AN EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION OF THE DIRICHLET PROBLEM WITH THE DATA IN MORREY SPACES

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Abstract. Let $n - 2 < \lambda < n$, $f$ be a function in Morrey spaces $L^{1,\lambda}(\Omega)$, and the equation

$$\begin{cases}
    Lu = f, \\
    u \in W^{1,2}_0(\Omega),
\end{cases}$$

be a Dirichlet problem, where $\Omega$ is a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, and $L$ is a divergent elliptic operator. In this paper, we prove the existence and uniqueness of this Dirichlet problem by directly using the Lax-Milgram Lemma and the weighted estimation in Morrey spaces.

Keywords: Morrey spaces, Dirichlet problem, elliptic equations.

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1. INTRODUCTION

Let $\Omega$ be a bounded and open subset of $\mathbb{R}^n$, where $n \geq 3$, and $l$ be the diameter of $\Omega$. For every $a \in \Omega$ and $r > 0$, we define

$$B(a, r) = \{y \in \mathbb{R}^n : |y - a| < r\},$$

and

$$\Omega(a, r) = \Omega \cap B(a, r) = \{y \in \Omega : |y - a| < r\}.$$

The Morrey spaces $L^{p, \lambda}(\Omega)$ is defined to be the set of all functions $f \in L^p(\Omega)$ which satisfy

$$\|f\|_{L^{p, \lambda}(\Omega)} = \sup_{a \in \Omega, r > 0} \left(\frac{1}{r^\lambda} \int_{\Omega(a, r)} |f(y)|^p dy\right)^{\frac{1}{p}} < \infty,$$

for $1 \leq p < \infty$ and $0 \leq \lambda \leq n$. This Morrey spaces were introduced by C. B. Morrey [1] and still attracted the attention of many researcher to investigate its inclusion properties or application in partial differential equation [2, 3, 4, 5, 6, 7, 8].

Let $W^{1,2}(\Omega)$ be the Sobolev space equipped by the norm

$$\|u\|_{W^{1,2}(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \sum_{i=1}^n \|\frac{\partial u}{\partial x_i}\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} = \left(\int_{\Omega} |u|^2 + \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{1}{2}}$$

for every $u \in W^{1,2}(\Omega)$. The closure of $C_c^\infty(\Omega)$ under the Sobolev norm is denoted by $W^{1,2}_0(\Omega)$. It is well known that $W^{1,2}_0(\Omega)$ is a Hilbert space, that is, the Sobolev norm is generated by an inner or scalar product on $W^{1,2}_0(\Omega)$.

We consider the following second order divergent elliptic operator

$$Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{i,j} \frac{\partial u}{\partial x_i}),$$  \hspace{1cm} (1)

where $u \in W^{1,2}_0(\Omega)$,

$$a_{i,j} \in L^\infty(\Omega), \hspace{1cm} i,j = 1, \ldots, n,$$

and there exists $\nu > 0$ such that

$$\nu |\xi|^2 \leq \sum_{i,j=1}^n a_{i,j}(x)\xi_i \xi_j,$$  \hspace{1cm} (3)

doing for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and for almost every $x \in \Omega$.

Let $f \in L^{1,2}(\Omega)$. In this paper, we will investigate the existence and uniqueness of the weak solution to the equation

$$\begin{cases}
Lu = f, \\
u \in W^{1,2}_0(\Omega),
\end{cases}$$  \hspace{1cm} (4)

where $L$ is defined by (1) and the $\lambda$ satisfies a certain condition. The Eq. (4) is called the Dirichlet problem.

The function $u \in W^{1,2}_0(\Omega)$ is called the weak solution of the Dirichlet problem (4) if

$$\int_{\Omega} \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx = \int_{\Omega} f(x) \phi(x) dx,$$  \hspace{1cm} (5)

for every $\phi \in W^{1,2}_0(\Omega)$.

Recently, Tumalun and Tuerah [9], continue the work of Di Fazio [10, 11], proved that the weak solution of gradient of (4) belongs to some weak Morrey spaces by assuming $f \in L^{1,\lambda}(\Omega)$ for $0 < \lambda < n - 2$. Notice that, the result in [9], generalized by themselves in [12]. In [9, 10, 11], the authors used a representation of the weak solution, which involves the Green function [13], and proved that this representation satisfies (5) to show the existence of the weak solution of (4). Cirmi et al. [14] proved that the weak solution of (4) exists and unique, and its gradient belongs to some Morrey spaces, where they assumed $f \in L^{1,\lambda}(\Omega)$ for $n - 2 < \lambda < n$. The proof of the existence and uniqueness of the weak solution, which is done by Cirmi et. al, used an approximation method.
By assuming $f \in L^{1,\lambda}(\Omega)$, for $n - 2 < \lambda < n$, in this paper we will give a direct proof of the existence and uniqueness of the weak solution of the Dirichlet problem (4). Our method uses a functional analysis tool, i.e. the Lax-Millgram lemma, combining with a weighted embeddings in Morrey and Sobolev spaces.

2. RESEARCH METHODS

The constant $C = C(\alpha, \beta, \ldots, \gamma)$, which appears throughout this paper, denotes that it is dependent on $\alpha, \beta, \ldots, \gamma$. The value of this constant may vary from line to line whenever it appears in the theorems or proofs.

Our method relies on functional analysis tools, that is Lax-Milgram lemma, that we will state in this section. We start by write down some properties related to Lax-Milgram lemma.

Let $H$ be a Hilbert space with norm $\| \cdot \|$ and $B: H \times H \rightarrow \mathbb{R}$ be a bilinear mapping. The map $B$ is called continuous if there exists a constant $C_1 > 0$ such that

$$ |B(u, w)| \leq C_1 \| u \| \| w \|, $$

for all $u, w \in H$, and called coercive if there exists a constant $C_2 > 0$ such that

$$ B(u, u) \geq C_2 \| u \|^2, $$

for all $u \in H$.

The following lemma is known as Lax-Milgram lemma and $H$ is the Hilbert space with norm $\| \cdot \|$. We refer to [15] for its proof.

**Lemma 1** (Lax-Milgram). Let $B: H \times H \rightarrow \mathbb{R}$ be a continuous and coercive bilinear mapping. Then, for every bounded linear functional $F: H \rightarrow \mathbb{R}$, there exists a unique element $u \in H$ such that

$$ F(w) = B(u, w), $$

for every $w \in H$.

We associate the operator $L$ with the mapping $B: W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega) \rightarrow \mathbb{R}$ defined by the formula

$$ B(u, \phi) = \int_\Omega \sum_{ij=1}^n a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial \phi(x)}{\partial x_j} \, dx. $$

(6)

For $n - 2 < \lambda < n$ and $f \in L^{1,\lambda}(\Omega)$, we define $F_f: W^{1,2}_0(\Omega) \rightarrow \mathbb{R}$ by the formula

$$ F_f(\phi) = \int_\Omega f(x)\phi(x) \, dx. $$

(7)

By the linearity of the weak derivative and integration, it is easy to show that $B$, defined by (6), is a bilinear mapping. Notice that, according to (5), (6), and (7), $u \in W^{1,2}_0(\Omega)$ is a weak solution of (4) if

$$ B(u, \phi) = F_f(\phi), $$

for every $\phi \in W^{1,2}_0(\Omega)$.

Now we state the following two theorems regarding to the estimation for any functions in $W^{1,2}_0(\Omega)$, that we will need later. The first theorems called Poincaré’s inequality (see [15] for its proof) and the second theorem called sub representation formula (see [16] for its proof).

**Theorem 1** (Poincaré’s Inequality). If $u \in W^{1,2}_0(\Omega)$, then there exists a positive constant $C = C(l)$ such that

$$ \int_\Omega |u|^2 \leq C \int_\Omega |\nabla u|^2. $$

**Theorem 2** (Sub Representation Formula). If $u \in W^{1,2}_0(\Omega)$, then there exists a positive constant $C = C(n)$ such that

$$ |u(x)| \leq C \int_\Omega \frac{|\nabla u(y)|}{|x - y|^{n-1}} \, dy, $$

for a. e. $x \in \Omega$. 

We close this section by stating the following Theorem which slightly modified from [17].

**Theorem 3.** Let \( n - 2 < \lambda < n \). If \( f \in L^{1,\lambda}(\Omega) \), then there exists a positive constant \( C = C(n, \lambda, l) \) such that
\[
\int_{\Omega} \frac{|f(x)|}{|z - x|^{n-2}} \, dx \leq C \| f \|_{L^{1,\lambda}}
\]
for every \( z \in \Omega \).

### 3. RESULTS AND DISCUSSION

To start our discussion, we prove that the bilinear mapping \( B \) defined by (6) is continuous and coercive.

**Lemma 2.** The mapping \( B \) defined by (6) is continuous and coercive.

**Proof.** Let \( u \in W_0^{1,2}(\Omega) \). We first prove the coercivity property. By using (3) and then Poincaré’s inequality, we have
\[
B(u, u) = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial u(x)}{\partial x_j} \, dx 
\geq \nu \int_{\Omega} |\nabla u|^2 
\geq \nu \frac{1}{2} \int_{\Omega} |\nabla u|^2 + C \int_{\Omega} |u|^2 
\geq C \left( \| u \|_{W^{1,2}(\Omega)}^2 \right)^2,
\]
where \( C = C(\nu, l) \) is a positive constant.

Now, we prove the continuity property. Let \( u, \phi \in W_0^{1,2}(\Omega) \). Note that,
\[
A = \sum_{i,j=1}^{n} \| a_{i,j} \|_{L^\infty(\Omega)}
\]
according to (2). By using Hölder’s inequality, we have
\[
|B(u, \phi)| \leq \int_{\Omega} \sum_{i,j=1}^{n} \| a_{i,j} \|_{L^\infty(\Omega)} \left| \frac{\partial u(x)}{\partial x_i} \right| \left| \frac{\partial \phi(x)}{\partial x_j} \right| \, dx 
\leq A \int_{\Omega} |\nabla u(x)| \| \nabla \phi(x) \| \, dx 
\leq A \left( \int_{\Omega} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \phi(x)|^2 \, dx \right)^{\frac{1}{2}} 
\leq A \| u \|_{W^{1,2}(\Omega)} \| \phi \|_{W^{1,2}(\Omega)}.
\]
This completes the proof. \( \square \)

We need the theorem below to prove that the function \( F_f \) defined by (7) is a bounded linear functional. This theorem states about a weighted estimation in Morrey spaces where the weight in Sobolev spaces. The proof of this theorem was given in [11]. However, the given proof did not complete. Here we give the complete proof.

**Theorem 4.** Let \( n - 2 < \lambda < n \). If \( f \in L^{1,\lambda}(\Omega) \), then there exists a positive constant \( C = C(n, \lambda, l) \) such that
\[
\int_{\Omega} |fu| \leq C \| f \|_{L^{1,\lambda}(\Omega)} \| u \|_{W^{1,2}(\Omega)}
\]
for every \( u \in W_0^{1,2}(\Omega) \).

**Proof.** Let \( u \in W_0^{1,2}(\Omega) \). According to the sub representation formula of \( u \) and Hölder’s inequality, we have
\[
\int_{\Omega} |f(x)u(x)| \, dx \leq C(n) \int_{\Omega} |f(x)| \left( \int_{\Omega} \frac{|\nabla u(y)|}{|x - y|^{n-1}} \, dy \right) \, dx
\]
\[
= C(n) \int_{\Omega} |\nabla u(y)| \left( \int_{\Omega} \frac{|f(x)|}{|x - y|^{n-1}} \, dx \right) \, dy
\]
\[
\leq C(n) \| \nabla u \|_{L^2(\Omega)} \left( \int_{\Omega} \left( \int_{\Omega} \frac{|f(x)|}{|x - y|^{n-1}} \, dx \right)^2 \, dy \right)^{\frac{1}{2}}. \tag{9}
\]

Notice that
\[
\int_{\Omega} |f(z)| \, dz \leq C(n, \lambda, l) \| f \|_{L^{1,\lambda}(\Omega)}. \tag{10}
\]

Hence
\[
\int_{\Omega} \left( \int_{\Omega} \frac{|f(x)|}{|x - y|^{n-1}} \, dx \right)^2 \, dy = \int_{\Omega} \left( \int_{\Omega} \frac{|f(z)|}{|x - y|^{n-1}} \, dx \right) \left( \int_{\Omega} \frac{|f(x)|}{|x - y|^{n-1}} \, dx \right) \, dy
\]
\[
= \int_{\Omega} \int_{\Omega} |f(z)||f(x)| \left( \int_{\Omega} \frac{1}{|x - y|^{n-1}|x - y|^{n-1}} \, dx \right) \, dy \, dz
\]
\[
\leq C(n) \int_{\Omega} \int_{\Omega} |f(z)||f(x)| \left( \int_{\Omega} \frac{1}{|x - y|^{n-1} |x - y|^{n-1}} \, dy \right) \, dz
\]
\[
= C(n) \int_{\Omega} |f(z)| \left( \int_{\Omega} \frac{|f(x)|}{|x - y|^{n-1}} \, dx \right) \, dz
\]
\[
\leq C(n, \lambda, l) \| f \|_{L^{1,\lambda}(\Omega)} \int_{\Omega} |f(z)| \, dz
\]
\[
\leq C(n, \lambda, l) \left( \| f \|_{L^{1,\lambda}(\Omega)} \right)^2. \tag{11}
\]

by virtue of Theorem 3 and (10). Combining (9) and (11), we obtain
\[
\int_{\Omega} |f(x)u(x)| \, dx \leq C(n) \| \nabla u \|_{L^2(\Omega)} \left( \int_{\Omega} \left( \int_{\Omega} \frac{|f(x)|}{|x - y|^{n-1}} \, dx \right)^2 \, dy \right)^{\frac{1}{2}}
\]
\[
\leq C(n, \lambda, l) \| \nabla u \|_{L^2(\Omega)} \| f \|_{L^{1,\lambda}(\Omega)}
\]
\[
\leq C(n, \lambda, l) \| u \|_{W^{1,2}(\Omega)} \| f \|_{L^{1,\lambda}(\Omega)}.
\]

The theorem is proved. \(\square\)

From Theorem 4, we obtain the following corollary.

**Corollary 1.** Let \( n - 2 < \lambda < n \). The mapping \( F_f: W^{1,2}_0(\Omega) \to \mathbb{R} \) defined by (7) is a bounded linear functional.

**Proof.** According to the linearity of integration, \( F_f \) is a linear functional on \( W^{1,2}_0(\Omega) \). For every \( u \in W^{1,2}_0(\Omega) \), Theorem 4 gives us
\[
|F_f(u)| \leq \int_{\Omega} |fu| \leq C \| u \|_{W^{1,2}(\Omega)},
\]
where the positive constant \( C = C(n, \lambda, l, \| f \|_{L^{1,\lambda}(\Omega)}) \). This means \( F_f \) is also bounded and the proof is complete. \(\square\)
Combining Lemma 2, Corollary 1, and Lax-Milgram lemma, we now state the following existence and uniqueness of the weak solution of the Dirichlet problem (4).

**Theorem 5.** Let $n - 2 < \lambda < n$ and $f \in L^{1,\lambda}(\Omega)$ in Dirichlet problem (4). Then there exists a unique element $u \in W^{1,2}_0(\Omega)$ which is the weak solution of the Dirichlet problem (4).

### 4. CONCLUSIONS

The Dirichlet problem (4) has a unique weak solution by assuming the data belongs some Morrey spaces. This fact can be proved by using functional analysis tool, i.e. the Lax-Milgram lemma, combining with the weighted embedding in Morrey spaces where the weight in Soblev spaces.

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