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QUOTIENT SEMINEAR-RINGS OF THE ENDOMORPHISM OF SEMINEAR-RINGS

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Abstract. *A seminear-ring is a generalization of ring. In ring theory, if R is a ring with the multiplicative identity, then the endomorphism R −module R is isomorphic to R. Let S be a seminear-ring. Here, we can construct the set of endomorphism from S to itself denoted by $\overline{End(S)}$. We show that if S is a seminear-ring, then $\overline{End(S)}$ is also a seminear-ring over addition and composition function. We will apply the congruence relation to get the quotient seminear-ring endomorphism. Furthermore, we show the relation between c-ideal and congruence relations. So, we can construct the quotient seminear-ring endomorphism with a c-ideal.*

Keywords: *seminear-ring, congruence relation, c-ideal, quotient seminear-ring, quotient seminear-ring endomorphism.*

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1. INTRODUCTION

The theory of rings grew out of studying two particular classes of rings and polynomial rings in n variables over the real or complex numbers and the integers of an algebraic number field. For the first, David Hilbert (1862-1943) introduced the term ring, but it was not until the second decade of the twentieth century that an entirely abstract definition appeared. Emmy Noether gave the theory of commutative rings a firm axiomatic foundation. From the ring theory, we can construct a homomorphism of ring to preserve the ring structure from one ring to another. The associative ring $End(A) = Hom(A, A)$ consists of all the homomorphism of a ring A into itself. The $End(A)$ forms a ring where the multiplication in $End(A)$ is the composition functions and the addition in $End(A)$ is the addition function. Furthermore, 1_A is the unit element of the ring $End(A)$ [1], [2].

The set of all positive integers, with two binary operations addition and multiplication, constructed another algebraic structure called semirings. A semiring is a non-empty set R with two binary operations addition and multiplication such that R under addition is a commutative monoid, R under multiplicative is a monoid and holds left and right distributive law [3]. The notion of semirings is a generalization of rings introduced by Vandiver in 1935. Several researchers have characterized the many types of properties in semirings. In [4], they represented some results on the ideal theory of commutative semirings with non-zero identity analogues to commutative rings with non-zero identity. Moreover, they also studied quotient semirings analogues to quotient rings.

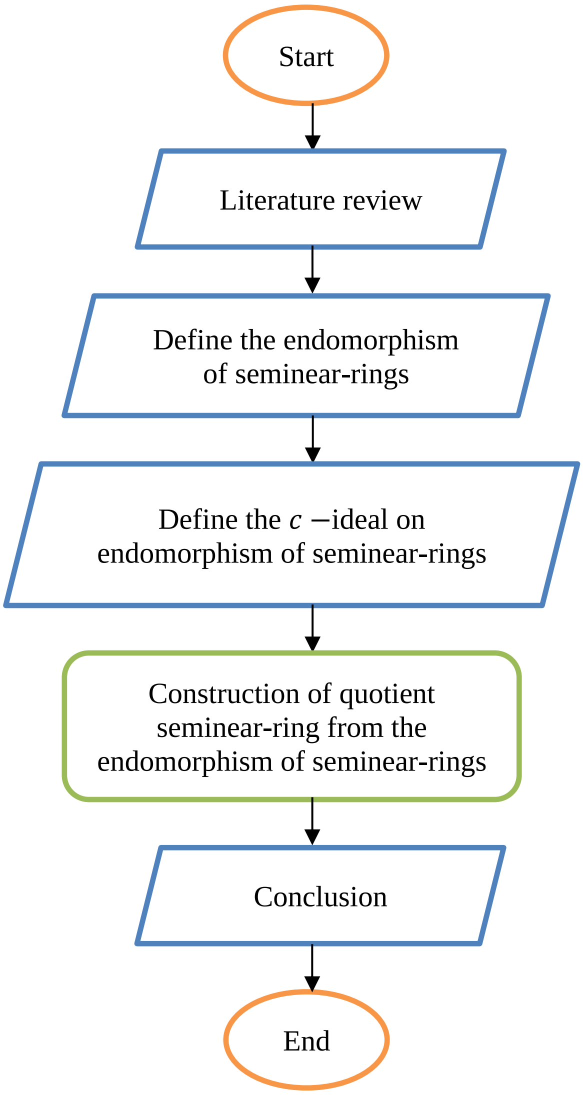
From the set of all natural numbers with two binary operations addition and multiplication, we can construct an algebraic structure called seminear-rings. The notion of seminear-rings which is a generalization of semirings introduced and discussed by Rootselar in 1967 [5]. Seminear-rings are a common generalization of semirings and near-rings [6]. Right seminear-ring is an algebraic system $(S, +, \cdot)$ with two binary associative operations and one distributive law, for all $x, y, z \in S, (x + y)z = xz + yz$ [7]. If we replace the distributive law with $x(y + z) = xy + xz$, S is called a left seminear-ring.

Let $\mathfrak{R}$ be an equivalence relation on a seminear-ring S. Then, the equivalence class corresponding to the element $s \in S$ is denoted by $s\mathfrak{R}$. Furthermore, if $\mathfrak{R}$ is a congruence relation on a seminear-ring S, then $s\mathfrak{R}$ is called a congruence class corresponding to the element $s \in S$. Moreover, the set of all congruence classes denoted by $S/\mathfrak{R}$ formed a quotient seminear-ring together with addition operation and multiplication operation [8], [9], [10]. Throughout this paper, by a seminear-ring, we mean a right seminear-ring. From the idea of quotient seminear-ring, we will investigate the quotient seminear-ring from the endomorphism of seminear-ring. Besides, the endomorphism of the seminear-ring itself is a seminear-ring under the addition and composition function and is denoted by $\left(\overline{End(S)}, +, \circ\right)$.

The concept of quotient seminear-rings have done by other authors in [6], [11], and [12]. Then we generalized the concept of quotient seminear-rings from the endomorphism of seminear-rings on this paper. From the endomorphism of seminear-ring $\left(\overline{End(S)}, +, \circ\right)$, we can construct a quotient seminear-ring from $\left(\overline{End(S)}, +, \circ\right)$ with $\mathfrak{R}$ as a congruence relation on $\overline{End(S)}$. Furthermore, we can call it a quotient seminear-ring from the endomorphism of seminear-ring denoted by $\left(\overline{End(S)}/\mathfrak{R}, +, \circ\right)$. An ideal I on seminear-ring S was introduced [13], [14], [15]. Moreover, it follows from the c-ideal I on seminear-ring S, we can construct a quotient seminear-ring $(S/I, +, \cdot)$. We can generalize the ideal $\overline{End(I)}$ on the endomorphism of seminear-ring $\overline{End(S)}$ based on [16]. It is interesting to show that the quotient seminear-ring from the endomorphism of seminear-ring where c-ideal $\overline{End(I)}$ as a congruence relation on $\overline{End(S)}$ and denoted by $\left(\overline{End(S)}/\overline{End(I)}, +, \circ\right)$.

2. RESEARCH METHODS

In this research, we used the following stages, to begin:



The first stage of studying the quotient seminear-ring from the endomorphism of seminear-ring is to study group theory, ring theory, semiring theory, near-ring theory, and the seminear-ring theory. We studied from several literature such as [**1**], [**2**], [**3**], [**5**], [**6**], [**7**], [**8**], [**9**]. Next, we studied the congruence relation from [**4**], [**10**] to construct the quotient seminear-ring from [**11**]. Furthermore, the concept of endomorphism from the seminear-ring can be studied from the generalization of the endomorphism from the ring theory. Then we have a result that the endomorphism from the seminear-ring is also a seminear-ring. When we studied the quotient seminear-ring and the relation from [**13**], we continued to generalize the quotient of seminear-ring to the quotient of the endomorphism from seminear-ring.

Moreover, we also studied the ideal on seminear-ring [**12**], [**14**], [**15**] and the generalization of ideal on endomorphism of seminear-ring based on [**16**]. The theory of c-ideal, from the ideal on seminear-ring, can be brought to be a congruence relation and a quotient seminear-ring. It follows from the theory of c-ideal that one can obtain the quotient from endomorphism of seminear-ring with c-ideal as a congruence relation.

3. RESULTS AND DISCUSSION

A seminear-ring is a non-empty set S together with two binary operations $'+'$ and $'\cdot'$ (called addition and multiplication) satisfying $(S,+)$ is a semigroup, $(S,\cdot)$ is a semigroup, and for all $a,b,c \in S$ holds $(a+b)\cdot c = (a\cdot c)+(b\cdot c)$. We give the following example of a seminear-ring.

Example 1. Let $\mathbb{N}$ be the set of all natural numbers. Define an ordinary addition of numbers on $\mathbb{N}$ and $a \cdot_L b = a$, for all $a,b \in \mathbb{N}$. Then $(\mathbb{N},+,\cdot_L)$ is a seminear-ring.

Furthermore, the set $\mathbb{N}$ from Example 1. under ordinary addition operation $+$ and right multiplication operation $\cdot_R$ can not be a seminear-ring. Because for $1,2,3 \in \mathbb{N}$ holds $(1+2) \cdot_R 3 = 3$ and $(1 \cdot_R 3) + (2 \cdot_R 3) = 3 + 3 = 6$. So that, $(1+2) \cdot_R 3 \neq (1 \cdot_R 3) + (2 \cdot_R 3)$. Moreover, we give another example of a seminear-ring.

Example 2. Let $\mathbb{Z}_4 = \{0,1,2,3\}$. Define addition operation $(+_\mathbb{Z})$ and multiplication operation $(\cdot_\mathbb{Z})$ on $\mathbb{Z}_4$ i.e.

Table 1. Addition Operation

$+_\mathbb{Z}$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 2. Multiplication Operation

$\cdot_\mathbb{Z}$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	0	0
3	0	1	0	1

The set $\mathbb{Z}_4$ is a seminear-ring under the addition operation $(+_\mathbb{Z})$ and multiplication operation $(\cdot_\mathbb{Z})$.

3.1. Endomorphism of Seminear-rings

Let $(S,+,\cdot)$ and $(T,\oplus,\odot)$ be two seminear-rings. A mapping $f: S \to T$ is called a homomorphism if and only if $f(a+b) = f(a) \oplus f(b)$ and $f(a \cdot b) = f(a) \odot f(b)$. It should be noted that if $T = S$, then the homomorphism f is known as an endomorphism. We denote the endomorphism $f: S \to S$ by $\overline{End(S)}$ is a seminear-ring under the addition and composition function. This result is important, which will lead us to construct the quotient seminear-ring from the endomorphism of the seminear-ring.

Proposition 3. *Let $(S,+,\cdot)$ be a seminear-ring and two binary operations on $\overline{End(S)}$. If for any $f,g \in \overline{End(S)}$, for all $x \in S$ we defined $(f+g)(x) = f(x) + g(x)$ and $(f \circ g)(x) = f(g(x))$, then $(\overline{End(S)},+,\circ)$ is a seminear-ring.*

Proof.

1. Let $f,g \in \overline{End(S)}$ and for all $r \in S$ then $(f+g)(r) = f(r) + g(r) \in S$. Therefore, for every $f,g \in \overline{End(S)}$ we get $f+g \in \overline{End(S)}$. Let $f_1, g_1, f_2, g_2 \in \overline{End(S)}$ where $f_1 = f_2$ and $g_1 = g_2$ and $r \in S$ then

$$\begin{aligned}(f_1+g_1)(r) &= f_1(r) + g_1(r)\\ &= f_2(r) + g_2(r)\\ &= (f_2+g_2)(r)\end{aligned} \tag{1}$$

On the other hand, holds

$$\begin{aligned}(f_1 \circ g_1)(r) &= f_1(g_1(r))\\ &= f_2(g_2(r))\\ &= (f_2 \circ g_2)(r)\end{aligned} \tag{2}$$

So, $\overline{End(S)}$ is well-defined under the addition and composition function.

2. Let $f, g, h \in \overline{End(S)}$ and for all $r \in S$ holds

$$\begin{aligned}\big((f+g)+h\big)(r) &= (f+g)(r)+h(r)\\ &= \big(f(r)+g(r)\big)+h(r)\\ &= f(r)+(g(r)+h(r))\\ &= f(r)+(g+h)(r)\\ &= (f+(g+h))(r)\end{aligned} \tag{3}$$

and

$$\begin{aligned}\big((f\circ g)\circ h\big)(r) &= (f\circ g)(h(r))\\ &= \Big(f\Big(g\big(h(r)\big)\Big)\Big)\\ &= f(r)\circ(g(h(r))\\ &= f(r)\circ(g\circ h)(r)\\ &= (f\circ(g\circ h))(r)\end{aligned} \tag{4}$$

Then $\overline{End(S)}$ associative under addition and composition function.

3. Let $f, g, h \in \overline{End(S)}$ and for all $r \in S$ holds

$$\begin{aligned}\big((f+g)\circ h\big)(r) &= (f+g)(h(r))\\ &= f\big(h(r)\big)+g(h(r))\\ &= (f\circ h)(r)+(g\circ h)(r)\\ &= ((f\circ h)+(g\circ h))(r)\end{aligned} \tag{5}$$

Hence, $\overline{End(S)}$ holds the right distributive law.
It follows that $(\overline{End(S)}, +, \circ)$ is a seminear-ring and is called endomorphism of seminear-ring. □

Based on Proposition 3., we give an example of an endomorphism of seminear-ring $(\overline{End(S)}, +, \circ)$.

Example 4. It follows from Example 2. that $(\mathbb{Z}_4, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}})$ is a seminear-ring. Define f, g, h is a function from $\mathbb{Z}_4$ to $\mathbb{Z}_4$ where:

$f: \mathbb{Z}_4 \to \mathbb{Z}_4$	$g: \mathbb{Z}_4 \to \mathbb{Z}_4$	$h: \mathbb{Z}_4 \to \mathbb{Z}_4$
$0 \mapsto 0$	$0 \mapsto 0$	$0 \mapsto 0$
$1 \mapsto 1$	$1 \mapsto 0$	$1 \mapsto 1$
$2 \mapsto 2$	$2 \mapsto 0$	$2 \mapsto 0$
$3 \mapsto 3$	$3 \mapsto 0$	$3 \mapsto 1$

Moreover, f, g, h is a homomorphism of seminear-ring. Therefore, $f, g, h \in \overline{End(\mathbb{Z}_4)}$.

The relation $\mathfrak{R}$ is the equivalence relation on a non-empty set of seminear-ring $\overline{End(S)}$. The equivalence class corresponding to $f \in \overline{End(S)}$ is denoted by $f\mathfrak{R} \coloneqq \{g \in \overline{End(S)} | (f, g) \in \overline{End(S)}\}$. Suppose $\mathfrak{R}$ is a congruence relation on a seminear-ring $(\overline{End(S)}, +, \circ)$, then $f\mathfrak{R}$ is the congruence classes of f in $\overline{End(S)}$ and is denoted by $\overline{End(S)}/\mathfrak{R}$ where $\overline{End(S)}/\mathfrak{R} \coloneqq \{f\mathfrak{R} | f \in \overline{End(S)}\}$. Now, we state the result of $(\overline{End(S)}/\mathfrak{R}, +, \circ)$ is a quotient seminear-ring.

Theorem 5. *Let $(\overline{End(S)}, +, \circ)$ be a seminear-ring and $\mathfrak{R}$ a congruence relation. Define $\overline{End(S)}/\mathfrak{R} \coloneqq \{f\mathfrak{R} | f \in \overline{End(S)}\}$ with two binary operations such that for all $f\mathfrak{R}, g\mathfrak{R} \in \overline{End(S)}/\mathfrak{R}$ holds $f\mathfrak{R} + g\mathfrak{R} = (f+g)\mathfrak{R}$ and $f\mathfrak{R} \circ g\mathfrak{R} = (f \circ g)\mathfrak{R}$, then $(\overline{End(S)}/\mathfrak{R}, +, \circ)$ is a seminear-ring which is called a quotient seminear-ring from endomorphism of seminear-ring.*

Proof.

1. Let $f\mathfrak{R}, g\mathfrak{R}, h\mathfrak{R}, i\mathfrak{R} \in \overline{End(S)}/\mathfrak{R}$ where $f\mathfrak{R} = g\mathfrak{R}$ and $h\mathfrak{R} = i\mathfrak{R}$. From Definition 3.1 and Lemma 4.2 in [10] then $(f,g),(h,i) \in \mathfrak{R}$ and $(f,g) + (h,i) = (f+h, g+i) \in \mathfrak{R}$ which means $(f+h)\mathfrak{R} = (g+i)\mathfrak{R}$ and $(f,g) \circ (h,i) = (f \circ h, g \circ i) \in \mathfrak{R}$ which means $(f \circ h)\mathfrak{R} = (g \circ i)\mathfrak{R}$. Consequently,
$$\begin{aligned} f\mathfrak{R} + h\mathfrak{R} &= (f+h)\mathfrak{R} \\ &= (g+i)\mathfrak{R} \\ &= g\mathfrak{R} + i\mathfrak{R} \end{aligned} \tag{6}$$
and
$$\begin{aligned} f\mathfrak{R} \circ h\mathfrak{R} &= (f \circ h)\mathfrak{R} \\ &= (g \circ i)\mathfrak{R} \\ &= g\mathfrak{R} \circ i\mathfrak{R} \end{aligned} \tag{7}$$
Hence, $\overline{End(S)}/\mathfrak{R}$ well-defined under the addition and composition function.
2. Let $f\mathfrak{R}, g\mathfrak{R}, h\mathfrak{R} \in \overline{End(S)}/\mathfrak{R}$, then
$$\begin{aligned} (f\mathfrak{R} + g\mathfrak{R}) + h\mathfrak{R} &= (f+g)\mathfrak{R} + h\mathfrak{R} \\ &= ((f+g)+h)\mathfrak{R} \\ &= \big(f + (g+h)\big)\mathfrak{R} \\ &= f\mathfrak{R} + (g+h)\mathfrak{R} \\ &= f\mathfrak{R} + (g\mathfrak{R} + h\mathfrak{R}) \end{aligned} \tag{8}$$
and
$$\begin{aligned} (f\mathfrak{R} \circ g\mathfrak{R}) \circ h\mathfrak{R} &= (f \circ g)\mathfrak{R} \circ h\mathfrak{R} \\ &= ((f \circ g) \circ h)\mathfrak{R} \\ &= (f \circ (g \circ h))\mathfrak{R} \\ &= f\mathfrak{R} \circ (g \circ h)\mathfrak{R} \\ &= f\mathfrak{R} \circ (g\mathfrak{R} \circ h\mathfrak{R}) \end{aligned} \tag{9}$$
3. Let $f\mathfrak{R}, g\mathfrak{R}, h\mathfrak{R} \in \overline{End(S)}/\mathfrak{R}$, then
$$\begin{aligned} (f\mathfrak{R} + g\mathfrak{R}) \circ h\mathfrak{R} &= (f+g)\mathfrak{R} \circ h\mathfrak{R} \\ &= \big((f+g) \circ h\big)\mathfrak{R} \\ &= (f \circ h + g \circ h)\mathfrak{R} \\ &= (f \circ h)\mathfrak{R} + (g \circ h)\mathfrak{R} \\ &= (f\mathfrak{R} \circ h\mathfrak{R}) + (g\mathfrak{R} \circ h\mathfrak{R}) \end{aligned} \tag{10}$$
Therefore, $\overline{End(S)}/\mathfrak{R}$ holds the right distributive law.

It follows that $(\overline{End(S)}/\mathfrak{R}, +, \circ)$ is a seminear-ring. Furthermore, $\overline{End(S)}/\mathfrak{R}$ is a quotient of seminear-ring.
□

3.2. The C-ideal of Endomorphism of Seminear-rings

This section introduces the notions of c-ideal and quotient seminear-ring from the endomorphism of seminear-ring. These concepts are analogues to c-ideal and quotient seminear-ring. We give the sufficient condition on the ideal over the endomorphism of seminear-ring.

Proposition 6. *Let $(S, +, \cdot)$ and $(\overline{End(S)}, +, \circ)$ be a seminear-ring. If $I \subseteq S$ ideal in S and $f(S) \subseteq I$, for all $f \in \overline{End(S)}$, then $\overline{End(I)}$ is an ideal on $\overline{End(S)}$.*

Proof.

Let $f, g \in \overline{End(I)}$ where $f, g: I \to I$ is a homomorphism on S, and I is an ideal on S. Then, for all $x \in S$ holds $(f+g)(x) = f(x) + g(x)$. Since $f(x), g(x) \in I$ and I ideal in S, then $f(x) + g(x) \in I$. Therefore $(f+g)(x) \in I, \forall x \in S$, hence $f + g \in \overline{End(I)}$. Furthermore, for all $f \in \overline{End(I)}$ and $\alpha \in \overline{End(S)}$, then $\forall x \in S$ holds $(f\alpha)(x) = f(\alpha(x))$. It follows from $\forall \alpha \in \overline{End(S)}, \alpha(S) \subseteq I$, then $\alpha(x) \in I, \forall x \in S$. Hence $f(\alpha(x)) \in I$, then for all $x \in S$ holds $(f\alpha)(x) \in I$ and $f\alpha \in \overline{End(I)}$. It follows that $\overline{End(I)}$ is ideal on $\overline{End(S)}$. □

Furthermore, to investigate the form of the quotient seminear-ring, we give the properties of c-ideal as a congruence relation, given the sufficient condition of c-ideal over the endomorphism of seminear-ring.

Proposition 7. *Let $(S, +, \cdot)$ and $(\overline{End(S)}, +, \circ)$ be a seminear-ring. If $I \subseteq S$ c-ideal in S and $f(S) \subseteq I$, for all $f \in \overline{End(S)}$, then $\overline{End(I)}$ is a c-ideal on $\overline{End(S)}$.*

Proof.

Let $f, g \in \overline{End(S)}$, there exist $\alpha, \beta \in \overline{End(I)}$.Then for all $x \in S$ holds $(f + g + \alpha)(x) = f(x) + g(x) + \alpha(x)$. Since for all $f \in \overline{End(S)}, f(S) \subseteq I$, then $\forall x \in S, \alpha(x) \in I$ and I is a c-ideal on S, then there exist $\beta(x) \in I, \forall x \in S$, such that $(f + g + \alpha)(x) = f(x) + g(x) + \alpha(x) = \beta(x) + g(x) + f(x) = (\beta + g + f)(x)$. It follows that $\overline{End(I)}$ is c-ideal on $\overline{End(S)}$. □

We give the relation between congruence relation and c-ideal to construct the endomorphism of the quotient seminear-ring.

Proposition 8. *Let $(S, +, \cdot)$ and $(\overline{End(S)}, +, \circ)$ be a seminear-ring and $\overline{End(I)} \subseteq \overline{End(S)}$. If $\overline{End(I)}$ is a c-ideal on $\overline{End(S)}$, then $\overline{End(I)}$ defines congruence relation $\mathfrak{R}$ on $\overline{End(S)}$ where for all $(f, g) \in \mathfrak{R}$ if and only if $\exists \alpha, \beta \in \overline{End(I)}$ such that $f + \alpha = \beta + g$.*

Proof.

Let $f \in \overline{End(S)}$ and $\alpha \in \overline{End(I)}$. As $\overline{End(I)}$ is a c-ideal in $\overline{End(S)}$ then exist $\beta_1, \beta_2 \in \overline{End(S)}$ such that for all $x \in S$ holds $(f + \alpha + \beta_1)(x) = f(x) + \alpha(x) + \beta_1(x)$. Since $\alpha(x), \beta_1(x) \in I$ and I is a c-ideal on S then $f(x) + \alpha(x) + \beta_1(x) = \beta_2(x) + \alpha(x) + f(x)$. Thus, $\alpha(x) + \beta_1(x) = \gamma_1(x), \beta_2(x) + \alpha(x) = \gamma_2(x) \in \overline{End(I)}$, for all $x \in S$ such that $f(x) + \gamma_1(x) = \gamma_2(x) + f(x)$. Then $(f, f) \in \mathfrak{R}$ else $\mathfrak{R}$ reflexive. Let $(f, g) \in \mathfrak{R}$ for $f, g \in \overline{End(S)}$ then exist $\rho_1, \rho_2 \in \overline{End(I)}$ such that $f + \rho_1 = \rho_2 + g$. Since $\overline{End(I)}$ is a c-ideal on $\overline{End(S)}$, then exist $\theta_1, \theta_2, \theta_3, \theta_4 \in \overline{End(I)}$ such that $f + \rho_1 + \theta_1 = \theta_2 + \rho_1 + f$ and $g + \rho_2 + \theta_3 = \theta_4 + \rho_2 + g$ and for all $x \in S$ holds

$$\begin{aligned}(g + \rho_2 + \theta_3)(x) &= g(x) + \rho_2(x) + \theta_3(x) \\ &= \theta_4(x) + \rho_2(x) + g(x)\end{aligned} \tag{11}$$

Since $f(x) + \rho_1(x) = \rho_2(x) + g(x)$ then,

$$\begin{aligned}g(x) + \rho_2(x) + \theta_3(x) + \theta_1(x) &= \theta_4(x) + f(x) + \rho_1(x) + \theta_1(x) \\ &= \theta_4(x) + \theta_2(x) + \rho_1(x) + f(x)\end{aligned} \tag{12}$$

For $\rho_2(x) + \theta_3(x) + \theta_1(x) = \alpha(x), \theta_4(x) + \theta_2(x) + \rho_1(x) = \beta(x) \in \overline{End(I)}$ then $g(x) + \alpha(x) = \beta(x) + f(x)$. So, $(g, f) \in \mathfrak{R}$ or else $\mathfrak{R}$ is simetric. Let $f, g, h \in \overline{End(S)}$ where $(f, g) \in \mathfrak{R}$ then exist $\alpha, \beta \in \overline{End(I)}$ such that $f + \alpha = \beta + g$ and $(g, h) \in \mathfrak{R}$ then exist $\gamma, \rho \in \overline{End(I)}$ such that $g + \gamma = \rho + h$. Then for all $x \in S$ holds,

$$\begin{aligned}f(x) + \alpha(x) + \gamma(x) &= \beta(x) + g(x) + \gamma(x) \\ &= \beta(x) + \rho(x) + h(x) \\ f(x) + \omega_1(x) &= \omega_2(x) + h(x)\end{aligned} \tag{13}$$

for some $\omega_1(x) = \alpha(x) + \gamma(x), \omega_2(x) = \beta(x) + \rho(x) \in \overline{End(I)}$ then $(f, h) \in \mathfrak{R}$, so $\mathfrak{R}$ transitive. Thus, $\mathfrak{R}$ is an equivalence relation. Furthermore, we show that $\mathfrak{R}$ is a compatible relation. Let $(f, g), (p, q) \in \mathfrak{R}$ for all $f, g, p, q \in \overline{End(S)}$ there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \overline{End(I)}$ such that for all $x \in S$ holds $(f + \alpha_1)(x) = f(x) + \alpha_1(x) = \alpha_2(x) + g(x) = (\alpha_2 + g)(x)$ and $(\rho + \alpha_3)(x) = \rho(x) + \alpha_3(x) = \alpha_4(x) + q(x) = (\alpha_4 + q)(x)$. Then for all $x \in S$ holds

$$\begin{aligned} f(x) + p(x) + \alpha_3(x) + \alpha_1(x) &= \alpha_2(x) + p(x) + \alpha_3(x) + g(x) \\ &= \alpha_2(x) + \alpha_4(x) + q(x) + g(x) \end{aligned} \tag{14}$$

Then for some $\alpha_5 \in \overline{End(I)}$ holds $f(x) + p(x) + \alpha_3(x) + \alpha_1(x) + \alpha_5(x) = \alpha_2(x) + \alpha_4(x) + q(x) + g(x) + \alpha_5(x)$. Since $\overline{End(I)}$ is a c-ideal on $\overline{End(S)}$ then $q(x) + g(x) + \alpha_5(x) = \alpha_5(x) + g(x) + q(x)$. We have

$$\begin{gathered} f(x) + p(x) + \alpha_3(x) + \alpha_1(x) + \alpha_5(x) = \alpha_2(x) + \alpha_4(x) + \alpha_5(x) + g(x) + q(x) \\ f(x) + p(x) + \gamma_1(x) = \gamma_2(x) + g(x) + q(x) \end{gathered} \tag{15}$$

for some $\alpha_3(x) + \alpha_1(x) + \alpha_5(x) = \gamma_1(x), \alpha_2(x) + \alpha_4(x) + \alpha_5(x) = \gamma_2(x) \in \overline{End(I)}$. So that, $(f + p, g + q) \in \mathfrak{R}$. Moreover, for all $x \in S$ then

$$\begin{aligned} f(x)p(x) + f(x)\alpha_3(x) &= f(x)\alpha_4(x) + f(x)q(x) \\ f(x)p(x) + f(x)\alpha_3(x) + \alpha_1(x)q(x) &= f(x)\alpha_4(x) + f(x)q(x) + \alpha_1(x)q(x) \\ &= f(x)\alpha_4(x) + \big(\alpha_2(x) + g(x)\big)q(x) \\ &= f(x)\alpha_4(x) + \alpha_2(x)q(x) + g(x)q(x) \end{aligned} \tag{16}$$

Since there exists $\gamma_1(x), \gamma_2(x) \in \overline{End(I)}$ where $\gamma_1(x) = f(x)\alpha_3(x) + \alpha_1(x)q(x)$, then we have $f(x)p(x) + \gamma_1(x) = \gamma_2(x) + g(x)q(x)$. In other words, $(fp, gq) \in \mathfrak{R}$. Thus, $\mathfrak{R}$ is a compatible relation. So, we have $\mathfrak{R}$ is a congruence relation. □

Let $\mathfrak{R}$ be a congruence relation on seminear-ring S. It follows from Proposition 8. the c-ideal on endomorphism of seminear-ring $\overline{End(I)}$ represents the congruence relation on endomorphism seminear-ring $\overline{End(S)}$. Assume that

$$\overline{End(S)}/\overline{End(I)} = \{f + \overline{End(I)} | f \in \overline{End(S)}\} \tag{17}$$

then, we have $\overline{End(S)}/.\overline{End(I)}$ is a seminear-ring over the additive and the composition function operation. Furthermore, we called $(\overline{End(S)}/\overline{End(I)}, +, \circ)$ as quotient seminear-ring from endomorphism of seminear-ring.

Theorem 9. *Let $(\overline{End(S)}, +, \circ)$ be a seminear-ring and $\overline{End(I)}$ c-ideal on $\overline{End(S)}$. Define $\overline{End(S)}/\overline{End(I)} = \{f + \overline{End(I)} | f \in \overline{End(S)}\}$ with two binary operation such that for all $f + \overline{End(I)}, g + \overline{End(I)} \in \overline{End(S)}/\overline{End(I)}$ holds $\big(f + \overline{End(I)}\big) + \big(g + \overline{End(I)}\big) = (f + g) + \overline{End(I)}$ and $\big(f + \overline{End(I)}\big) \circ \big(g + \overline{End(I)}\big) = (f \circ g) + \overline{End(I)}$, then $(\overline{End(S)}/\overline{End(I)}, +, \circ)$ is a seminear-ring which is called a quotient seminear-ring from endomorphism of seminear-rings.*

Proof.

The proof analogue with Theorem 5. □

4. CONCLUSIONS

Let $(S, +, \cdot)$ be a seminear-ring. Seminear ring S is a common generalization from semiring and near-ring. If we have the other semiring $(T, \oplus, \odot)$, then the map $f: S \to T$ is a homomorphism of seminear-ring. Furthermore, if $S = T$ and the map $f: S \to S$ is a homomorphism, we have an endomorphism on S and are denoted by $\overline{End(S)}$. Based on our investigation, we have some points as follow:

1. The endomorphism of the seminear-ring $\overline{End(S)}$ is a seminear-ring under the addition and composition function.

2. Let $\mathfrak{R}$ be a congruence relation on a seminear-ring S, then $f\mathfrak{R}$ as the congruence classes in $\overline{End(S)}$ and is denoted by $\overline{End(S)}/\mathfrak{R}$ where $\overline{End(S)}/\mathfrak{R} \coloneqq \{f\mathfrak{R} | f \in \overline{End(S)}\}$.
3. The set $\overline{End(S)}/\mathfrak{R}$ under the addition and composition function is a seminear-ring and is called by quotient seminear-ring from the endomorphism of seminear-ring.
4. The c-ideal on endomorphism of seminear-ring $\overline{End(I)}$ represents the congruence relation on endomorphism of seminear-ring $\overline{End(S)}$.
5. We have defined $\overline{End(S)}/\overline{End(I)} = \{f + \overline{End(I)} | f \in \overline{End(S)}\}$, then $\overline{End(S)}/\overline{End(I)}$ is a seminear-ring over the additive and the composition function operation. Furthermore, we called $(\overline{End(S)}/\overline{End(I)}, +, \circ)$ as the quotient seminear-ring from endomorphism of seminear-rings.

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