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FLUID FLOW MODELING WITH FREE SURFACE

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Abstract. *This study discussed modelling fluid flow with a free surface and a submerged obstacle in the fluid flow. To build the mathematical model, we assumed the fluid is incompressible, steady-state, and irrotational. Firstly, we used Newton's second law, the law of mass conservation and the law of conservation of momentum to obtain the general Navier-Stokes equation. Then, the Euler-free surface equation and the Bernoulli equation were designed before making a free surface representation and linearizing the wave equation to obtain a fluid flow model. The resulting mathematical model is a Laplace equation with boundary conditions in the fluid.*

Keywords: *incompressible, irrotational, steady-state.*

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1. INTRODUCTION

Differential equations are a branch of mathematics that play a role in solving problems related to the real world. It deals with Analysis [2], Algebra [3], Geometry [4], and others. Most problems in differential equations are finding exact (analytic) solutions from mathematical models derived from real problems. [5].

Fluid modelling is one of the mathematical models obtained from the behaviour of a fluid. A fluid is a substance that can flow in the form of a liquid or gas [6]. A liquid is a substance with a certain volume that fills the container in which the liquid is placed.

In physics, a free surface is a fluid surface with an interface between two homogeneous fluids, such as water and air [7]. Unlike liquids, gases cannot form a free surface by themselves. A liquid will have a free surface if the volume of the liquid is less than the volume of a container.

This research was conducted to obtain a model of the fluid flow and free surface conditions when an object is given to the fluid flow. We assume in this research that the fluid is incompressible [8], ideal [9], irrotational [10], and steady-state [11]. The process of obtaining a model of fluid flow and free surface conditions begins by reconstructing the general Navier-Stokes equation [12] from Newton's second law [13], designing the Euler-free surface equation and Bernoulli equation [14], and representing the free surface as well as the linearizing wave equation [15].

2. RESEARCH METHODS

Newton's second law in terms of momentum can be written as:

$$\sum \vec{F} = \frac{\Delta \vec{p}}{\Delta t} = \frac{\Delta m\vec{v}}{\Delta t}, \tag{1}$$

where $\vec{p}$ is momentum and t represents time. To obtain the Navier-Stokes equation from equation (1), first, equation (1) was changed in the form of the differential operator D. The second step was to derive the equation using the chain rule. Third, the change in momentum of the fluid particles partitioned and represented as a cube was determined. Fourth, each force that affects the fluid particles was deduced based on the change in momentum obtained. After obtaining the Navier-Stokes equation for an incompressible fluid, the fifth step was to assume that the fluid was an ideal fluid by ignoring the fluid's viscosity so that the Euler-free surface equation was obtained. Then, the sixth step was to assume that the fluid has an irrotational flow, namely by proving that the vorticity in the flow is 0. Seventh, the Laplace equation was obtained based on the Helmholtz-Hodge theorem [16], which states that each vector can be expressed into two parts, namely, the part where the divergence is zero and the part where the curl is zero. The eighth step was to determine the boundary conditions on the fluid's free surface, resulting in the Bernoulli system of equations for water waves. Finally, a flow model and fluid-free surface conditions were obtained by linearizing the dynamic and kinetic-free surface conditions in the Bernoulli equation system. The completion steps in this paper are referred to [15].

3. RESULTS AND DISCUSSION

3.1. The Navier-Stokes Equation on a Free Surface

The general equation used in this study is the Navier-Stokes equation. The Navier-Stokes equation in fluid mechanics is a partial differential equation describing an incompressible fluid's flow. The equation is a generalization of the equations made by the Swiss mathematician Leonhard Euler in the 18th century to describe the flow of incompressible and frictionless fluids. In 1821, French engineer Claude-Louis Navier introduced the element of viscosity (friction) for a more realistic and much more difficult problem of viscous fluids. Throughout the mid-19th century, the English physicist and mathematician Sir George Gabriel Stokes developed this research. However, complete solutions were obtained only for the case of simple two-dimensional flows. To obtain the Navier-Stokes equation, firstly we use Newton's second law in Equation (1).

For very small values of Δt, Equation (1) can be written as:

$$\sum \vec{F} = D_t(mv)$$
$$= \frac{\partial mv}{\partial t} + \frac{\partial mv}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial mv}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial mv}{\partial x_3}\frac{\partial x_3}{\partial t}. \tag{2}$$

After obtaining Equation (2), the next step was to determine the change in momentum of the fluid particles. We divided the fluid into small parts. Let us see one partition as in Figure 1.

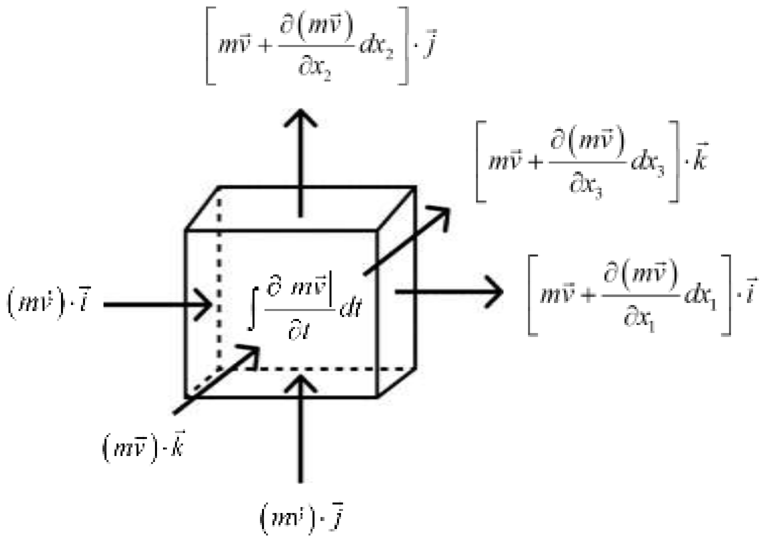

Figure 1. Fluid Particles

From Figure 1, the change in momentum of the fluid particles over time is obtained as follows:

$$\frac{\Delta p}{\Delta t} = \rho\left(\frac{\partial|\vec{v}|}{\partial t} + \left[\frac{\partial(\vec{v})}{\partial x_1}v_{x_1}\right]\cdot\vec{\imath} + \left[\frac{\partial(\vec{v})}{\partial x_2}v_{x_2}\right]\cdot\vec{\jmath} + \left[\frac{\partial(\vec{v})}{\partial x_3}v_{x_3}\right]\cdot\vec{k}\right)dx_1dx_2dx_3. \tag{3}$$

Three forces always act on fluids, namely gravity (g), pressure (p), and viscosity (μ) so that the Navier-Stokes equation can be written as follows:

$$\rho g_{x_1} - \frac{\partial P}{\partial x_1} + \mu\left(\frac{\partial^2 v_{x_1}}{\partial {x_1}^2} + \frac{\partial^2 v_{x_1}}{\partial {x_2}^2} + \frac{\partial^2 v_{x_1}}{\partial {x_3}^2}\right) = \rho\left(\frac{\partial v_{x_1}}{\partial t} + \frac{\partial v_{x_1}}{\partial x_1}v_{x_1} + \frac{\partial v_{x_1}}{\partial x_2}v_{x_2} + \frac{\partial v_{x_1}}{\partial x_3}v_{x_3}\right)$$
$$\rho g_{x_2} - \frac{\partial P}{\partial x_2} + \mu\left(\frac{\partial^2 v_{x_2}}{\partial {x_1}^2} + \frac{\partial^2 v_{x_2}}{\partial {x_2}^2} + \frac{\partial^2 v_{x_2}}{\partial {x_3}^2}\right) = \rho\left(\frac{\partial v_{x_2}}{\partial t} + \frac{\partial v_{x_2}}{\partial x_1}v_{x_1} + \frac{\partial v_{x_2}}{\partial x_2}v_{x_2} + \frac{\partial v_{x_2}}{\partial x_3}v_{x_3}\right)$$
$$\rho g_{x_3} - \frac{\partial P}{\partial x_3} + \mu\left(\frac{\partial^2 v_{x_3}}{\partial {x_1}^2} + \frac{\partial^2 v_{x_3}}{\partial {x_2}^2} + \frac{\partial^2 v_{x_3}}{\partial {x_3}^2}\right) = \rho\left(\frac{\partial v_{x_3}}{\partial t} + \frac{\partial v_{x_3}}{\partial x_1}v_{x_1} + \frac{\partial v_{x_3}}{\partial x_2}v_{x_2} + \frac{\partial v_{x_3}}{\partial x_3}v_{x_3}\right). \tag{4}$$

3.2 Modelling the Neumann-Kelvin Equation

The previously obtained Navier-Stokes equation can only be used in modelling simple fluid flows where there is no object in the flow and also does not have conditions on the free surface as seen in Figure 2.

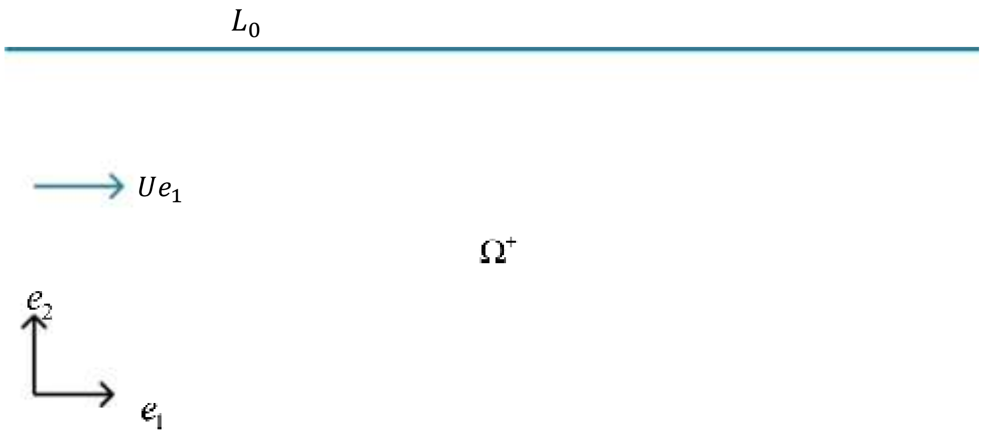

Figure 2. Simple Fluid Flow

However, in this study, the fluid flow modelled has an obstacle submerged in the flow as well as conditions on the free surface as can be seen in Figure 3.

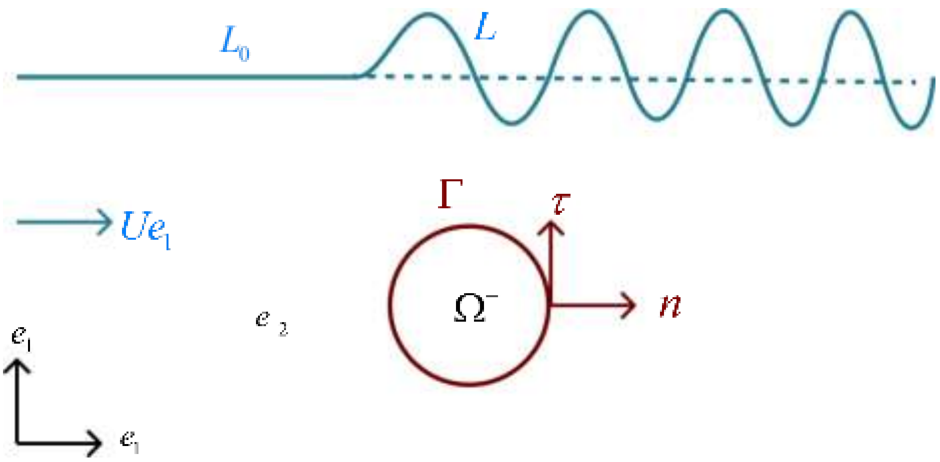

Figure 3. Fluid Flow and An Object

Notation Ω^+ represents the domain of the fluid, and Ω^- expresses the object's domain. The boundary of Ω^+ is a free surface L and the object's surface is denoted as Γ. Vector $\vec{n} = (n_1, n_2)$ represents the normal vector outward, whereas $\vec{\tau}$ expresses the tangential vector. Notation L_0 represents a horizontal line on the surface of the water and represents the velocity of the fluid in the direction e_1.

The Navier-Stokes equation in Equation (4) for a two-dimensional space can be rewritten as:

$$\rho\left(\frac{\partial|\vec{v}|}{\partial t}+\left[\frac{\partial(\vec{v})}{\partial x_1}v_{x_1}\right]+\left[\frac{\partial(\vec{v})}{\partial x_2}v_{x_2}\right]\right)=\mu\left(\frac{\partial^2(\vec{v})}{\partial {x_1}^2}+\frac{\partial^2(\vec{v})}{\partial {x_2}^2}\right)-\nabla Pdxdy+\rho gdxdy$$

$$=\mu\left(\frac{\partial^2(\vec{v})}{\partial {x_1}^2}+\frac{\partial^2(\vec{v})}{\partial {x_2}^2}\right)-\nabla\cdot PId+\rho ge_2$$

$$
\begin{aligned}
&= \left(\mu(\Delta V) - \nabla \cdot PId\right) + \rho g e_2 \\
&= \left(\frac{\mu}{2}(2\Delta V) - \nabla \cdot PId\right) + \rho g e_2 \\
&= \left(\frac{\mu}{2}(\Delta V + \Delta V) - \nabla \cdot PId\right) + \rho g e_2 \\
&= \left(\mu \nabla \cdot \left(\frac{\nabla V + \nabla V}{2}\right) - \nabla \cdot PId\right) + \rho g e_2 \\
&= \nabla \cdot \left(\mu \left(\frac{\nabla V + \nabla V}{2}\right) - \nabla \cdot PId\right) + \rho g e_2 \\
&= \nabla \cdot \sigma - \rho g e_2.
\end{aligned}
$$

Hence, we obtain:

$$
\rho\left(\frac{\partial|\vec{v}|}{\partial t} + \left[\frac{\partial(\vec{v})}{\partial x_1} v_{x_1}\right] + \left[\frac{\partial(\vec{v})}{\partial x_2} v_{x_2}\right]\right) = \nabla \cdot \sigma - \rho g e_2. \tag{5}
$$

For an incompressible fluid, Equation (5) can be written as:

$$
\begin{aligned}
&\rho\,(\partial_t V + (V \cdot \nabla)V) = \nabla \cdot \sigma - \rho g e_2, \\
&\nabla \cdot V = 0,
\end{aligned} \quad \text{on } \Omega^+ \tag{6}
$$

3.3 Ideal Fluid Assumption

Further, we assume that the fluid is ideal. An ideal fluid is a fluid that has no viscosity and is incompressible. Viscosity is a frictional force that occurs due to friction between layers contained in the fluid. If the viscosity possessed by the fluid in Equation (5) is neglected (assuming an ideal fluid), the stress tensor [15] can be written as:

$$
\sigma = PId \tag{7}
$$

Substituting Equation (7) into Equation (6), we obtain the Euler-free surface equation:

$$
\begin{aligned}
&\rho\,(\partial_t V + (V \cdot \nabla)V) = \nabla P - \rho g e_2, && \text{on } \Omega^+ \\
&\nabla \cdot V = 0, && \text{on } \Omega^+
\end{aligned} \tag{8}
$$

3.4 Irrotational Flow Assumption

Furthermore, we assume fluid flow is irrotational. Irrotational flow occurs when the velocity in each fluid layer is the same, which causes the fluid particles not to rotate. Irrotational flow can also be defined as a flow with vorticity equal to zero. Vorticity is a vector quantity that shows the rotational rate of the fluid particles. To get the equation of vorticity, the curl is taken from Equation (6):

$$
\begin{aligned}
\nabla \times \left(\rho(\partial_t V + (V \cdot \nabla)V)\right) &= \nabla \times (\nabla P - \rho g e_2) \\
\nabla \times \left((\partial_t V + (V \cdot \nabla)V)\right) &= \frac{1}{\rho} \nabla \times (\nabla P - \rho g e_2) \\
\nabla \times \partial_t V + \nabla \times \left((V \cdot \nabla)V\right) &= \frac{1}{\rho} \nabla \times (\nabla P - \rho g e_2).
\end{aligned} \tag{9}
$$

Because $e_2 = \nabla x_2$ then Equation (9) becomes:

$$
\begin{aligned}
\partial_t \nabla \times V + \nabla \times \left((V \cdot \nabla)V\right) &= \frac{1}{\rho} \nabla \times (\nabla P - \rho g \nabla x_2) \\
\partial_t \nabla \times V + \nabla \times \left((V \cdot \nabla)V\right) &= \frac{1}{\rho} \nabla \times (\nabla P - \nabla \rho g x_2) \\
\partial_t \nabla \times V + \nabla \times \left((V \cdot \nabla)V\right) &= \frac{1}{\rho} \nabla \times \nabla (P - \rho g x_2),
\end{aligned} \tag{10}
$$

assuming that $\nabla \times V = \omega$, we obtain the result of $(V \cdot \nabla)V$:

$$
\begin{aligned}
(V\cdot\nabla)V &= \left(v_1 i + v_2 j + v_3 k \cdot \frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_3} k\right)V \\
&= \left(\frac{\partial}{\partial x_1} v_1 + \frac{\partial}{\partial x_2} v_2 + \frac{\partial}{\partial x_3} v_3\right) v_1 i + v_2 j + v_3 k \\
&= \left(\frac{\partial v_1}{\partial x_1} v_1 + \frac{\partial v_1}{\partial x_2} v_2 + \frac{\partial v_1}{\partial x_3} v_3\right) i + \left(\frac{\partial v_2}{\partial x_1} v_1 + \frac{\partial v_2}{\partial x_2} v_2 + \frac{\partial v_2}{\partial x_3} v_3\right) j \\
&\quad + \left(\frac{\partial v_3}{\partial x_1} v_1 + \frac{\partial v_3}{\partial x_2} v_2 + \frac{\partial v_3}{\partial x_3} v_3\right) k \\
&= \left(\frac{\partial v_1}{\partial x_1} v_1 + \frac{\partial v_2}{\partial x_1} v_2 + \frac{\partial v_3}{\partial x_1} v_3\right) i + \left(\frac{\partial v_1}{\partial x_2} v_1 + \frac{\partial v_2}{\partial x_2} v_2 + \frac{\partial v_3}{\partial x_2} v_3\right) j \\
&\quad + \left(\frac{\partial v_1}{\partial x_3} v_1 + \frac{\partial v_2}{\partial x_3} v_2 + \frac{\partial v_3}{\partial x_3} v_3\right) k \\
&\quad - \left(\begin{array}{l}
\left(\left(\frac{\partial v_2}{\partial x_1} v_2 - \frac{\partial v_1}{\partial x_2} v_2\right) - \left(\frac{\partial v_1}{\partial x_3} v_3 - \frac{\partial v_3}{\partial x_1} v_3\right)\right) i \\
+\left(\left(\frac{\partial v_3}{\partial x_2} v_3 - \frac{\partial v_2}{\partial x_3} v_3\right) - \left(\frac{\partial v_2}{\partial x_1} v_1 - \frac{\partial v_1}{\partial x_2} v_1\right)\right) j \\
+\left(\left(\frac{\partial v_1}{\partial x_3} v_1 - \frac{\partial v_3}{\partial x_1} v_1\right) - \left(\frac{\partial v_3}{\partial x_2} v_2 - \frac{\partial v_2}{\partial x_3} v_2\right)\right) k
\end{array}\right) \\
&= \frac{1}{2}\nabla|V|^2 - V\times(\nabla\times V) \\
&= \frac{1}{2}\nabla|V|^2 - V\times\omega. \qquad (11)
\end{aligned}
$$

Take the curl on both sides of Equation (11), then it is obtained:

$$
\begin{aligned}
\nabla\times\big((V\cdot\nabla)V\big) &= \nabla\times\nabla\left(\tfrac{1}{2}|V|^2\right) - \nabla\times(V\times\omega) \\
\nabla\times\big((V\cdot\nabla)V\big) &= -\nabla\times(V\times\omega). \qquad (12)
\end{aligned}
$$

Assuming that $\omega = \omega_1 i + \omega_2 j + \omega_3 k$, Equation (12) becomes:

$$
\begin{aligned}
\nabla\times\big((V\cdot\nabla)V\big) &= -\nabla\times(V\times\omega) \\
&= -\big(\nabla\times(v_1 i + v_2 j + v_3 k \times \omega_1 i + \omega_2 j + \omega_3 k)\big) \\
&= -\left(\nabla\times\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}\right) \\
&= -\left(\begin{array}{l} \nabla\times(v_2\omega_3 - v_3\omega_2) i + (v_3\omega_1 - v_1\omega_3) j \\ +(v_1\omega_2 - v_2\omega_1) k \end{array}\right) \\
&= -\left(\frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_3} k \times (v_2\omega_3 - v_3\omega_2) i + (v_3\omega_1 - v_1\omega_3) j + (v_1\omega_2 - v_2\omega_1) k\right)
\end{aligned}
$$

$$= -\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ (v_2\omega_3 - v_3\omega_2) & (v_3\omega_1 - v_1\omega_3) & (v_1\omega_2 - v_2\omega_1) \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial x_3}(v_3\omega_1 - v_1\omega_3) - \frac{\partial}{\partial x_2}(v_1\omega_2 - v_2\omega_1)\right)i + \left(\frac{\partial}{\partial x_1}(v_1\omega_2 - v_2\omega_1) - \frac{\partial}{\partial x_3}(v_2\omega_3 - v_3\omega_2)\right)j$$

$$+ \left(\frac{\partial}{\partial x_2}(v_2\omega_3 - v_3\omega_2) - \frac{\partial}{\partial x_1}(v_3\omega_1 - v_1\omega_3)\right)k$$

$$= \left(\left(\frac{\partial v_1}{\partial x_2}\omega_2 + \frac{\partial \omega_2}{\partial x_2}v_1\right) - \left(\frac{\partial v_2}{\partial x_2}\omega_1 + \frac{\partial \omega_1}{\partial x_2}v_2\right)\right)i - \left(\left(\frac{\partial v_3}{\partial x_3}\omega_1 + \frac{\partial \omega_1}{\partial x_3}v_3\right) - \left(\frac{\partial v_1}{\partial x_3}\omega_3 + \frac{\partial \omega_3}{\partial x_3}v_1\right)\right)i$$

$$+ \left(\left(\frac{\partial v_2}{\partial x_3}\omega_3 + \frac{\partial \omega_3}{\partial x_3}v_2\right) - \left(\frac{\partial v_3}{\partial x_3}\omega_2 + \frac{\partial \omega_2}{\partial x_3}v_3\right)\right)j - \left(\left(\frac{\partial v_1}{\partial x_3}\omega_2 + \frac{\partial \omega_2}{\partial x_3}v_1\right) - \left(\frac{\partial v_2}{\partial x_3}\omega_1 + \frac{\partial \omega_1}{\partial x_3}v_2\right)\right)j$$

$$+ \left(\left(\frac{\partial v_3}{\partial x_1}\omega_1 + \frac{\partial \omega_1}{\partial x_1}v_3\right) - \left(\frac{\partial v_1}{\partial x_1}\omega_3 + \frac{\partial \omega_3}{\partial x_1}v_1\right)\right)k - \left(\left(\frac{\partial v_2}{\partial x_3}\omega_3 + \frac{\partial \omega_3}{\partial x_3}v_2\right) - \left(\frac{\partial v_3}{\partial x_3}\omega_2 + \frac{\partial \omega_2}{\partial x_3}v_3\right)\right)k$$

$$= \left(\frac{\partial v_1}{\partial x_1}\omega_1 + \frac{\partial v_2}{\partial x_2}\omega_1 + \frac{\partial v_3}{\partial x_3}\omega_1\right)i + \left(\frac{\partial v_1}{\partial x_1}\omega_2 + \frac{\partial v_2}{\partial x_2}\omega_2 + \frac{\partial v_3}{\partial x_3}\omega_2\right)j$$

$$+ \left(\frac{\partial v_1}{\partial x_1}\omega_3 + \frac{\partial v_2}{\partial x_2}\omega_3 + \frac{\partial v_3}{\partial x_3}\omega_3\right)k - \left(\frac{\partial v_1}{\partial x_1}\omega_1 + \frac{\partial v_1}{\partial x_2}\omega_2 + \frac{\partial v_1}{\partial x_3}\omega_3\right)i$$

$$- \left(\frac{\partial v_2}{\partial x_1}\omega_1 + \frac{\partial v_2}{\partial x_2}\omega_2 + \frac{\partial v_2}{\partial x_3}\omega_3\right)j - \left(\frac{\partial v_3}{\partial x_1}\omega_1 + \frac{\partial v_3}{\partial x_2}\omega_2 + \frac{\partial v_3}{\partial x_3}\omega_3\right)k$$

$$- \left(\frac{\partial \omega_1}{\partial x_1}v_1 + \frac{\partial \omega_2}{\partial x_2}v_1 + \frac{\partial \omega_3}{\partial x_3}v_1\right)i - \left(\frac{\partial \omega_1}{\partial x_1}v_2 + \frac{\partial \omega_2}{\partial x_2}v_2 + \frac{\partial \omega_3}{\partial x_3}v_2\right)j$$

$$- \left(\frac{\partial \omega_1}{\partial x_1}v_3 + \frac{\partial \omega_2}{\partial x_2}v_3 + \frac{\partial \omega_3}{\partial x_3}v_3\right)k + \left(\frac{\partial \omega_1}{\partial x_1}v_1 + \frac{\partial \omega_1}{\partial x_2}v_2 + \frac{\partial \omega_1}{\partial x_3}v_3\right)i$$

$$+ \left(\frac{\partial \omega_2}{\partial x_1}v_1 + \frac{\partial \omega_2}{\partial x_2}v_2 + \frac{\partial \omega_2}{\partial x_3}v_3\right)j + \left(\frac{\partial \omega_3}{\partial x_1}v_1 + \frac{\partial \omega_3}{\partial x_2}v_2 + \frac{\partial \omega_3}{\partial x_3}v_3\right)k$$

$$= -(\omega \cdot \nabla)V + \omega(\nabla \cdot V) + (V \cdot \nabla)\omega - V(\nabla \cdot \omega).$$

Hence, we obtain:

$$\nabla \times \left((V \cdot \nabla)V\right) = -(\omega \cdot \nabla)V + \omega(\nabla \cdot V) + (V \cdot \nabla)\omega - V(\nabla \cdot \omega). \tag{13}$$

Because $\nabla \cdot V = 0$ at Equation (6) and $\nabla \cdot \omega = 0$ at Equation (13), then Equation (13) becomes:

$$\nabla \times (V \cdot \nabla)V = -(\omega \cdot \nabla)V + (V \cdot \nabla)\omega, \tag{14}$$

by substituting Equation (14) into Equation (9), then we get:

$$\partial_t \nabla \times V + \nabla \times \left(\left(V \cdot \nabla \right) V \right) = \frac{1}{\rho} \nabla \times \nabla (P - \rho g x_2)$$

$$\partial_t \nabla \times V + \nabla \times \left(\left(V \cdot \nabla \right) V \right) = 0$$

$$\partial_t \nabla \times V + \left(-\left(\omega \cdot \nabla \right) V + \left(V \cdot \nabla \right) \omega \right) = 0$$

$$\partial_t \nabla \times V + \left(V \cdot \nabla \right) \omega = \left(\omega \cdot \nabla \right) V$$

$$\partial_t \omega + \left(V \cdot \nabla \right) \omega = \left(\omega \cdot \nabla \right) V, \tag{15}$$

from the vorticity equation (15), if $\omega = 0$ is given at a time, then $\omega = 0$ is applied for all time.

The Helmholtz-Hodge theorem states that every vector can be expressed into two parts, namely the part where the divergence is zero and the part where the curl is zero [16]. If the curl of a vector $\vec{F}$ is equal to zero ($\nabla \times \vec{F} = 0$) so $\vec{F}$ can be written as the gradient of a scalar ($\vec{F} = \nabla \Phi$). If the divergence of a vector $\vec{F}$ is equal to zero ($\nabla \cdot \vec{F} = 0$) so $\vec{F}$ can be written as a curl of a vector ($\vec{F} = \nabla \times \vec{A}$). Because $\omega = 0$, there is a scalar Φ so that:

$$V = \nabla \Phi, \tag{16}$$

Because V is also incompressible in Ω^+, then we obtain:

$$\nabla \cdot V = 0$$

$$\nabla \cdot \nabla \Phi = 0$$

$$\nabla^2 \Phi = 0$$

$$\Delta \Phi = 0. \tag{17}$$

Equation (17) is also known as Laplace's equation. Further, Equations (16) and (11) are substituted into Equation (8) to obtain:

$$\rho\left(\partial_t V+(V\cdot\nabla)V\right)=\nabla P-\rho g e_2$$

$$\left(\partial_t V+(V\cdot\nabla)V\right)=\frac{\nabla P}{\rho}-g e_2$$

$$\left(\partial_t V+\left(\frac{1}{2}\nabla|V|^2-V\times(\nabla\times V)\right)\right)=\frac{\nabla P}{\rho}-g\nabla x_2$$

$$\left(\partial_t\nabla\Phi+\left(\frac{1}{2}\nabla|\nabla\Phi|^2-\nabla\Phi\times(\omega)\right)\right)=\frac{\nabla P}{\rho}-g\nabla x_2$$

$$\left(\partial_t\nabla\Phi+\frac{1}{2}\nabla|\nabla\Phi|^2\right)=\frac{\nabla P}{\rho}-g\nabla x_2$$

$$\left(\partial_t\nabla\Phi+\frac{1}{2}\nabla|\nabla\Phi|^2-\frac{\nabla P}{\rho}+g\nabla x_2\right)=0$$

$$\nabla(\partial_t\Phi+\frac{1}{2}|\nabla\Phi|^2-\frac{1}{\rho}P+gx_2)=0$$

$$\partial_t\Phi+\frac{1}{2}|\nabla\Phi|^2-\frac{1}{\rho}P+gx_2=c, \tag{18}$$

where c is a constant. Equation (18) is also known as Bernoulli's equation. In equation (18), the value of P pressure is still unknown, except at the surface (where the dynamic free surface condition is applicable). Because P = 0 at the L surface, the dynamic free surface condition is obtained as follows:

$$\partial_t\Phi+\frac{1}{2}|\nabla\Phi|^2+gx_2=c. \tag{19}$$

3.5 Boundary Conditions on Free Surfaces

From now on, L will be considered as a function describing the free surface of the fluid:

$$L_t=\{(x_1,\eta(t,x_1));x_1\in\mathbb{R}\}, \tag{20}$$

where $\eta(t,x_1)$ is the height of the wave that is formed over time. Let us see any point $(x_1(t),x_2(t))$ on the free surface carried by the fluid flow, from the kinematic free surface conditions we obtain:

$$\frac{d}{dt}x_1(t)=V_1(t,x_1(t),x_2(t)), \tag{21}$$

$$\frac{d}{dt}x_2(t)=V_2(t,x_1(t),x_2(t)), \tag{22}$$

with the definition of η in Equation (20), $x_2(t)=\eta(t,x_1(t))$ is obtained in equation (22). The velocity $V_2(t,x_1(t),x_2(t))$ is the first derivative of $\eta(t,x_1(t))$**.** It can be written as follows:

$$V_2\left(t,x_1(t),\eta\left(t,x_1(t)\right)\right)=\frac{d}{dt}\eta\left(t,x_1(t)\right)$$

$$=\partial_t\eta\left(t,x_1(t)\right)+\frac{d}{dt}x_1(t)\partial_1\eta\left(t,x_1(t)\right)$$

$$=\partial_t\eta\left(t,x_1(t)\right)+V_1\left(t,x_1(t),\eta\left(t,x_1(t)\right)\right)\partial_1\eta\left(t,x_1(t)\right). \tag{23}$$

Equation (23) can be rewritten as a kinematic boundary condition:

$$\partial_t\eta(t,x_1(t)) + V_1\left(t,x_1(t),\eta(t,x_1(t))\right)\partial_1\eta(t,x_1(t)) = V_2(t,x_1,\eta(t,x_1)), \tag{24}$$

where $t \in \mathbb{R}^+$ and $x_1 \in \mathbb{R}$. By this representation, Bernoulli's system of equations for water waves can be written as:

$$\begin{aligned}
&\Delta\Phi = 0, && \text{on } \Omega^+\\
&\partial_t\Phi(t,x_1,\eta(t,x_1)) + \frac{1}{2}\left|\nabla\Phi(t,x_1,\eta(t,x_1))\right|^2 = -g\eta(t,x_1) + c, && \text{for } x_1 \in \mathbb{R}\\
&\partial_t\eta(t,x_1) + \partial_1\Phi(t,x_1,\eta(t,x_1))\partial_1\eta(t,x_1) = \partial_2\Phi(t,x_1,\eta(t,x_1)), && \text{for } x_1 \in \mathbb{R}\\
&\partial_n\Phi = 0, && \text{on } \Gamma
\end{aligned} \tag{25}$$

If it is far from the object, the velocity is assumed to be uniform and horizontal:

$$V \to Ue_1, \text{ for } |x| \to \infty \tag{26}$$

by substituting (16) into Equation (26), we obtain:

$$\nabla\Phi \to Ue_1, \text{ for } |x| \to \infty \tag{27}$$

With $e_1 = \nabla x_1$, Equation (27) can be written as:

$$\nabla\Phi \to U\nabla x_1,$$

$$\nabla\Phi \to \nabla U x_1, \tag{28}$$

subtracting the left and right sides of the term (28) by $\nabla U x_1$, we get:

$$\nabla\Phi - \nabla U x_1 \to \nabla U x_1 - \nabla U x_1$$

$$\nabla(\Phi - Ux_1) \to 0, \text{ for } |x| \to \infty \tag{29}$$

By changing the variable $\Psi = \Phi - Ux_1$ then Equation (29) can be written:

$$\nabla\Psi \to 0, \text{ for } |x| \to \infty \tag{30}$$

substitute $\Phi = \Psi + Ux_1$ to the system of equations (25) under the conditions of the first and second free surfaces, we get:

$$\begin{aligned}
&\Delta\Psi = 0, && \text{on } \Omega^+\\
&\partial_t\Psi(t,x_1,\eta(t,x_1)) + \frac{1}{2}\left|\nabla\Psi(t,x_1,\eta(t,x_1)) + Ue_1\right|^2 = -g\eta(t,x_1) + c, && \text{for } x_1 \in \mathbb{R}\\
&\partial_t\eta(t,x_1) + \left(\partial_1\Psi(t,x_1,\eta(t,x_1)) + U\right)\partial_1\eta(t,x_1) = \partial_2\Psi(t,x_1,\eta(t,x_1)), && \text{for } x_1 \in \mathbb{R}\\
&\partial_n\Psi = -Un\cdot e_1, && \text{on } \Gamma\\
&|\nabla\Psi| \to 0, && \text{for } |x| \to \infty
\end{aligned} \tag{31}$$

3.6 Linearization of the Wave Equation

See equation (23), the function can be approximated using the Taylor series against η, hence we obtain:

$$\begin{aligned}\Psi(t,x_1,\eta(t,x_1)) &= \Psi(t,x_1,0)+\partial_2\Psi(t,x_1,0)(\eta(t,x_1)-0)+\frac{1}{2}\partial_2^2\Psi(t,x_1,0)(\eta(t,x_1)-0)^2\\ &\quad+\frac{1}{3!}\partial_2^3\Psi(t,x_1,0)(\eta(t,x_1)-0)^3+\ldots+\frac{1}{n!}\partial_2^n\Psi(t,x_1,0)(\eta(t,x_1)-0)^n+\ldots\\ &=\Psi(t,x_1,0)+\partial_2\Psi(t,x_1,0)\eta(t,x_1)+o(\eta(t,x_1))\\ &=\Psi(t,x_1,0)+o(\|(\eta,\Psi)\|).\end{aligned} \tag{32}$$

Equation (32) is substituted into the system of Equations (31) for the first and second free surface conditions by keeping only the first-order derivatives so that we get:

$$\begin{aligned} &\Delta\Psi=0, && \text{on } \Omega^+\\ &\partial_t\Psi+U\partial_1\Psi=-g\eta, && \text{on } L_0\\ &\partial_t\eta+U\partial_1\eta=\partial_2\Psi, && \text{on } L_0\\ &\partial_n\Psi=Un\cdot e_1, && \text{on } \Gamma\\ &|\nabla\Psi|\to 0, && \text{for } |x|\to\infty\end{aligned} \tag{33}$$

Steady-state is a condition which states that no change occurs with time. By setting the steady state condition, we have $\partial_t\Psi=\partial_t\eta=0$. Further, the first and second free surface conditions at (33), become:

$$U\partial_1\Psi=-g\eta, \quad \text{on } L_0 \tag{34}$$

$$U\partial_1\eta=\partial_2\Psi, \quad \text{on } L_0 \tag{35}$$

Differentiate Equation (34) with respect to x_1. Further, Equation (35) is substituted to Equation (34), hence the free surface boundary condition is obtained as:

$$\partial_{11}^2\Psi+v\partial_2\Psi=0, \text{ on } L_0 \tag{36}$$

where $v=\frac{g}{U^2}$.

Based on Equations (36) and (33), the system of equations of the free surface and fluid flow conditions is obtained as follows:

$$\begin{aligned} &\Delta\Psi=0, && \text{on } \Omega^+\\ &\partial_{11}^2\Psi+v\partial_2\Psi=0, && \text{on } L_0\\ &\partial_n\Psi=Un\cdot e_1, && \text{on } \Gamma\\ &|\nabla\Psi|\to 0, && \text{for } |x|\to\infty\end{aligned} \tag{37}$$

4. CONCLUSIONS

Based on the results of the research that has been done, a mathematical model is obtained in the form of the Laplace equation $\Delta\Psi = 0$ on Ω^+ along with the boundary conditions in the fluid, the free surface condition, the condition at the obstacle surface and the condition at infinity, namely: $\partial_{11}^2\Psi + v\partial_2\Psi = 0$ at L_0, $\partial_n\Psi = Un\cdot e_1$ at Γ and $|\nabla\Psi| \to 0$ for $|x| \to \infty$.

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