all container units are empty and all processing units are not working. A processing unit can only start working to produce new products if the previous production process has been completed. If the processing unit has completed the production process and the product output portion of the processing unit is full, the product remains in the input section of the product container of the processing unit, or in other words during output the product container is full, the processing unit cannot start processing a new product. It is also assumed that each processing unit can immediately start working when all materials are available and the input of the storage unit is empty.

For example, defined

a. $u_i(\mathrm{k}+1)$ is the time when raw materials are entered into the machine for processing to-$(k+1)$,

b. $x_i(\mathrm{k})$ is the time when the i processor starts working for the k process, and

c. $x_i(\mathrm{k})$ is the time when the product leaves the machine for the k process,

for all $i = 1,2,\dots,n$ and $k \in \mathbb{N}_0$.

In serial manufacturing machine, there are 2 processing units namely P_1 and P_2 which are connected in series, as given in Figure 1. Between P_1 and P_2 there is a container unit with a limited capacity of N_1.

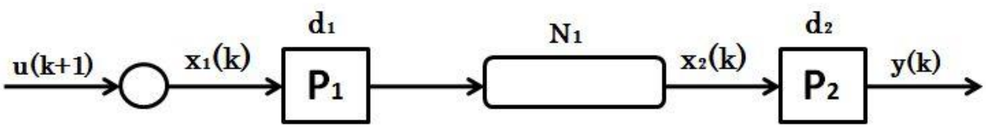

Figure 1. Serial Manufacturing Machine with Storage Unit

Time is obtained at P_1 and P_2 starts working for the process to $(k+1)$ and the time when the product leaves the machine for the k process is written using the max-plus algebra operation is

$$\begin{aligned} x_1(k+1) &= x_1(k) \otimes d_1 \oplus x_2(k-N_1) \oplus u(k+1) \\ x_2(k+1) &= x_1(k) \otimes {d_1}^{\otimes 2} \oplus x_2(k) \otimes d_2 \oplus x_2(k-N_1) \otimes d_1 \oplus u(k+1) \otimes d_1 \\ y(k) &= x_2(k) \otimes d_2. \end{aligned} \tag{1}$$

3.2. General Forms of Production System Equations

The equations in the serial manufacturing machine can be written in the form of a max-plus algebra matrix [9]–[11], the form of the equation system is obtained

$$\begin{aligned} x(k+1) &= \boldsymbol{A} \otimes x(k) \oplus \boldsymbol{B} \otimes u(k+1) \\ y(k) &= \boldsymbol{C} \otimes x(k) \end{aligned} \tag{2}$$

Furthermore, in (2) for example, it is assumed that if the time when the raw material is entered into the machine is the same as the time when the product leaves the machine or can be written ($u(k+1) = y(k)$), then the equation is obtained

$$x(k+1) = \overline{\boldsymbol{A}} \otimes x(k) \tag{3}$$

with $\overline{\boldsymbol{A}} = \boldsymbol{A} \oplus \boldsymbol{B} \otimes \boldsymbol{C}$.

3.3. Periodization Issues

The following is given the periodization problem to get the beginning time of a manufacturing machine so that the machine takes place periodically.

Based on Figure 1 in the Subsection 3.1, for example, given the required processing time for P_1, P_2, and P_3 each is $d_1 = 5$ and $d_2 = 4$ units of time. For example, P_1 has a container unit with a capacity of $N_1 = 3$, so it can be assumed that the travel time for semi-finished products from P_1 goes to P_2 is $t_1 = (d_1)^{\otimes(N_1-1)} = 5^{\otimes 2} = 10$ units of time. The following is given Figure 2 as an illustration.

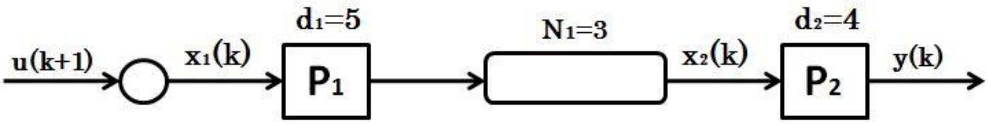

Figure 2. Example of A Serial Manufacturing Machine with Storage Unit

Based on Figure 2, the current time is obtained P_1 and P_2 starts working on the process for $(k+1)$ and the time when the product leaves the manufacturing machine for the k process using the max-plus algebra operation is as follows:

$$\begin{aligned} x_1(k+1) &= 5 \otimes x_1(k) \oplus u(k+1) \\ x_2(k+1) &= 20 \otimes x_1(k) \oplus 4 \otimes x_2(k) \oplus 15 \otimes u(k+1) \\ y(k) &= 4 \otimes x_2(k). \end{aligned} \tag{4}$$

If (4) is written in the form of a max-plus algebra matrix, it is obtained

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 5 & \varepsilon \\ 20 & 4 \end{pmatrix} \otimes x(k) \oplus \begin{pmatrix} 0 \\ 15 \end{pmatrix} \otimes u(k+1) \\ y(k) &= (\varepsilon \quad 4) \otimes x(k) \end{aligned} \tag{5}$$

with $x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}$. The equation system (5) is a form of (2). Furthermore, from (5), a matrix is obtained $\overline{\boldsymbol{A}} = \begin{pmatrix} 5 & 4 \\ 20 & 19 \end{pmatrix}$.

Next, using the Power algorithm is obtained by eigenvalues and eigenvectors from the matrix $\overline{\boldsymbol{A}}$ [5], [14] for (5) respectively $\lambda = 19$ and $\boldsymbol{v} = \begin{pmatrix} 5 \\ 20 \end{pmatrix}$. Furthermore, from the $\boldsymbol{v}$, it is found that if the serial manufacturing machine has a limited capacity container unit, then the beginning time is obtained so that the machine takes place periodically $\begin{pmatrix} 0 \\ 15 \end{pmatrix}$. For beginning time it means that the time when the P_1 and P_2 process start working for the first process each is the 0 time unit and 15 . For the next process that is $k = 2, 3, \ldots, n$ each processor works periodically with the period is the eigen value of the matrix $\overline{\boldsymbol{A}}$ that is 19 time unit.

3.4. Time Problem Output and Input

The following is given the time of the output problem which is to get the time when the product leaves the manufacturing machine with the time when the raw material is put into the machine. Furthermore, given the problem of time input that is to obtain the most recent time for raw materials to enter into the manufacturing machine based on the product for leaving the machine.

Based on the example in Subsection 3.1., the following is searched for the matrix $\boldsymbol{H}$ by using (5). Given $p = 4$. Defined vector output $Y = [y(1) \quad y(2) \quad y(3) \quad y(4)]^T$ and vector input $U = [u(1) \quad u(2) \quad u(3) \quad u(4)]^T$. Because it has been assumed that $x_i(0) = \varepsilon$ for $i = 1{,}2$, then obtained the matrix $\boldsymbol{H}$ as follows:

$$\boldsymbol{H} = \begin{pmatrix} 19 & \varepsilon & \varepsilon & \varepsilon \\ 24 & 19 & \varepsilon & \varepsilon \\ 29 & 24 & 19 & \varepsilon \\ 34 & 29 & 24 & 19 \end{pmatrix}.$$

Furthermore, if given the time when raw materials are entered into the machine for the process of 1, 2 , 3 , and 4 that is vector input $U = [0 \quad 15 \quad 31 \quad 48]^T$, then vector output is obtained $Y = \boldsymbol{H} \otimes U = (19 \quad 34 \quad 50 \quad 67)^T$. This means that when given the time when raw materials are put into the manufacturing machine for the 1, 2, 3, and 4 processes, the time units are 0, 15, 31, and 48, then the time when the product left the machine for the 1, 2, 3, and 4 processes, respectively the time units of 19, 34, 50, and 67.

Furthermore, for other cases, the following is determined the most recent time for raw materials to be entered into the manufacturing machine provided the time is given when the product leaves the machine for the process of 1, 2, 3, and 4 respectively, namely the 23, 40, 57 and 72 time unit. In this case it means being given a vector output $Y = (23 \quad 40 \quad 57 \quad 72)^T$. Furthermore,the largest subsolution is obtained $\widehat{U} = (4 \quad 21 \quad 38 \quad 53)^T$ so that it meets $\boldsymbol{H} \otimes \widehat{U} \leq_m Y$. The vector input $\widehat{U}$ is the last time for the raw material to be entered into the manufacturing machine with the vector given output Y. Furthermore, the following is calculated vector output $\widehat{Y}$ which corresponds to vector input $\widehat{U}$.

$$\widehat{Y} = \boldsymbol{H} \otimes \widehat{U} = (19 \quad 34 \quad 50 \quad 67)^T = Y$$

It appears that vector output $\widehat{Y}$ which is the fastest time for the product to leave the machine the same as vector output Y is the time given when the product leaves the machine, so the maximum difference between Y and $\widehat{Y}$ that is $\delta = 0$. Because the vector output $\widehat{Y} = Y$ is obtained so that the value $\delta = 0$, then the vector input $\widetilde{U} = \widehat{U}$ is obtained and vector output $\widetilde{Y} = \widehat{Y} = Y$. In other words, a minimum of the maximum difference between Y and $\widetilde{Y}$ is $\frac{\delta}{2} = \frac{0}{2} = 0$.

So, with the time given when the product leaves the machine for the 1, 2, 3, and 4 processes, the time units are 23, 40, 57, and 72, then the time when raw materials are included to the manufacturing machine for the 1, 2, 3, and 4 processes, each of which is a time unit of 4, 21, 38, and 53 is the last time for raw materials to be entered into the manufacturing machine. Furthermore, the time given when the product leaves the machine is also the fastest time for the product to leave the machine, because the maximum difference between the time given when the product leaves the machine and the actual time when the product leaves the machine is 0 time unit.

4. CONCLUSIONS

From the results of the discussion, it can be concluded that

a. The system of equation (2) is a form of a system of equations in general from a production system.

b. The beginning time and period of the manufacturing machine is obtained by determining the eigenvector and eigenvalue of the matrix $\bar{A}$ in the linear equation (3).

c. The time when the product leaves the manufacturing machine can be obtained if given the time when the raw material is put into the machine.

d. The time when raw materials are entered into the manufacturing machine can be obtained if given the time when the product leaves the machine.

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