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# MATHEMATICAL MODEL OF THREE SPECIES FOOD CHAIN WITH INTRASPECIFIC COMPETITION AND HARVESTING ON PREDATOR

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***Abstract.*** *This research develops a mathematical model of three species of food chains between prey, predator, and top predator by adding intraspecific competition and harvesting factors. Interaction between prey with predator and interaction between predator with top predator uses the functional response type II. Model formation begins with creating a diagram food chain of three species compartments. Then a nonlinear differential equation system is formed based on the compartment diagram. Based on this system four equilibrium points are obtained. Analysis of local stability at the equilibrium points by linearization shows that there is one unstable equilibrium point and three asymptotic stable local equilibrium points. Numerical simulations at equilibrium points show the same results as the results of the analysis. Then numerical simulations on several parameter variations show that intraspecific competition has little effect on population changes in predator and top predator. While the harvesting parameter predator affects the population of predator and top predator.*

***Keywords:*** *equilibrium stability, food chain model, numerical simulation, predator-prey model.*

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## 1. INTRODUCTION

Every living thing and its environment in an ecosystem interact with each other. The existence of these interactions will maintain the balance, stability, and productivity of an ecosystem. One form of interaction in an ecosystem is predation, which is the interaction between prey and predator[1]. One of the goals of this interaction is to stabilize the number of prey and predator populations in an ecosystem. Without predators, the amount of prey will increase and will damage the food chain below. Conversely, without prey, a predator will lose food for survival.

Predation in an ecosystem does not only involve two species but some predations involve more than two species. Predations involving more than two species are known as food chains[2]. The food chain is the process of eating and being eaten between living things. One example of the food chain is in the marine ecosystem. Predations in these ecosystems occur at trophic levels II, III, and IV [3]. At the trophic level II, a herbivorous animal, namely Zooplankton, preyed on a trophic level III (predator) like Sardines, and subsequently preyed on a trophic level IV (top predator) such as a dolphin. Living creatures at trophic level III namely Sardines is one type of fish that is widely traded as food so that the fishermen harvest it.

Each predator population (predator and top predator) in real life will interact with each other. Besides cooperating, it can happen that living things in predator populations will compete in capturing their prey. Competition between living things in a population is called intraspecific competition. This interaction can also affect efficiency in finding and killing prey known as the response function or consumption rate of predators. This intraspecific competition disruption can be formed with a type II response function, where it is assumed that each encounter between predators is wasted time, this is the same as the time process of handling prey [4].

Research on predation has been carried out, not only in the field of biology, in the field of mathematics, a Predator-Prey Model has also been developed. The Predator-Prey Model was first introduced by Alfred J. Lotka and Vito Volterra, so it is often called the Lotka-Volterra Model [5]. Some researchers have developed the Predator-Prey Model, one of which is the three-species food chain model. Research on the three-species food chain model was introduced by [6] and concluded that there was chaos in the dynamics of the model that might be common in the food chain. Then, [7] developed the research by showing the existence of chaos dynamics where no top species are found in the food chain in an equilibrium environment. Next, [8] modified the food chain model of three species from [7] using the Lyapunov Function.

Another modification related to the three-species food chain model was carried out by [9] who examined Horf bifurcation in a three-species food chain model with a time delay. [10] studied the food chain model of three species by providing nutrition and dividing it into three important regions, namely areas where predators will become extinct, areas of equilibrium are stable, and areas where there are stable boundary cycles. Then [11] have examined three species of food chain models with intraspecific competition in both predators developed from the research of [12] on the Predator-Prey Model of two species with intraspecific competition in predators.

In addition, [13] have examined the effects of predatory harvesting on the competitive model of two predators with one prey. Based on the harvesting at trophic level III (predators) in the marine ecosystem and intraspecific competition in predator populations, this research will develop a three-species food chain model from the model introduced by [11] with the addition of harvesting parameters examined by [13] and use type II response functions.

## 2. RESEARCH METHODS

### 2.1. Basic Predator-Prey Model

Predator-Prey Model is a model that describes the interaction between prey and predator populations. This model was first introduced by Lotka (1925) and Volterra (1926) so it is often referred to as the Lotka-Volterra Model. The Lotka-Volterra model is the simplest Predator-Prey model which involves two species, namely one predator species that preys on other species (prey).

Without predators, the growth of the prey population is increasing rapidly due to the growth of the prey population in proportion to the current population. Without prey, predators will become extinct. This is due to the assumption that the growth of predators depends on prey as their food source. The predation interaction between predators and prey influences the growth of predators and inhibits prey growth.

Notated $X$ and $Y$ are prey and predator populations with time $t$, the Lotka-Volterra model can be written

$$\begin{aligned}\frac{dX}{dt} &= bX - \gamma XY,\\ \frac{dY}{dt} &= -cY + \gamma XY,\end{aligned} \tag{1}$$

where $b$ is the rate of growth of prey ($b > 0$), c is the rate of death from predators ($c > 0$), and $\gamma$ is the coefficient of interaction between prey and predator[14].

### 2.2. Functional Response Type II

The response function in ecology is a function of the predatory consumption rate for different prey densities. Holling categorizes response functions into three types, one of these functions is the hyperbolic response function (type II) [15]. Hyperbolic response function (type II) is a response function where the rate of consumption increases but there is a continuous decrease. That is, the rate of consumption of predators increases with increasing prey population but will decrease when predators approach satiety. This is because when the prey population is small, some predator time is spent searching for prey. Whereas when the prey population is large, predators spend the time available to handle and digest prey not to look for it, consequently the consumption rate will be lower. This causes the consumption rate to reach half-saturation. This type describes a predator that moves actively looking for prey. The type II response function is represented as follows.

$$f(X) = \frac{aX}{d + X}, \tag{2}$$

where $f(X)$ is the predatory consumption rate, $a$ represents the interaction coefficient between prey and predator, $X$ is the number of population prey, and $d$ is the half-saturation constant, that is the number of population prey when the consumption rate per unit of prey reaches half of the maximum value.

## 3. RESULTS AND DISCUSSION

### 3.1. Model Formulation

The assumptions used to form the food chain model of three species with intraspecific competition and harvesting of predators are as follows. (1) Prey population is a group of species that fall prey to the predator population. (2) Predator population is a group of species that prey on prey and fall prey to top predator population. (3) Top predator population is a group of species that prey on predator population. (4) The interaction between prey, predator, and top predator follows the form of interaction in the food chain, so that the prey population cannot prey on the top predator population. (5) The top predator population does not prey on the prey population. (6) Prey population, predator population, and top predator population are closed, meaning that there is no migration in all three populations. (7) The prey population growth follows the logistical growth rate. (8) There is no type of food other than prey that is preyed on by predator and no other type of food besides predator that is preyed on by top predator. (9) Predator and the top predator in predation follow the type II response function. (10) Growth of predator population depends on predation of prey and growth of top predator population depends on predation of the predator. (11) There is intraspecific competition in living things in predator and top predator populations. (12) Predator population is species that can be harvested.

Variables and parameters for the assumption model used to form a three-species food chain model with intraspecific competition and harvesting on predators are presented in Table 1. Then the transfer diagram illustrating the relationship between prey, predator, and top predator is presented in Figure 1.

**Table 1. Lists the variables and parameters of a three-species food chain model with intraspecific competition and harvesting on predator**

| Symbol | Definition | Type | Condition | Unit |
|---|---|---|---|---|
| $\boldsymbol{X(t)}$ | Total population of prey at the $t$-time | Variable | $X(t) \geq 0$ | $taik$ |
| $\boldsymbol{Y\ (t)}$ | Total population of the predator at the $t$-time | Variable | $Y(t) \geq 0$ | $tail$ |
| $\boldsymbol{Z(t)}$ | Total population of the top predators at the $t$-time | Variable | $Z(t) \geq 0$ | $tail$ |
| $\boldsymbol{r}$ | Prey growth rate without being influenced by the environment | Parameter | $r > 0$ | $\frac{1}{day}$ |
| $\boldsymbol{K}$ | The environmental carrying capacity of the prey population | Parameter | $K > 0$ | $tail$ |
| $\boldsymbol{M_1}$ | Coefficient of interaction between prey and predator | Parameter | $M_1 \geq 0$ | $day$ |
| $\boldsymbol{M_2}$ | Coefficient of interaction between predator and top predator | Parameter | $M_2 \geq 0$ | $\frac{1}{day}$ |
| $\boldsymbol{A_1}$ | Half-saturation constant in predator | Parameter | $A_1 \geq 0$ | $tail$ |
| $\boldsymbol{A_2}$ | Half-saturation constant in top predator | Parameter | $A_2 \geq 0$ | $tail$ |
| $\boldsymbol{E_1}$ | Changing the consumption of prey to the birth of predator | Parameter | $E_1 \geq 0$ | $-$ |
| $\boldsymbol{E_2}$ | Changing the consumption of the predator to the birth of top predator | Parameter | $E_2 \geq 0$ | $-$ |
| $\boldsymbol{D_1}$ | Predator death rate | Parameter | $D_1 \geq 0$ | $\frac{1}{day}$ |
| $\boldsymbol{D_2}$ | Top Predator death rate | Parameter | $D_2 \geq 0$ | $\frac{1}{day}$ |
| $\boldsymbol{H_1}$ | The rate of competition between living things in predator population | Parameter | $H_1 \geq 0$ | $\frac{1}{tail\ day}$ |
| $\boldsymbol{H_2}$ | The rate of competition between living things in the top predator population | Parameter | $H_2 \geq 0$ | $\frac{1}{tail\ day}$ |
| $\boldsymbol{q}$ | Predator harvesting rate | Parameter | $q \geq 0$ | $\frac{tail}{day}$ |
| $\boldsymbol{W}$ | The effort required to harvest predator | Parameter | $W \geq 0$ | $\frac{1}{tail}$ |

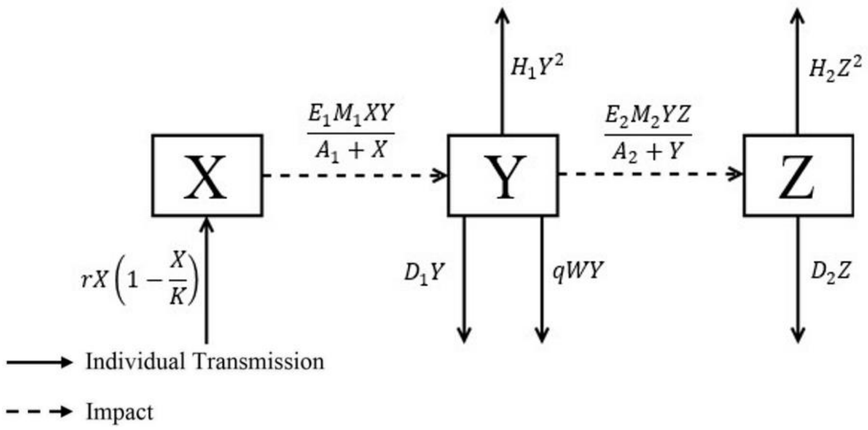


**Figure 1. Transfer diagram of the parameters of a three-species food chain model with intraspecific competition and predatory harvesting**

Based on the assumptions and Figure 1, a three species food chain model can be formed with intraspecific competition and harvesting on predators in the form of a nonlinear differential equation system

$$\begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X}{K}\right)-\frac{M_1XY}{A_1+X}, \\
\frac{dY}{dT} &= \frac{E_1M_1XY}{A_1+X}-D_1Y-\frac{M_2YZ}{A_2+Y}-H_1Y^2-qWY, \\
\frac{dZ}{dT} &= \frac{E_2M_2YZ}{A_2+Y}-D_2Z-H_2Z^2.
\end{aligned} \tag{3}$$

System (3) can be written in dimensionless form, referring to [11] with $x = \frac{X}{K}$, $y = \frac{Y}{KE_1}$, $z = \frac{Z}{KE_1E_2}$, dan $t = rT$ so that System (3) becomes

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{a_1xy}{1+b_1x}, \\ \frac{dy}{dt} &= \frac{a_1xy}{1+b_1x} - d_1y - \frac{a_2yz}{1+b_2y} - h_1y^2 - py, \\ \frac{dz}{dt} &= \frac{a_2yz}{1+b_2y} - d_2z - h_2z^2. \end{aligned} \tag{4}$$

where

$$a_1 = \frac{M_1KE_1}{rA_1},\ b_1 = \frac{K}{A_1},\ d_1 = \frac{D_1}{r},\ h_1 = \frac{H_1KE_1}{r}, a_2 = \frac{M_2KE_1E_2}{rA_2},$$
$$b_2 = \frac{KE_1}{A_2},\ d_2 = \frac{D_2}{r},\ h_2 = \frac{H_2KE_1E_2}{r}, p = \frac{qW}{r},$$

and

$$\Gamma = \{(x,\ y,\ z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ z \geq 0\}.$$

### 3.2. Model Analysis

Next will be found the equilibrium point of System (4). The equilibrium points in the three-species food chain model with intraspecific competition and predator harvesting is obtained if $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$.

So that the equilibrium points are obtained:

1. Equilibrium point $E_1 = (0, 0, 0)$;
2. Equilibrium point $E_2 = (1, 0, 0)$;
3. Equilibrium point $E_3 = (\tilde{x}, \tilde{y}, 0)$, where $\tilde{y} = \frac{(1-\tilde{x})(1+b_1\tilde{x})}{a_1}$ and $\tilde{x}$ is the root that satisfies the equation $\sum_{i=1}^{3} B_i x^i = 0$ with

$$\begin{aligned} B_0 &= -h_1 - a_1(d_1 + p), \\ B_1 &= a_1^2 - 2b_1h_1 + h_1 - a_1b_1(d_1 + p), \\ B_2 &= 2b_1h_1 - b_1^2h_1, \\ B_3 &= b_1^2h_1. \end{aligned}$$

4. Equilibrium point $E_4 = (\hat{x}, \hat{y}, \hat{z})$, where $\hat{y} = \frac{(1-\hat{x})(1+b_1\hat{x})}{a_1}$, $\hat{z} = \frac{(a_2-b_2d_2)\hat{y}-d_2}{h_2(1+b_2\hat{y})}$, and $\hat{x}$ is the root that satisfies the equation $\sum_{i=1}^{7} G_i x^i = 0$ with

$$G_0 = -a_1^2a_2(a_2 - b_2d_2) + a_1^3a_2d_2 - h_1h_2(a_1 + b_2)^2 - a_1h_2(d_1 + p)(a_1 + b_2)^2,$$

$$\begin{aligned} G_1 = {} & a_1^2h_2(a_1 + b_2)^2 - a_1^2a_2b_1(a_2 - b_2d_2) - a_1^2a_2(a_2 - b_2d_2)(b_1 - 1) + a_1^3a_2b_1d_2 - \\ & 2a_1b_2h_1h_2(b_1 - 1) - 2b_2^2h_1h_2(b_1 - 1) - b_1h_1h_2(a_1 + b_2)^2 - h_1h_2(b_1 - 1) \\ & (a_1 + b_2)^2 - 2a_1b_2h_2(a_1 + b_2)(b_1 - 1)(d_1 + p) - a_1b_1h_2(d_1 + p)(a_1 + b_2)^2, \end{aligned}$$

$$\begin{aligned} G_2 = {} & 2a_1^2b_2h_2(b_1 - 1)(a_1 + b_2) - a_1^2a_2b_1(a_2 - b_2d_2)(b_1 - 1) + 2b_1b_2h_1h_2(a_1 + b_2) \\ & -b_2^2h_1h_2(b_1 - 1)^2 - 2b_1b_2h_1h_2(b_1 - 1)(a_1 + b_2) - 2b_2h_1h_2(b_1 - 1)^2(a_1 + b_2) \\ & -b_1h_1h_2(b_1 - 1)(a_1 + b_2)^2 + b_1h_1h_2(a_1 + b_2)^2 + 2a_1b_1b_2h_2(d_1 + p)(a - 1 + b_2) - \\ & a_1b_2^2h_2(b_1 - 1)^2(d_1 + p) - 2a_1b_1b_2h_2(a_1 + p)(b_1 - 1)(d_1 + p), \end{aligned}$$

$$\begin{aligned} G_3 = {} & a_1^2b_2^2h_2(b_1 - 1)^2 - 2a_1^2b_1b_2h_2(a_1 + b_2) + a_1^2a_2b_2^2(a_2 - b_2d_2) + 2b_1b_2^2h_1h_2 \\ & (b_1 - 1) + 2b_1^2b_2h_1h_2(a_1 + b_2) - b_1b_2^2h_1h_2(b_1 - 1)^2 + 4b_1b_2h_1h_2(a_1 + b_2) \\ & (b_1 - 1) - b_2^2h_1h_2(b_1 - 1)^3 - 2b_1b_2h_1h_2(a_1 + b_2)(b_1 - 1)^2 + b_1^2h_1h_2 \\ & (a_1 + b_2)^2 + a_1b_1b_2^2h_2(b_1 - 1)(d_1 + p) + 2a_1b_1^2b_2h_2(a_1 + b_2)(d_1 + p), \end{aligned}$$

$$\begin{aligned}G_4 =\ & 2b_1^2b_2^2h_1h_2(b_1-1) - 2a_1^2b_1b_2^2h_2(b_1-1) - b_1^2b_2^2h_1h_2 + 2b_1b_2^2h_1h_2(b_1-1)^2 \\ & +2b_1^2b_2h_1h_2(a_1+b_2)(b_1-1) - b_1b_2^2h_1h_2(b_1-1)^3 - 2b_1^2b_2h_1h_2(a_1+b_2) \\ & +b_1b_2^2h_1h_2(b_1-1) + 2b_1^2b_2h_1h_2(a_1+b_2)(b_1-1) - a_1b_1^2b_2^2h_2(d_1+p) \\ & +2a_1b_1^2b_2^2h_1h_2(b_1-1)(d_1+p),\end{aligned}$$

$$\begin{aligned}G_5 =\ & a_1^2b_1^2b_2^2h_2 - b_1^3b_2^2h_1h_2 - 3b_1^2b_2^2h_1h_2(b_1-1) + 2b_1^2b_2^2h_1h_2(b_1-1)^2 \\ & -2b_1^3b_2h_1h_2(a_1+b_2) + b_1^2b_2^2h_1h_2(b_1-1) - a_1b_1^3b_2^2h_2(d_1+p)\end{aligned}$$

$$G_6 = b_1^3b_2^2h_1h_2 - 3b_1^3b_2^2h_1h_2(b_1-1),$$

$$G_7 = b_1^4b_2^2h_1h_2.$$

The stability of the equilibrium point is investigated from the linearization results in System (4) around the equilibrium point. Before linearizing, a Jacobian matrix of System (4) is formed around the equilibrium point $E = (x, y, z)$

$$J_{(f(x,y,z))} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}_{(x,y,z)} \tag{5}$$

where $\frac{dx}{dt} = f_1(x,y,z)$, $\frac{dy}{dt} = f_2(x,y,z)$, and $\frac{dz}{dt} = f_3(x,y,z)$. The matrix elements of $J_{(f(x,y,z))}$ is

$$\frac{\partial f_1}{\partial x} = 1 - 2x - \frac{a_1y}{1+b_1x} + \frac{a_1b_1xy}{(1+b_1x)^2},$$

$$\frac{\partial f_1}{\partial y} = -\frac{a_1x}{(1+b_1x)},$$

$$\frac{\partial f_1}{\partial z} = 0,$$

$$\frac{\partial f_2}{\partial x} = \frac{a_1y}{1+b_1x} + \frac{a_1b_1xy}{(1+b_1x)^2},$$

$$\frac{\partial f_2}{\partial y} = \frac{a_1x}{(1+b_1x)} - d_1 - \frac{a_2z}{1+b_2y} + \frac{a_2b_2yz}{(1+b_2y)^2} - 2h_1y - p,$$

$$\frac{\partial f_2}{\partial z} = -\frac{a_2y}{1+b_2y},$$

$$\frac{\partial f_3}{\partial x} = 0,$$

$$\frac{\partial f_3}{\partial y} = \frac{a_2z}{1+b_2y} - \frac{a_2b_2yz}{(1+b_2y)^2},$$

$$\frac{\partial f_3}{\partial z} = \frac{a_2y}{1+b_2y} - d_2 - 2h_2z.$$

**Theorem 1.** *The equilibrium point $E_1 = (0,0,0)$ is unstable.*

***Proof***. The equilibrium point $E_1$ is substituted to the matrix elements of $J_{(f(x,y,z))}$, obtained by the matrix $J(f(E_1))$ is

$$J(f(E_1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(d_1+p) & 0 \\ 0 & 0 & -d_2 \end{pmatrix}.$$

So, the characteristic equation for $J(f(E_1))$ is

$$(\lambda - 1)(\lambda + (d_1+p))(\lambda + d_2) = 0.$$

Then, the eigenvalues obtained are $\lambda_1 = 1$, $\lambda_2 = -(d_1+p)$, dan $\lambda_3 = -d_2$. Because $d_1, p, d_2 > 0$ then $\lambda_2, \lambda_3 < 0$. However, $\lambda_1 > 0$. So, the equilibrium point $E_1 = (0,0,0)$ is unstable.

□

**Theorem 2.** *If $a_1 < (d_1+p)(1+b_1)$, then the equilibrium point $E_2 = (1,0,0)$ is locally asymptotically stable.*

***Proof***. The equilibrium point $E_2$ is substituted to the matrix elements of $J_{(f(x,y,z))}$, obtained by the matrix $J(f(E_2))$ that is

$$J(f(E_2)) = \begin{pmatrix} -1 & -\frac{a_1}{1+b_1} & 0 \\ 0 & \frac{a_1}{1+b_1} - d_1 - p & 0 \\ 0 & 0 & -d_2 \end{pmatrix}$$

So, the characteristic equation for $J(f(E_2))$ is

$$(\lambda + 1)\left(\lambda - \left(\frac{a_1}{1+b_1} - d_1 - p\right)\right)(\lambda + d_2) = 0$$

Then, the eigenvalues obtained are $\lambda_1 = -1$, $\lambda_2 = \frac{a_1}{1+b_1} - d_1 - p$, and $\lambda_3 = -d_2$. Because $d_2 > 0$, consequently $\lambda_3 < 0$. For the equilibrium point $E_2$ to be locally asymptotically stable, it must be $\lambda_2 < 0$. That is,

$$\frac{a_1}{1+b_1} - d_1 - p < 0$$

$$\Leftrightarrow \quad a_1 < (d_1 + p)(1 + b_1).$$

So, the equilibrium point $E_2 = (1, 0, 0)$ is locally asymptotically stable if $a_1 < (d_1 + p)(1 + b_1)$.

□

**Theorem 3.** *If* $\tilde{x} > \frac{b_1 - 1}{2b_1}$ *dan* $\tilde{y} < \frac{d_2}{a_2 - b_2 d_2}$, *then the equilibrium point* $E_3 = (\tilde{x}, \tilde{y}, 0)$ *is locally asymptotically stable.*

***Proof.*** The equilibrium point $E_3$ is substituted to the matrix elements of $J_{(f(x,y,z))}$, obtained by the matrix $J(f(E_3))$ that is

$$J(f(E_3)) = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & g_{23} \\ 0 & 0 & g_{33} \end{pmatrix},$$

where

$$g_{11} = 1 - 2\tilde{x} - \frac{a_1 \tilde{y}}{1 + b_1 \tilde{x}} + \frac{a_1 b_1 \tilde{x} \tilde{y}}{(1 + b_1 \tilde{x})^2}, \quad g_{12} = -\frac{a_1 \tilde{x}}{1 + b_1 \tilde{x}}, \quad g_{23} = -\frac{a_2 \tilde{y}}{1 + b_2 \tilde{y}},$$

$$g_{21} = \frac{a_1 \tilde{y}}{1 + b_1 \tilde{x}} + \frac{a_1 b_1 \tilde{x} \tilde{y}}{(1 + b_1 \tilde{x})^2}, \quad g_{22} = -h_1 \tilde{y}, \quad g_{33} = \frac{a_2 \tilde{y}}{1 + b_2 \tilde{y}} - d_2.$$

So, the characteristic equation for $J(f(E_3))$ is

$$(\lambda - g_{33})[\lambda^2 - (g_{22} + g_{11})\lambda + g_{11}g_{22} - g_{12}g_{21}] = 0.$$

Then the eigenvalues satisfy the characteristic equation is $\lambda_1 = g_{33}$, $\lambda_{2,3} = \frac{(g_{11} + g_{22}) \pm \sqrt{(g_{11} + g_{22})^2 - 4(g_{11}g_{22} - g_{12}g_{21})}}{2}$. Because $a_1, a_2, b_1, b_2, d_1, d_2, h_1, p > 0$, so that the equilibrium point $E_3$ is locally asymptotically stable local must be $\lambda_1 < 0$ and $\lambda_{2,3} < 0$. Then, $g_{33} < 0$, $(g_{11} + g_{22}) < 0$ and $(g_{11}g_{22} - g_{12}g_{21}) > 0$. So $g_{11} < 0$, $g_{22} < 0$, $g_{12} < 0$, and $g_{21} > 0$. Based on $g_{33} < 0$ and $g_{11} < 0$, obtained $\tilde{y} < \frac{d_2}{a_2 - b_2 d_2}$ dan $\tilde{x} > \frac{b_1 - 1}{2b_1}$. So the equilibrium point$E_3 = (\tilde{x}, \tilde{y}, 0)$ is locally asymptotically stable if $\tilde{x} > \frac{b_1 - 1}{2b_1}$ and $\tilde{y} < \frac{d_2}{a_2 - b_2 d_2}$.

□

**Theorem 4.** *If* $a_1 > \frac{(1 - 2\hat{x})(1 + b_1 \hat{x})^2}{\hat{y}}$ *and* $a_2 < \frac{h_1(1 + b_2 \hat{y})^2}{b_2 \hat{z}}$, *then the equilibrium point* $E_4 = (\hat{x}, \hat{y}, \hat{z})$ *is locally asymptotically stable.*

***Proof.*** The equilibrium point $E_4$ is substituted to the matrix elements of $J_{(f(x,y,z))}$, obtained by the matrix $J(f(E_4))$ that is

$$J(f(E_4)) = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix},$$

Where:

$$m_{11} = 1 - 2\hat{x} - \frac{a_1\hat{y}}{1+b_1\hat{x}} + \frac{a_1b_1\hat{x}\hat{y}}{(1+b_1\hat{x})^2}, \qquad m_{23} = -\frac{a_2\hat{y}}{1+b_2\hat{y}},$$

$$m_{12} = -\frac{a_1\hat{x}}{1+b_1\hat{x}}, \qquad m_{32} = \frac{a_2\hat{z}}{1+b_2\hat{y}} - \frac{a_2b_2\hat{y}\hat{z}}{(1+b_2\hat{y})^2},$$

$$m_{21} = \frac{a_1\hat{y}}{1+b_1\hat{x}} + \frac{a_1b_1\hat{x}\hat{y}}{(1+b_1\hat{x})^2}, \qquad m_{33} = -h_2\hat{z}.$$

$$m_{22} = \frac{a_2b_2\hat{y}\hat{z}}{(1+b_2\hat{y})^2} - h_1\hat{y},$$

So, the characteristic equation for $J(f(E_4))$ is

$$\lambda^3 - (m_{11}+m_{22}+m_{33})\lambda^2 + (m_{22}m_{33} - m_{23}m_{32} + m_{11}m_{22} + m_{11}m_{33} - m_{12}m_{21})\lambda$$
$$-m_{11}m_{22}m_{33} + m_{11}m_{23}m_{32} + m_{12}m_{21}m_{33} = 0$$

or

$$\lambda^3 + C\lambda^2 + D\lambda + E = 0,$$

where $C = m_{11}+m_{22}+m_{33}$, $D = m_{22}m_{33} - m_{23}m_{32} + m_{11}m_{22} + m_{11}m_{33} - m_{12}m_{21}$, and $E = -m_{11}m_{22}m_{33} + m_{11}m_{23}m_{32} + m_{12}m_{21}m_{33}$.

Determination sign eigenvalues $\lambda_1$, $\lambda_2$, dan $\lambda_3$ in the equation using Routh-Hurwitz criterion, that is

$$H = \begin{pmatrix} C & 1 & 0 \\ E & D & C \\ 0 & 0 & E \end{pmatrix},$$

so that the determinant obtained from the Routh-Hurwitz Matrix is

$$\Delta_1 = |C| = C, \qquad \Delta_2 = \begin{vmatrix} C & 1 \\ E & D \end{vmatrix} = CD - E, \qquad \Delta_3 = \begin{vmatrix} C & 1 & 0 \\ E & D & C \\ 0 & 0 & E \end{vmatrix} = E(CD-E).$$

For the characteristic equation to have a real part of the negative eigenvalues it must be $\Delta_1 > 0$, $\Delta_2 > 0$, and $\Delta_3 > 0$, where

$$CD - E = -(m_{11})^2m_{22} - (m_{11})^2m_{33} + m_{11}m_{12}m_{21} - (m_{22})^2m_{33} - m_{11}(m_{22})^2$$
$$-2m_{11}m_{22}m_{33} + m_{12}m_{21}m_{22} + m_{22}m_{23}m_{32} - m_{22}(m_{33})^2 - m_{11}(m_{33})^2$$
$$+m_{23}m_{32}m_{33}.$$

So $m_{11} < 0, m_{12} < 0$, $m_{21} < 0$, $m_{22} < 0$, $m_{23} < 0$, $m_{32} > 0$, and $m_{33} < 0$. Based on $m_{11} < 0$ and $m_{22} < 0$ obtained $a_1 > \frac{(1-2\hat{x})(1+b_1\hat{x})^2}{\hat{y}}$ and $a_2 < \frac{h_1(1+b_2\hat{y})^2}{b_2\hat{z}}$. So the equilibrium point $E_4 = (\hat{x}, \hat{y}, \hat{z})$ is locally asymptotically stable if $a_1 > \frac{(1-2\hat{x})(1+b_1\hat{x})^2}{\hat{y}}$ and $a_2 < \frac{h_1(1+b_2\hat{y})^2}{b_2\hat{z}}$.

□

### 3.3. Model Simulation

Numerical simulations are carried out using the Maple 18 program. Numerical simulations are performed at each of the equilibrium points in System (3) and simulations on variations of several parameters. Numerical simulations at the equilibrium points are performed to determine the behavior of the System (3) around the equilibrium points, while simulations on variations of several parameters are carried out to find out how changes in these parameters affect predator population and top predator population.

The parameter values in this simulation are taken around the parameters of several studies of the Predator-Prey model, that is [6], [11], [13], [16]–[20]. Then for the population of prey (Zooplankton) using research data at Kartini Beach, Jepara City [21]. The parameter values used are $K = 8{,}545777778 \times 10^{13}$ tail; $E_1 = 0$; $E_2 = 0$; $a_1 = 4{,}288$; $a_2 = 2{,}76$; $b_1 = 2{,}312$; $b_2 = 2{,}198$; $d_1 = 0{,}97$; $d_2 = 0{,}353$; $h_1 = 0{,}05$; $h_2 = 0{,}05$; $p = 0{,}152$; and the initial value taken is $x(0) = 0{,}47$; $y(0) = 0{,}61$; $z(0) = 0{,}28$.

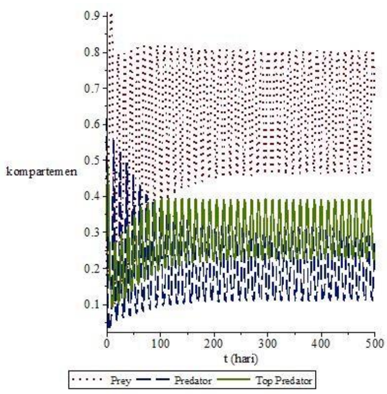


**Figure 2. Simulation of System (1) to the equilibrium point $E_1$**

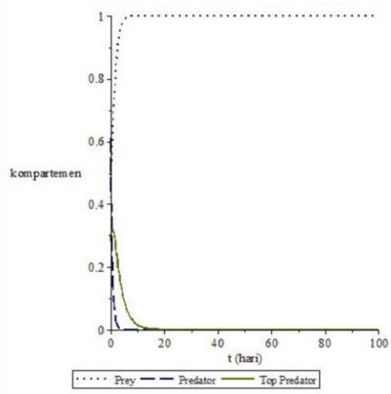


**Figure 3. Simulation of System (1) to the equilibrium point $E_2$**

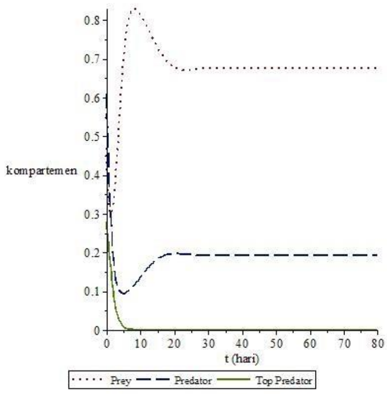


**Figure 4. Simulation of System (1) to the equilibrium point $E_3$**

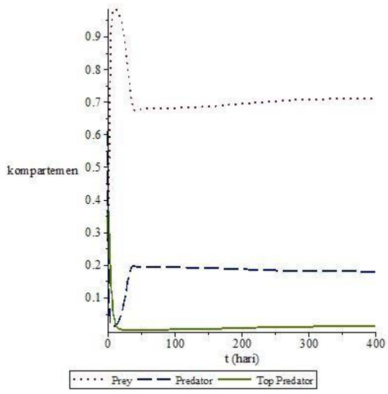


**Figure 5. Simulation of System (1) to the equilibrium point $E_4$**

Numerical simulations are carried out at the equilibrium point $E_1 = (0, 0, 0)$ by reducing the mortality rate of predators to $d_1 = 0{,}36$, the results of numerical simulations are shown in Figure 2. Figure 2 shows the instability where the population of prey, predator, and top predator is away from the equilibrium point $(0{,}0{,}0)$. That is, the ecosystem does not occur simultaneously in all three populations.

If the coefficient of interaction between prey and predator lowered to $a_1 = 0{,}5$ conditions in Theorem 1 are met, so that the three populations is shown in Figure 3. Figure 3 shows System (5) towards the equilibrium point $E_2 = (1, 0, 0)$, wherein the predator population is extinct or decreases and is stable at point 0. As a result, the top predator population decreases to extinction and is stable at point 0, while the prey population increases and is stable at point 1.

If the top predator mortality rate is increased $d_2 = 1{,}055$ all three populations will go to the equilibrium point $E_3 = (\tilde{x}, \tilde{y}, 0)$ where $\tilde{x} = 0{,}6769937155$ and $\tilde{y} = 0{,}1932319916$. Then the conditions in Theorem 2 are fulfilled and the results of numerical simulations for the equilibrium point $E_3$ are shown in Figure 4. The population of top predators decreases until it reaches point 0 and is stable at that point, which means it is experiencing extinction. As a result, the predator population increased and stabilized at point 0,1932319916. While the prey population declined and stabilized at the point 0,6769937155.

If the parameter values used in this study, the equilibrium point $E_4 = (\hat{x}, \hat{y}, \hat{z})$ with $\hat{x} = 0{,}7107453472$, $\hat{y} = 0{,}1783046657$, and $\hat{z} = 0{,}01114087784$. Then the conditions in Theorem 3 are fulfilled and the numerical simulation results for the equilibrium point $E_4$ are shown in Figure 5. The prey population decreases to 0,672, then increases to 0,7107453472 and is stable at that point. The population of predators increases to 0,196, then slightly decreases until it reaches 0,1783046657 and is stable at that point. The top predator population decreases and is at the point 0,003, then rises to the point 0,01114087784 and is stable at that point.

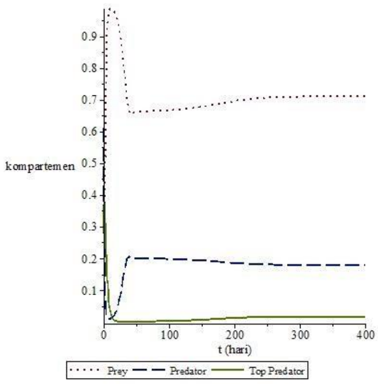


**Figure 6. Simulation of parameter variations $h_1 = 0, h_2 = 0$**

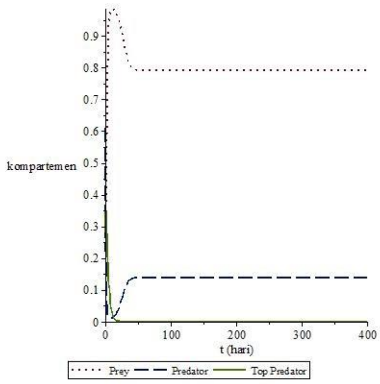


**Figure 7. Simulation of parameter variations $h_1 = 0,56, h_2 = 0$**

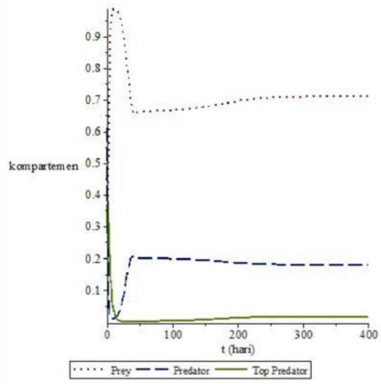


**Figure 8. Simulation of parameter variations $h_1 = 0, h_2 = 0,0004$**

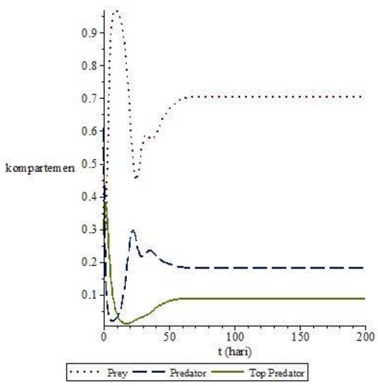


**Figure 9. Simulation of parameter variations $p = 0$**

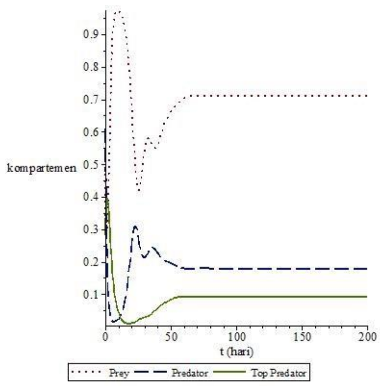


**Figure 10. Simulation of parameter variations $h_1 = 0, h_2 = 0$, and $p = 0$**

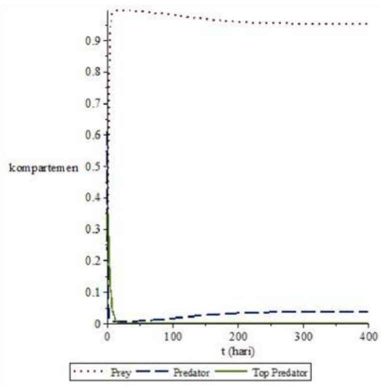


**Figure 11. Simulation of parameter variations $p = 0,304$**

Then a numerical simulation is carried out by varying parameters $h_1$ (rate of competition between living things in predator population), $h_2$ (rate of competition between living things in top predator population), and $p$ (predator harvesting rate). Figure 6 shows when $h_1 = 0$, $h_2 = 0$, the population of prey, predator, and top predator goes to the same equilibrium point as in Figure 5. However, going towards stability shows a greater increase or decrease compared to Figure 5. In Figure 7 shows when $h_1 = 0,56$, $h_2 = 0$, the top predator population is extinct. Figure 8 shows when $h_1 = 0$, $h_2 = 0,0004$, the population of prey, predator, and top predator experiences the same stability as in Figure 6.

Furthermore, Figure 9 shows when $p = 0$, the population of prey decreases until it reaches 0,456, then increases until it reaches 0,71 and is stable at that point. The predator population increases until it reaches the point 0,295, then decreases until it reaches the point 0,18 and is stable at that point. The population of top predators increases until it reaches a point of 0,087 and is stable at that point. Figure 10 shows when $h_1 = 0$, $h_2 = 0$, and $p = 0$, the population of prey decreases until it reaches point 0,422 and then increases until it reaches the point of 0,71 and is stable at that point. The predator population increases until it reaches 0,312, then decreases until it reaches 0,18 and is stable at that point. The population of top predators increases, until it reaches the point 0,095 and is stable at that point. In Figure 11 when p is enlarged twice to $p = 0,304$, the predator population approaches extinction, causing the top predator population to experience extinction.

## 4. CONCLUSIONS

1. Based on the assumptions set in this study, a three-species food chain mathematical model was formed with intraspecific competition and harvesting on predators, namely in System (3).
2. The system has four equilibrium points namely $E_1 = (0,0,0)$, $E_2 = (1,0,0)$, $E_3 = (\tilde{x}, \tilde{y}, 0)$, and $E_4 = (\hat{x}, \hat{y}, \hat{z})$. Stability analysis was performed at the four equilibrium points, the $E_1$ equilibrium point was

found to be unstable, while the equilibrium points $E_2$, $E_3$, dan $E_4$ were locally asymptotically stable with certain conditions.

3. Numerical simulations were carried out with the Maple 18 program using parameter values taken from research related to the Predator-Prey model. Numerical simulations at the equilibrium points show the same results as the results of the analysis. Numerical simulations with variations of several parameters show the parameters$h_1$ and $h_2$ have little effect on changes in predator and top predator populations. At the $h_1$ and $h_2$ values which are quite low, System (3) towards a stable point experiences a smaller increase or decrease than without intraspecific competition. This is due to the reduction in predator and top predator populations due to intraspecific competition. While the predator harvest parameters affect the predator population and top predator.

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