THE HARMONIC INDEX AND THE GUTMAN INDEX OF COPRIME GRAPH OF INTEGER GROUP MODULO WITH ORDER OF PRIME POWER

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Abstract. *In the field of mathematics, there are many branches of study, especially in graph theory, mathematically a graph is a pair of sets, which consists of a non-empty set whose members are called vertices and a set of distinct unordered pairs called edges. One example of a graph from a group is a coprime graph, where a coprime graph is defined as a graph whose vertices are members of a group and two vertices with different x and y are neighbors if only if (|x|,|y|)=1. In this study, the author discusses the Harmonic Index and Gutman Index of Coprime Graph of Integer Group Modulo n. The method used in this research is a literature review and analysis based on patterns formed from several case studies for the value of n. The results obtained from this study are the coprime graph of the group of integers modulo n has the harmonic index of $2(\frac{n-1}{n})$ and the Gutman index $(n-1)(2n-3)$ for $n = p^k$ where p is prime and k is a natural number.*

Keywords: *Harmonic Index, Gutman Index, Coprime Graph, Integer Group Modulo.*

1. INTRODUCTION

Graph theory is used to represent discrete objects and the relationship between these objects. The visual representation of a graph is to represent objects as dots, circles, or points, while the relationship between objects is represented by lines. In recent years, a graph is used to visualize the algebraic structure such as groups or rings. Some of the graphs visualizing groups are the coprime graph of dihedral groups [1] and quaternion groups [2] on the dihedral group [3]. This coprime graph is introduced by Ma [4], and later the dual of the coprime graph, called non-coprime, that introduced by Mansoori [5], which also studied integer modulo [6] and dihedral group [7]. Some other graphs visualize are the power graph of groups [8][9], and the intersection graph of groups [10].

The coprime graph of groups given by Ma et. al is defined as a graph whose vertices are members of a group, and two different vertices x and y are adjacent if only if $(|x|, |y|) = 1$ [4]. Later, Juliana et. al gives some properties of the coprime graph of integer modulo group [11]. One of many important things in a graph is the topological index a graph, Hua et. al stated that atom-bond connectivity relies on the topological index of a graph [12]. In this article, we studied two topological indexes of the coprime graph for integer modulo, which are the Harmonic index and the Gutman index.

2. RESEARCH METHODS

This study conducts a literature review to achieve new knowledge from a recent terminology in the graph representation of the algebraic structure. First, we divide the problem into several cases and choose some examples to get a pattern and construct a conjecture from it. And by deductive proof, we prove the conjecture.

3. RESULTS AND DISCUSSION

This research will be discussed on the Harmonic Index and Gutman Index of Coprim Graph of Integer Group Modulo n, for n is a prime power.

3.1. Basic Terminology

These are some definitions and theorems to use in this research. The order of an element of groups is defined as follows

Definition 1 [11]
If G is a group with identity e and $x \in G$, the order of x is the power of natural number such that $x^k = e$ and write $|x| = k$.

The group representation in this study is the coprime graph, this graph is defined as follows.

Definition 2 [1]
The coprime graph of G group, denoted by Γ_G is a graph whose vertices are elements of G and two distinct vertices u and v are adjacent if and only if $(|x|, |y|) = 1$.

The degree of the graph is defined as follows.

Definition 3 [13]
The degree of a graph is the number of edges that are incident to the vertex. It is annotated as $deg(a)$ for a any vertex.

And the distance of the graph will be defined below.

Definition 4 [13]
The distance of a graph is the number of edges in a shortened path connecting them.

The first index that we study is the harmonic index, but first, we will define the Randic indexes of a graph.

Definition 5 [14]

Let G be a connected graph with $V(G)$ as a set of vertices and $E(G)$ as a set of edges. Randic index definition as,

$$R(G) = \sum_{uv \epsilon E(G)} (\deg(u) \deg(v))^{-\frac{1}{2}}$$

And the harmonic index will be defined as follows.

Definition 6 [15]

The harmonic index of the graph is denoted by $H(G)$ defined as follows.

$$H(G) = \sum_{uv \epsilon E(G)} \frac{2}{\deg(u) + \deg(v)}$$

With $deg(u)$ is the degree of vertices u that is the number of vertices $u \neq v$ with adjacent node u.

The second index is the Gutman index, which we defined as follows.

Definition 7 [14]

The Gutman Index of a graph G or denoted by $Gut(G)$ is defined as

$$Gut(G) = \sum_{\{u,v\} \epsilon V(G)} deg(u)deg(v)d(u,v)$$

With deg (u), deg (v) is the degree of u and v, and $d(u,v)$ is the distance of vertices u and v in graph G.

And last, we will give you a theorem that is very important through this article.

Theorem 1 [11]

If $n = p^k$ is for some prime p and $k \epsilon \mathbb{N}$, then the coprime graph of $\mathbb{Z}_n$ is a complete bipartite graph.

Proof. See [11]

3.2. The Harmonic Index of the Coprim Graph of Integer Modulo Group

In this research, the discussion will be focused on the harmonic index of the coprime graph of integer group modulo n with $n = p^k$ is for some prime p and $k \epsilon \mathbb{N}$.

Example 1

Let $\Gamma_{\mathbb{Z}_n}$ is coprim graph of $\Gamma_{\mathbb{Z}_5}$. Based on Theorem 1, $\Gamma_{\mathbb{Z}_5}$ is a complete bipartite graph. As a result, the degree of a node identity is 4 and another degree of a node is 1. So that is a harmonic index of $\Gamma_{\mathbb{Z}_5}$ is as follows.

$$H(\Gamma_{\mathbb{Z}_5}) = \sum_{uv \epsilon E(G)} \frac{2}{\deg(u) + \deg(v)}$$

$$H(\Gamma_{\mathbb{Z}_5}) = \frac{2}{\deg(0) + \deg(1)} + \frac{2}{\deg(0) + \deg(2)} + \frac{2}{\deg(0) + \deg(4)} + \frac{2}{\deg(0) + \deg(4)}$$

$$H(\Gamma_{\mathbb{Z}_5}) = \frac{2}{4+1} + \frac{2}{4+1} + \frac{2}{4+1} + \frac{2}{4+1} = \frac{8}{5}$$

With the same steps, we obtained the harmonic index for $n = 2,3,5,7,8,9$ as shown in the table bellows.

Table 1. The Harmonic Index

n	Harmonic Index
2	1
3	$4/3$
4	$3/2$
5	$8/5$
7	$12/7$
8	$7/4$
9	$16/9$

From these cases, we get some pattern of the harmonic index of the coprime graph of integer group modulo n, and this conjecture is true as stated in the theorem bellows.

Theorem 2

Suppose $\Gamma_{\mathbb{Z}_n}$ coprime graph of the integer group modulo n, If $n = p^k$ is for a p prime number and $k\epsilon\mathbb{N}$ then

$$H\left(\Gamma_{\mathbb{Z}_n}\right) = \frac{2n-2}{n}.$$

Proof. Suppose $\Gamma_{\mathbb{Z}_n}$ coprime graph of $\mathbb{Z}_n$ group. Take $n = p^k$, where p is a prime number and $k\epsilon\mathbb{N}$. By definition 3, we obtained $\deg(e) = n - 1$, and $\deg(v) = 1$ for each $v\epsilon\mathbb{Z}_n\backslash\{e\}$ and $uv\epsilon E(G)$ if and only if $\{e\} \subset \{u, v\}$ and $u \neq v$. Obtained harmonic index of the coprime graph of the group $\mathbb{Z}_n$ as follows,

$$H\left(\Gamma_{\mathbb{Z}_n}\right) = \sum_{uv\epsilon E(G)} \frac{2}{\deg(u) + \deg(v)}$$

$$H\left(\Gamma_{\mathbb{Z}_n}\right) = \sum_{uv\epsilon E(G)} \frac{2}{\deg(e) + \deg(v)}$$

$$H\left(\Gamma_{\mathbb{Z}_n}\right) = (n - 1)\left(\frac{2}{(n-1) + 1}\right) = \left(\frac{2n-2}{n}\right)$$

So, we proved that the harmonic index of the coprime graph of integer group modulo n with $n = p^k$ is $\left(\frac{2n-2}{n}\right)$. ∎

3.3.   The Gutman Index of The Coprime Graph of Integer Modulo Group

In this research, the discussion will be focused on the Gutman index of the coprime graph of integer group modulo n with $n = p^k$ is for some prime p and $k\epsilon\mathbb{N}$.

Example 2

Let $\Gamma_{\mathbb{Z}_n}$ is coprim graph of $\Gamma_{\mathbb{Z}_5}$ with $\Gamma_{\mathbb{Z}_5} = \{0,1,2,3,4\}$. Based on theorem 1, $\Gamma_{\mathbb{Z}_5}$ is a complete bipartite graph. As the result, the degree of a node identity is 4 and another degree of a node is 1 as well as researcher divide by 2 cases. First with the identity we have $(n - 1)$ combined and we have the distance of all combined with identity is 1, second without identity we have $= \frac{(n-1)(n-2)}{(2)}$ combine and the distance without identity is 2. Therefore by definition, the result of the Gutman index of $\Gamma_{\mathbb{Z}_5}$ is as follows:

$$Gut\left(\Gamma_{\mathbb{Z}_5}\right) = \sum_{\{u,v\}\epsilon V(G)} deg(u)deg(v)d(u,v)$$

$$Gut\left(\Gamma_{\mathbb{Z}_5}\right) = \sum_{\{u\}\epsilon v(\mathbb{Z}_5)\backslash\{e\}} deg(u)deg(v)d(u,v) + \sum_{\{u,v\}\epsilon v(\mathbb{Z}_5)\backslash\{e\}} deg(u)deg(v)d(u,v)$$

$$Gut\left(\Gamma_{\mathbb{Z}_5}\right) = (4)(4)(1)(1) + (6)(1)(1)(2) = 28$$

With the same steps, obtained the Gutman index for $n = 2,3,5,7,8,9$ as shown in table 2.

Tabel 2. The Gutman index

n	Gutman Index
2	1
3	6
4	15
5	28
7	66
8	91
9	120

From these cases, we had some pattern of the Gutman index of the coprime graph of integer group modulo n as stated in the next theorem.

Theorem 3

Suppose $\Gamma_{\mathbb{Z}_n}$ coprime graph of the integer group modulo n, If $n = p^k$ is for a p prime number and $k\epsilon\mathbb{N}$ then
$Gut(\Gamma_{\mathbb{Z}_n}) = (n-1)(2n-3)$

Proof. Suppose $\Gamma_{\mathbb{Z}_n}$ coprime graph of $\mathbb{Z}_n$ group. Take $n = p^k$ a p prime number and $k\epsilon\mathbb{N}$. By definition 4, obtained $\deg(e) = n-1$ dan $\deg(v) = 1$ and based on the definition of 5 and the pattern formed, obtained $d(e, v) = 1$ and $d(u, v) = 2$ if u and v are not equal to e, obtained that many pairs of two different vertices on the $\Gamma_{\mathbb{Z}_n}$ are as follows,

for $d(e, u)$

$$C_1^{n-1} = \frac{(n-1)!}{((n-1)-1)!\,(1)!} = (n-1)$$

for $d(u, v)$

$$C_2^{n-1} = \frac{(n-1)!}{((n-1)-2)!\,(2)!} = \frac{(n-1)(n-2)}{(2)}$$

Then we have

$$Gut(\Gamma_{\mathbb{Z}_n}) = \sum_{\{u,v\}\epsilon V(G)} deg(u)deg(v)d(u,v)$$

$$Gut(\Gamma_{\mathbb{Z}_n}) = \sum_{\{e,v\}\epsilon v(\mathbb{Z}_n)\backslash\{e\}} deg(e)deg(v)d(u,v) + \sum_{\{u,v\}\epsilon v(\mathbb{Z}_n)\backslash\{e\}} deg(u)deg(v)d(u,v)$$

$$Gut(\Gamma_{\mathbb{Z}_n}) = (n-1)(n-1)(1)(1) + \left(\left(\frac{(n-1)(n-2)}{(2)}\right)\right)(1)(1)(2)$$

$$Gut(\Gamma_{\mathbb{Z}_n}) = (n-1)(2n-3)$$

Then we proved the Gutman index of the coprime graph for integer group modulo n with $n = p^k$ is $(n-1)(2n-3)$.∎

4. CONCLUSIONS

Based on the research that has been carried out, the results were obtained that the Harmonic index and the Gutman index of the coprime graph of the modulo integer group n with $n = p^k$ for a p prime number and $k\epsilon\mathbb{N}$ successively is $2\left(\frac{n-1}{n}\right)$ and $(n-1)(2n-3)$.

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