BAREKENG: Journal of Mathematics and Its Applications
March 2023 Volume 17 Issue 1 Page 0009–0020
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol17iss1pp0009-0020

A NUMERICAL STUDY OF SUBSTANCE SPREAD IN THE POOL FROM TWO POINT SOURCES

Nurcahya Yulian Ashar [1], **Susilo Hariyanto** [2*]

[1,2] *Department of Mathematics, Faculty of Sciences and Mathematics, University Diponegoro*
Jl. Prof. Jacob No. 50275, Tembalang, Kota Semarang, Jawa Tengah, Indonesia.

*Corresponding author's e-mail: * **susilohariyanto@lecturer.undip.ac.id***

Article History:
Received: 30th June 2022
Revised: 6th November 2022
Accepted: 5th January 2023

Keywords:
Diffusion-Convection Model;
Dual Reciprocity Method;
K-Epsilon;
Modified Navier-Stokes Equation;
Turbulent Model.

ABSTRACT

Problems related to the purification of holding pool or reservoirs become an interesting discussion in real events. In this paper, the author modelled the distribution of the substance/purifier in a pool model with turbulent water flow using the diffusion-convection equation. The Dual Reciprocity Method is applied to the diffusion-convection equation, whose derivation discussed in this paper. This method is chosen because the problem cannot be solved analytically, so it must be solved numerically. The Dual Reciprocity Method has good flexibility in problems of water infiltration, pollutant spread, and heat transfer. This paper also discussed velocity profile of turbulent flow from upcoming part of pool region. So before using DRM, numerical solution of turbulent flow by k-epsilon turbulent model is used. In numerical calculations, two source points are selected whose positions are combined to see the most effective way to make the substance/purifier evenly distributed in the pool.

How to cite this article:

N. Y. Ashar and S. Hariyanto., "A NUMERICAL STUDY OF SUBSTANCE SPREAD IN THE POOL FROM TWO POINT SOURCES," *BAREKENG: J. Math. & App.*, vol. 17, iss. 1, pp. 0009-0020, March, 2023.

1. INTRODUCTION

The discussion about the distribution of the substance in a path or region is the concern of researchers in this paper. In mathematical modeling, the diffusion-convection equation is very widely used to model the distribution of substance in a path or region with a certain flow. This model can be used to minimize research fund related to the distribution of substance in certain path/region. Before this model is applied to the problem, it is necessary to discuss its mathematical derivation. Samec [1] has derived the diffusion-convection equation into a mathematical model that can be used to solve the problem of the distribution of substance in a path/region. Previously, analytical solutions of the diffusion-convection equations have been solved by Polyanin [2] and Morales-Delgadoa et al. [3]. Since analytical solution can only be used for certain problems, it is necessary to use a numerical approach to the problem of dispersion of substance on a path/region.

Numerical method is one solution to observe the distribution of substance in the path/region. Several researchers have studied numerical methods to solve the diffusion-convection problem. Fajie et al. [4], Xingxing et al. [5], Mengxing et al. [6], Ji-Huan [7] and Boztosun et al. [8], the researcher solved the diffusion-confection problem but not with a single or many source point.

Problems that use one or more source points are very suitable if solved by the Dual Reciprocity Method (DRM). DRM is an extension of the Boundary Element Method (BEM). Both of these methods have been successfully used by several researchers to solve various problems. As an example, Clement and Lobo [9], Solekhudin and Ang [10], Solekhudin [11], Munadi et al. [12], Yun and Ang [13], Ashar [14], and Ashar and Solekhudin [15]. Researchers in [9], [10], [11], and [12] discussed water infiltration channels into soils. [13] discussed heat conduction in non-homogeneous solids. [14] and [15] discuss pollutant spread in curved paths with laminar water flow. In the previous results, DRM shows quite good flexibility by solving various problems over any domain shape bounded by a simple curved curve. In this paper, DRM is used to determine the distribution of substance in pool with turbulent water flow and two source points. Several combinations of two source points are used to find the right method for efficient dispersion of the substance.

2. RESEARCH METHODS

The mathematical model of the steady diffusion-convection problem will be shown. A derivation of DRM method to resolve the problem will also be shown. Steady diffusion-convection problem over region ω and bounded by simple closed curved κ are defined by

$$v_1 \frac{\partial S}{\partial x} + v_2 \frac{\partial S}{\partial y} - E\left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right) = G(x,y), \tag{1}$$

where S is substance/purifier concentration, v_1 and v_2 are fluid velocities in x and y directions, E is substance diffusion coefficient in certain fluids, and G is the sources. The problem with a source point is defined as follows

$$v_1 \frac{\partial S}{\partial x} + v_2 \frac{\partial S}{\partial y} - E\left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right) = G(x,y)\sigma(x,y;a,b), \tag{2}$$

where (a, b) is the coordinate of the sources, and σ is a Dirac delta function with the sources at (a, b). The DRM can solve **Equation (1)** and **Equation (2)** numerically by expressing the solution in the form of boundary integral equations as below

$$\begin{aligned} \lambda(\alpha,\beta)S(\alpha,\beta) = \int_\kappa \left[S(x,y)\frac{\partial \Xi(x,y;\alpha,\beta)}{\partial n} - \Xi(x,y;\alpha,\beta)\frac{\partial S(x,y)}{\partial n} \right] ds + \\ \frac{1}{E}\iint_\omega \Xi(x,y;\alpha,\beta)\left[v_1(x,y)\frac{\partial S(x,y)}{\partial x} + v_2(x,y)\frac{\partial S(x,y)}{\partial y} - G(x,y) \right] dx\,dy, \end{aligned} \tag{3}$$

and

$$\lambda(\alpha,\beta)S(\alpha,\beta)=\int_{\kappa}\left[S(x,y)\frac{\partial\Xi(x,y;\alpha,\beta)}{\partial n}-\Xi(x,y;\alpha,\beta)\frac{\partial S(x,y)}{\partial n}\right]ds+ \frac{1}{E}\iint_{\omega}\Xi(x,y;\alpha,\beta)\left[v_1(x,y)\frac{\partial S(x,y)}{\partial x}+v_2(x,y)\frac{\partial S(x,y)}{\partial y}\right]dx\,dy- \frac{\Xi(x,y;\alpha,\beta)G(x,y)}{E} \quad (4)$$

respectively. Here

$$\lambda(\alpha,\beta)=\begin{cases}1 & ,(\alpha,\beta)\in\omega\\ \frac{1}{2} & ,(\alpha,\beta)\text{ on the smooth part of }\kappa,\end{cases}$$

and

$$\Xi(x,y;\alpha,\beta)=\frac{1}{4\pi}ln\left[(x-\alpha)^2+(y-\beta)^2\right]$$

is the fundamental solution of two-dimensional Laplace's equation. From **Equation (3)** and **Equation (4)**, we get two systems of linear algebraic equations

$$\lambda^{(n)}S^{(n)}=\sum_{k=1}^{N}\left[S^{(n)}\int_{\kappa^{(k)}}\frac{\partial\Xi\left(x,y;x^{(n)},y^{(n)}\right)}{\partial n}ds-S_n^{(k)}\int_{\kappa^{(k)}}\Xi\left(x,y;x^{(n)},y^{(n)}\right)ds\right]+ \sum_{i=1}^{N+L}\left[v_1^{(n)}\mu_x^{(ni)}+v_2^{(n)}\mu_y^{(ni)}\right]S^{(i)}-\sum_{i=1}^{N+L}\mu^{(ni)}G\left(x^{(i)},y^{(i)}\right),$$

$$n=1,2,\ldots,N+L, \quad (5)$$

and

$$\lambda^{(n)}S^{(n)}=\sum_{k=1}^{N}\left[S^{(n)}\int_{\kappa^{(k)}}\frac{\partial\Xi\left(x,y;x^{(n)},y^{(n)}\right)}{\partial n}ds-S_n^{(k)}\int_{\kappa^{(k)}}\Xi\left(x,y;x^{(n)},y^{(n)}\right)ds\right]+ \sum_{i=1}^{N+L}\left[v_1^{(n)}\mu_x^{(ni)}+v_2^{(n)}\mu_y^{(ni)}\right]S^{(i)}-\Xi\left(a,b;x^{(n)},y^{(n)}\right)G(a,b),$$

$$n=1,2,\ldots,N+L, \quad (6)$$

are respectively derived. Here N is the number of segment of element on boundary $\kappa,\kappa^{(1)},\kappa^{(2)},\ldots,\kappa^{(N)}$ are the segments satisfy $\kappa=\kappa^{(1)}\cup\kappa^{(2)}\cup\ldots\cup\kappa^{(N)}$. Number L is the number of interior collocation point. Points $\left(x^{(1)},y^{(1)}\right),\left(x^{(2)},y^{(2)}\right),\ldots,\left(x^{(N)},y^{(N)}\right)$ are the midpoints of segments $\kappa^{(1)},\kappa^{(2)},\ldots,\kappa^{(N)}$, respectively. Points $\left(x^{(N+1)},y^{(N+1)}\right),\left(x^{(N+2)},y^{(N+2)}\right),\ldots,\left(x^{(N+L)},y^{(N+L)}\right)$, are the interior collocation points,

$$\lambda^{(n)} = \lambda\left(x^{(n)}, y^{(n)}\right),$$

$$S^{(n)} = S\left(x^{(n)}, y^{(n)}\right),$$

$$S_n^{(n)} = \left.\frac{\partial S(x,y)}{\partial n}\right|_{(x,y)=\left(x^{(n)},y^{(n)}\right)},$$

$$v_1^{(n)} = v_1\left(x^{(n)}, y^{(n)}\right),$$

$$v_2^{(n)} = v_2\left(x^{(n)}, y^{(n)}\right),$$

and

$$\mu_x^{(ni)} = \sum_{j=1}^{N+L} \mu^{(ni)} \left.\frac{\partial \rho\left(x,y;x^{(j)},y^{(j)}\right)}{\partial x}\right|_{(x,y)=\left(x^{(i)},y^{(i)}\right)} \times \rho^{-1}\left(x^{(i)},y^{(i)};\, x^{(j)},y^{(j)}\right),$$

$$\mu_y^{(ni)} = \sum_{j=1}^{N+L} \mu^{(ni)} \left.\frac{\partial \rho\left(x,y;x^{(j)},y^{(j)}\right)}{\partial y}\right|_{(x,y)=\left(x^{(i)},y^{(i)}\right)} \times \rho^{-1}\left(x^{(i)},y^{(i)};\, x^{(j)},y^{(j)}\right),$$

$$\mu^{(ni)} = \sum_{j=1}^{N+L} \psi^{(nj)} \rho^{-(ij)},$$

$$\psi^{(nj)} = \lambda\left(x^{(n)},y^{(n)}\right)\chi\left(x^{(n)},y^{(n)};\, x^{(j)},y^{(j)}\right) + \int_C \left[\Xi\left(x,y;\, x^{(n)},y^{(n)}\right) \frac{\partial \chi\left(x,y;\, x^{(j)},y^{(j)}\right)}{\partial n} - \chi\left(x,y;\, x^{(j)},y^{(j)}\right) \frac{\partial \Xi\left(x,y;\, x^{(n)},y^{(n)}\right)}{\partial n} \right] ds.$$

Where

$$\rho^{(kl)} = 1 + r^2\left(x^{(k)},y^{(k)};\, x^{(l)},y^{(l)}\right) + r^3\left(x^{(k)},y^{(k)};\, x^{(l)},y^{(l)}\right),$$

and

$$\chi\left(x,y;\, x^{(m)},y^{(m)}\right) = \frac{1}{4} r^2\left(x,y;\, x^{(m)},y^{(m)}\right) + \frac{1}{16} r^4\left(x,y;\, x^{(m)},y^{(m)}\right) + \frac{1}{25} r^5\left(x,y;\, x^{(m)},y^{(m)}\right).$$

Function r is defined as

$$r(x,y;\, a,b) = \sqrt{(x-a)^2 + (y-b)^2}.$$

By solving system of linear algebraic **Equation (5)** and **Equation (6)**, numerical solutions at collocation points may be obtained. Using these solutions, numerical solution at any $(\xi,\eta) \in \omega \cup \kappa$ can also obtained.

3. RESULTS AND DISCUSSION

DRM in Section 2 is applied to solve the diffusion-convection problem. First, this method will be applied to a problem that has an analytical solution. It is useful to see the accuracy of the DRM method in solving related problems. After that, DRM is applied to problems that do not have an analytical solution. The problem above is the problem of substance distribution in pool with turbulent water flow.

3.1 A Diffusion-Convection Problem with Analytic Solution

A diffusion-convection problem with an analytical solution is defined as follows

$$g(x,y) = v_1(x,y)\frac{\partial S(x,y)}{\partial x} + v_2(x,y)\frac{\partial S(x,y)}{\partial y} - E\left(\frac{\partial^2 S(x,y)}{\partial x^2} + \frac{\partial^2 S(x,y)}{\partial y^2}\right) \qquad (7)$$

with

$$v_1(x,y)=y, \quad v_2(x,y)=x, \quad E=1,$$

and

$$g(x,y)=4x^3y+4xy^3-12x^2-12y^2,$$

defined over a square region with boundary condition presented in **Figure 1**.

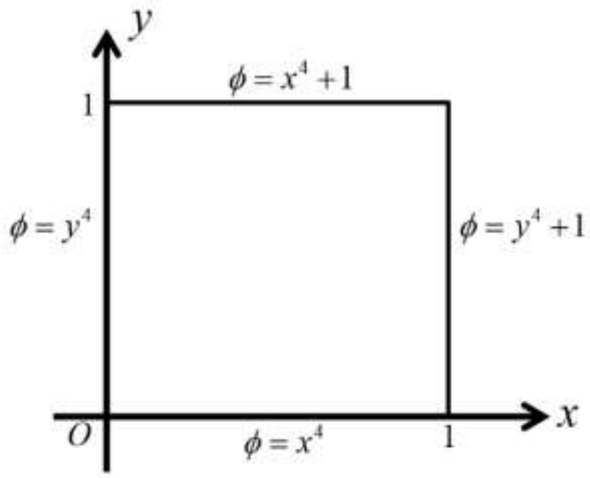


Figure 1. Region and boundary conditions of diffusion-convection equation (7)

Analytical solution of **Equation (7)** subject to boundary conditions in **Figure 1** is

$$\phi(x,y)=x^4+y^4.$$

Then DRM is applied by determining the number of segment elements and interior collocation points, namely Set A and Set B. In Set A, the number of segment elements is 96 and the number of interior collocation points is 81. In Set B, the number of segment elements is 216 and the number of interior collocation points is 200. Absolute error using Set A denoted by e_A and using Set B denoted by e_B . The result of the numerical calculation is shown in **Table 1**.

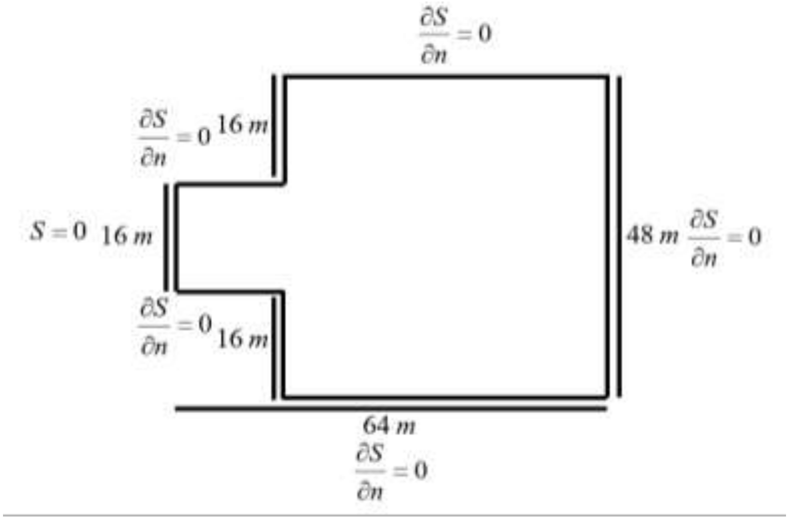
Table 1. Numerical and Analytical Solutions at Selected Points

Point	Analytic	Set A	Set B	e_A	e_B
(0.2, 0.2)	0.0032	0.00362	0.00334	0.00042	0.00014
(0.4, 0.2)	0.0272	0.02750	0.02730	0.00030	0.00010
(0.6, 0.2)	0.1312	0.13138	0.13126	0.00018	0.00006
(0.8, 0.2)	0.4112	0.41156	0.41132	0.00036	0.00012
(0.2, 0.4)	0.0272	0.02756	0.02732	0.00036	0.00012
(0.4, 0.4)	0.0512	0.05126	0.05122	0.00006	0.00002
(0.6, 0.4)	0.1552	0.15532	0.15524	0.00012	0.00004
(0.8, 0.4)	0.4352	0.43574	0.43538	0.00054	0.00018
(0.2, 0.6)	0.1312	0.13126	0.13122	0.00006	0.00002
(0.4, 0.6)	0.1552	0.15562	0.15534	0.00042	0.00014
(0.6, 0.6)	0.2592	0.25938	0.25926	0.00018	0.00006
(0.8, 0.6)	0.5392	0.53974	0.53938	0.00054	0.00018
(0.2, 0.8)	0.4112	0.41138	0.41126	0.00018	0.00006
(0.4, 0.8)	0.4352	0.43544	0.43528	0.00024	0.00008
(0.6, 0.8)	0.5392	0.53974	0.53938	0.00054	0.00018
(0.8, 0.8)	0.8192	0.81974	0.81938	0.00054	0.00018

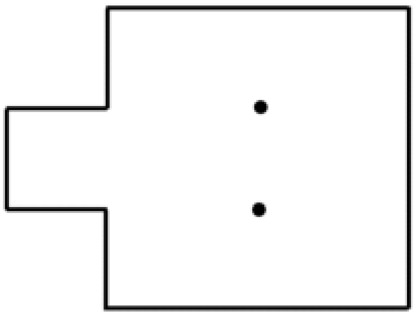
Table 1 shows the numerical result at selected points using DRM with Set A and Set B. **Table 1** also shows each absolute error resulted from numerical calculation. From the numerical result of **Table 1**, the absolute error is not more than 0.0005 for Set A and not more than 0.0003 for Set B. From this result, in general, the more collocation point, the more accurate the numerical solution, although with Set B it is also very accurate.

3.2 Steady Substance Concentration in the Pool with Two Point Sources

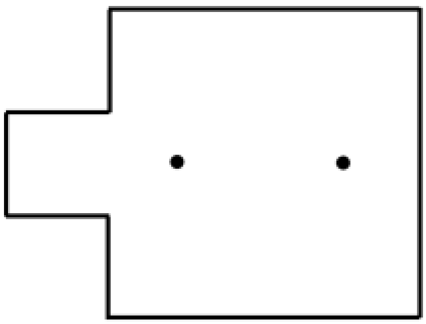
Furthermore, DRM is used to observe the distribution of substance in the pool with two point sources. Given a shallow path shaped like a pond with a width of 48 m. To check the influence of the position of the two sources, four cases are defined. Case 1 is a problem with the source points placed at the coordinates (40, 12), and (40, 36). Case 1 is a problem with the source points placed at the coordinates (40, 12), and (40, 36). Case 2 is a problem with the source points placed at the coordinates (28, 24), and (52, 24). Case 3 is a problem with the source points placed at the coordinates (28, 8), and (28, 40). Case 4 is a problem with the source points placed at the coordinates (52, 8), and (52, 40). The concentration value of each source is 25/107 gram/s and E = 11.75 m^2/s. The fourth case problem is illustrated in Figure 2.



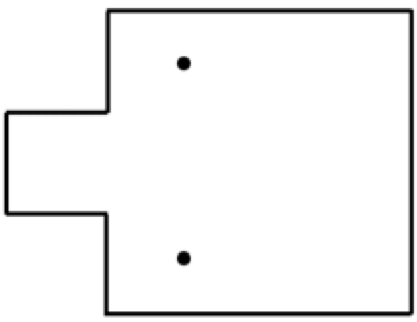
(a) Region and boundary condition



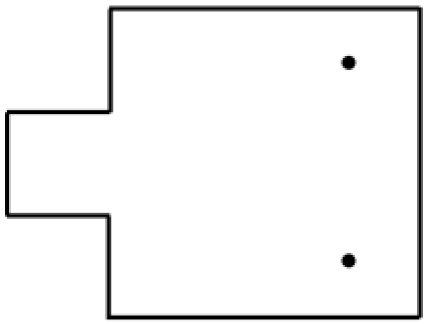
(b) Sources location in Case 1



(c) Sources location in Case 2



(d) Sources location in Case 3



(e) Sources location in Case 4

Figure 2. Region, boundary conditions, and sources location of all cases

In **Figure 2**, pool width 48 m is selected, and boundary condition in upstream part is S=0, and the others part boundary condition is $\frac{\partial S}{\partial n}=0$. Before using DRM, we search v_x and v_y on interior collocation points, with maximum incoming flow is 0.150 m/s. The k-epsilon turbulent model is obtained by solving a system of equations

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0,$$

$$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\gamma\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)$$

$$v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\gamma\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{v_t}{\sigma_k}\frac{\partial_k}{\partial x}\right)=\varepsilon-P_k$$

$$\frac{\partial}{\partial y}\left(\frac{v_t}{\sigma_k}\frac{\partial_k}{\partial y}\right)=\varepsilon-P_k$$

$$\frac{\partial}{\partial x}\left(\frac{v_t}{\sigma_\varepsilon}\frac{\partial_\varepsilon}{\partial x}\right)=C_{\varepsilon 2}\frac{\varepsilon^2}{k}-C_{\varepsilon 1}\frac{\varepsilon}{k}P_k$$

$$\frac{\partial}{\partial y}\left(\frac{v_t}{\sigma_\varepsilon}\frac{\partial_\varepsilon}{\partial y}\right)=C_{\varepsilon 2}\frac{\varepsilon^2}{k}-C_{\varepsilon 1}\frac{\varepsilon}{k}P_k$$

where $v_t=c_\mu\frac{k^2}{\varepsilon}, P_k=\frac{1}{2}v_t\left(\frac{\partial\overrightarrow{v_x}}{\partial y}+\frac{\partial\overrightarrow{v_y}}{\partial x}\right)$, $c_\mu=0.09$, $c_{\mu 1}=1.44$, $c_{\mu 2}=1.92$, $\sigma_k=1.00$, $\sigma_\varepsilon=1.30$, $\overrightarrow{v_i}$ is shear stress respect to i direction, ρ is density, γ is kinematic viscosity, and p is pressure. For the cases considered, the resulted velocity profiles are presented in **Figure 3**. These velocity profiles are obtained using *ANSYS 19.2*.

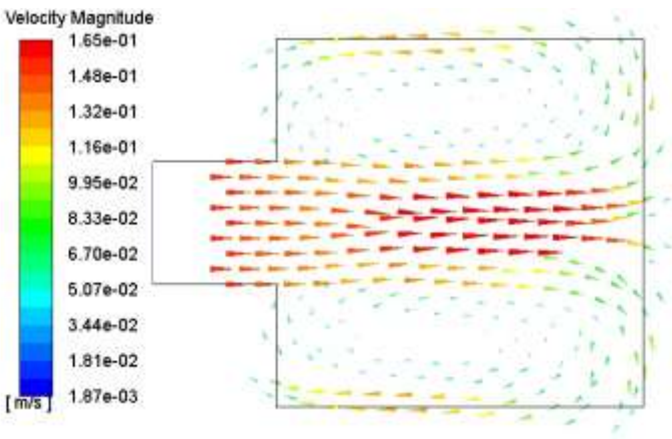


Figure 3. Velocity profile generated by *ANSYS 19.2.*

To check the effect of number collocation points, DRM is applied to Case 1 with different number of boundary collocations and interior collocations. In Set C, 300 boundary collocation points and 300 interior collocation points are selected. Meanwhile, in Set D, 360 boundary collocation points and 360 interior collocation points are selected. Moreover, in Set E, 400 boundary collocation points and 425 interior collocation points are selected. The numerical results for some points selected using the three sets are shown in **Table 2**.

Table 2. Numerical Solution at Selected Points for Case 1 using Set C, D, and E

Point	Set C	Set D	Set E
(5.8182, 24.0000)	0.036080	0.036360	0.036320
(11.6364, 24.0000)	0.035923	0.036483	0.036403
(17.4545, 24.0000)	0.035750	0.036100	0.036050
(23.2727, 24.0000)	0.035702	0.036122	0.036062
(29.0909, 24.0000)	0.035795	0.036215	0.036155
(34.9091, 24.0000)	0.003743	0.003813	0.003803
(40.7273, 24.0000)	0.007806	0.007946	0.007926
(46.5455, 24.0000)	0.001232	0.012952	0.012862
(52.3636, 24.0000)	0.017498	0.017568	0.017558
(58.1818, 24.0000)	0.023018	0.023508	0.023438

Table 2 shows a numerical solution of 10 selected points using Sets C, D, and E. It can be concluded from **Table 2** that the concentration of substance has relatively the same value. The overall value only has differences not more than 0.0007. So, for the implementation of DRM in the next problem, Set C is chosen as the number of collocation points. The surface plots of Case 1, Case 2, Case 3, and Case 4 are shown in **Figure 5**.

For further discussion, the behavior of the substance distribution along lines l_1 and l_2 is analyzed for all cases. Line l_1 is a vertical line that divides the main part of the pool into two equal parts. And line l_2 is a horizontal line that divides the pool into two equal parts. Illustrations of lines l_1 and l_2 are shown in **Figure 4**.

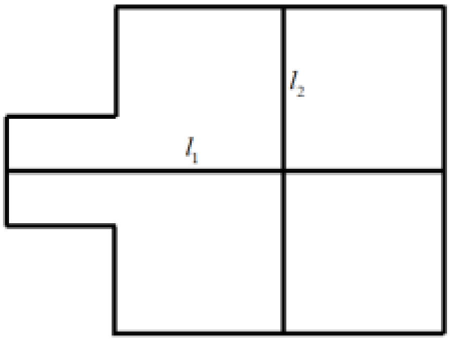


Figure 4. Line l_1 and l_2 formulation

From **Figures 5, 6, 7**, and **8**, it can be seen that the comparison of surface plots of Case 1, Case 2, Case 3, and Case 4. For all cases, the largest concentration of substance is concentrated at two source points. In Case 1, the source with coordinates (40, 12) has a higher concentration level than the source with coordinates (40, 36). In Case 2, the source with coordinate (52, 24) has a higher concentration level than the source with coordinate (28, 24). In Case 3, the source with coordinate (28, 8) has a higher concentration level than the source with coordinate (28, 40). And in Case 4, the source with coordinate (52, 8) has a higher concentration level than the source with coordinate (52, 40). It can be concluded, the area that gets a larger turbulent flow, will produce significant increasing in concentration substance.

Then to find the total concentration of the substance in each case along line l_1 and line l_2, $\int_{l_i} S(x,y)ds,\ i=1,2,$ is calculated. And to find the total concentration of substance in each case in all region, $\iint_{\omega} S(x,y)dxdy,$ is calculated. The results obtained in **Table 3** are as follows:

Table 3. Total concentration along l_1, l_2 and all region

Source	Along l_1	Along l_2	All Region
Case 1	280.08 mg	329.39 mg	27655.93 mg
Case 2	284.42 mg	275.62 mg	26060.82 mg
Case 3	198.65 mg	222.94 mg	20802.13 mg
Case 4	294.46 mg	320.61 mg	30557.70 mg

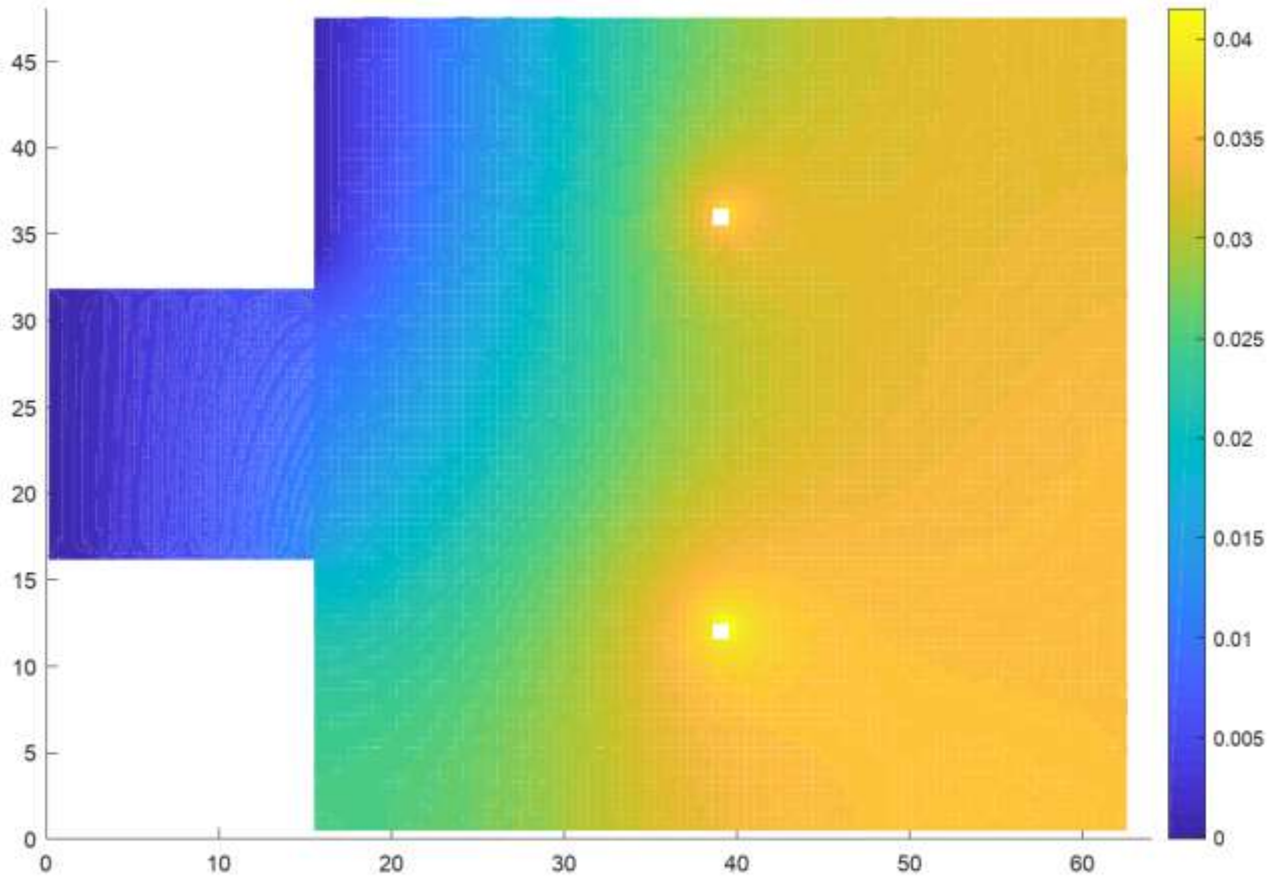


Figure 5. Surface plot of substance concentration for Case 1

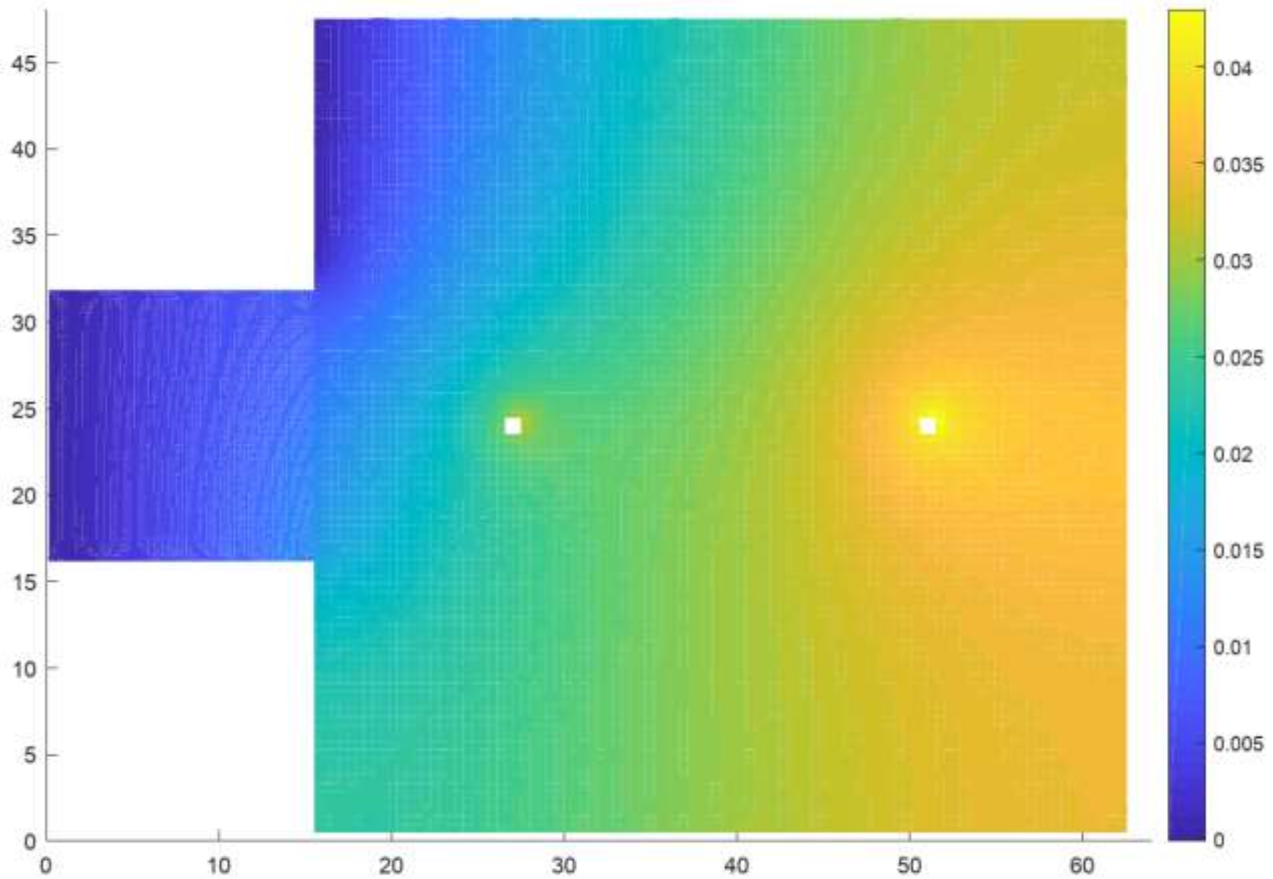


Figure 6. Surface plot of substance concentration for Case 2

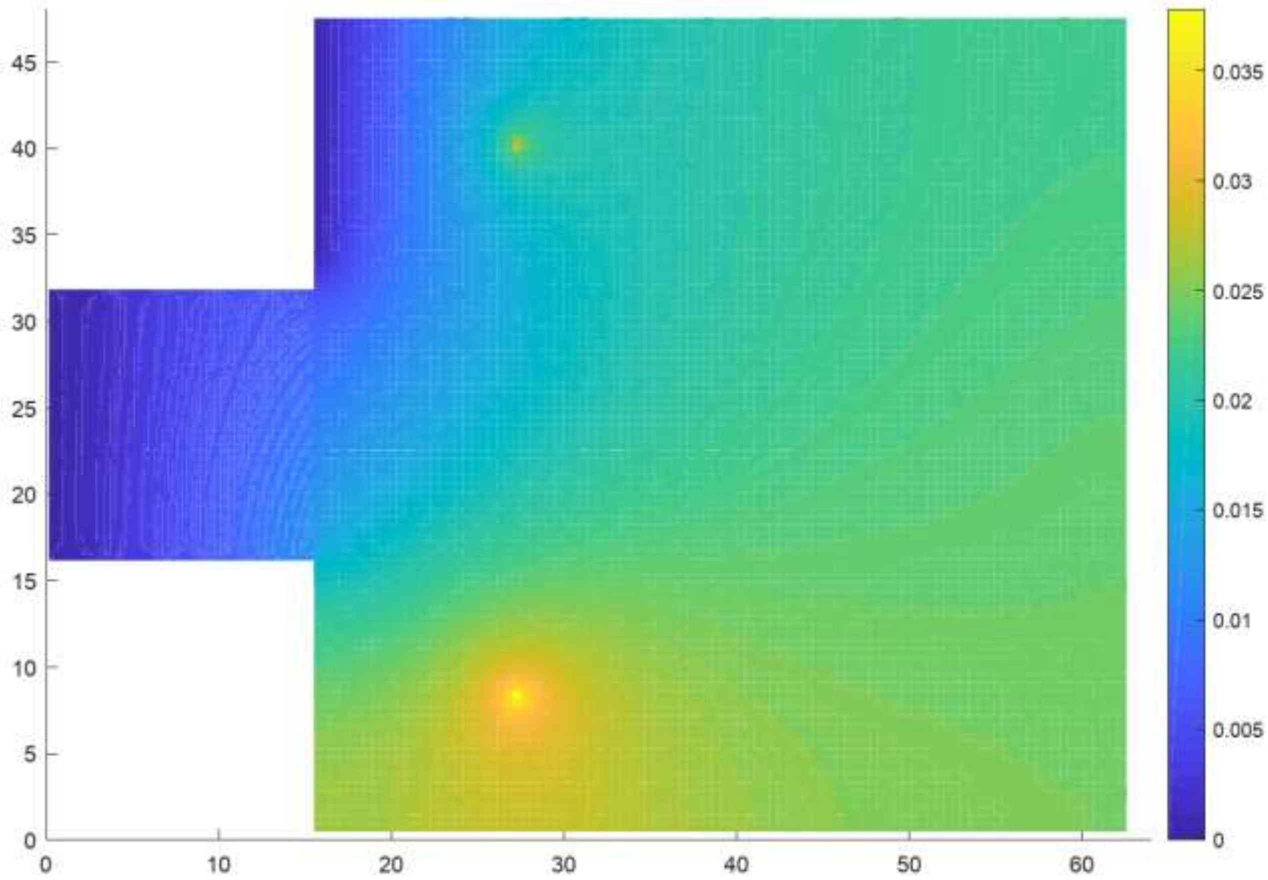


Figure 7. Surface plot of substance concentration for Case 3

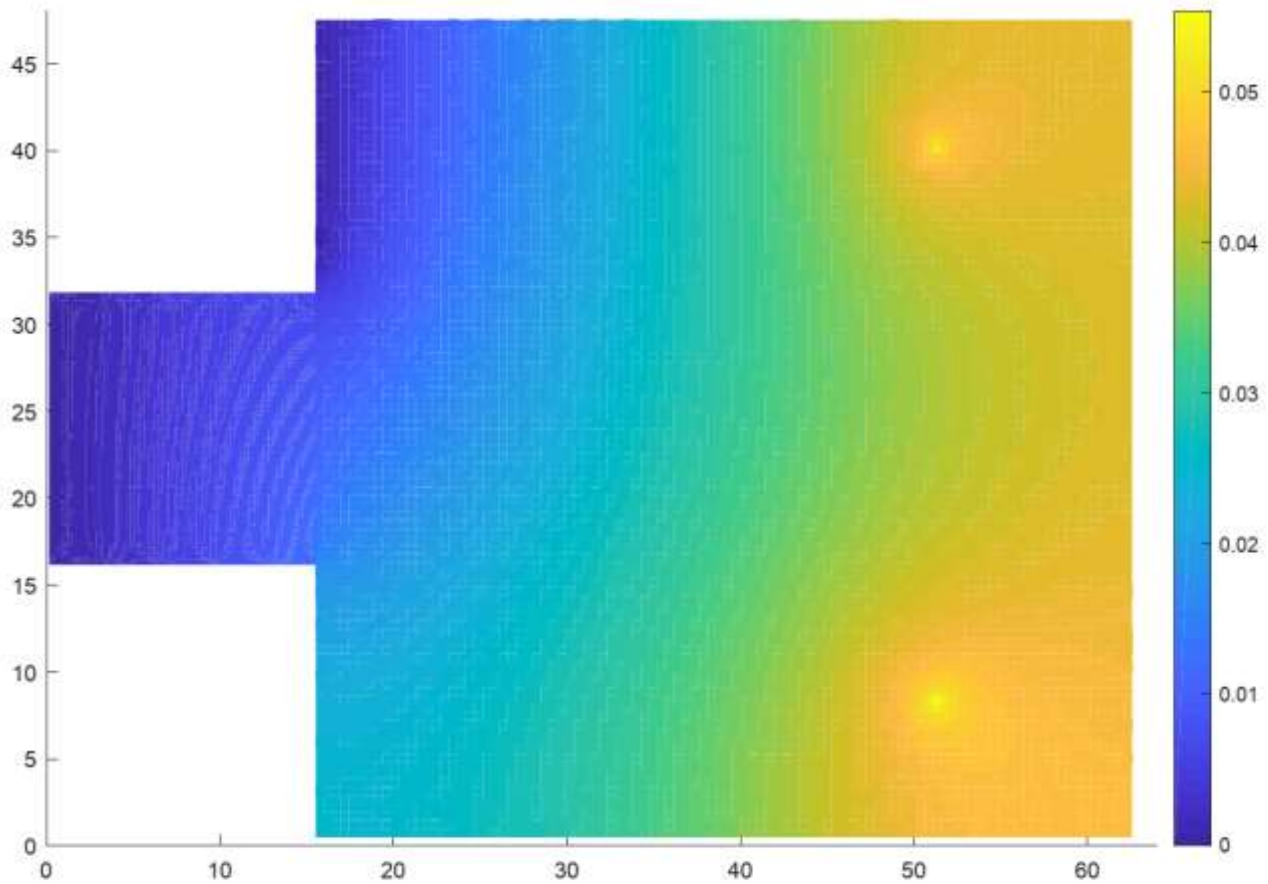


Figure 8. Surface plot of substance concentration for Case 4

From **Table 4**, it is found that Case 4 has the largest total concentration along line l_1 and Case 1 has the largest total concentration along line l_2. It is also found that the largest total concentration for all regions is owned by Case 4. So it can be concluded from the four cases discussed previously, the most effective way to disperse substance in the pool is by choosing a combination of source locations as in Case 4.

4. CONCLUSIONS

A mathematical model of the distribution of steady substance concentration in pool with turbulent water flow has been successfully developed**.** DRM is successfully implemented in the mathematical model that is formed. This method is used to investigate the combination of 2 point sources that are most effective in disperse substance in pool with turbulent water flow. To check the accuracy of numerical calculation using DRM, Set A and B are selected which are applied to problems that has analytical solution. In Set A, the number of boundary collocation points is 96 and the number of interior collocation points is 81. In Set B, the number of boundary collocation points is 216 and the number of interior collocation points is 200. Using Set A, the maximum absolute error is 0.0005, and using Set B, the maximum absolute error is 0.0003. It can be concluded that with Set B, the numerical result is more accurate than Set A. In general, the more the number of collocation points, the more accurate the numerical result**.** From the four surface plots, in general, the distribution pattern of the substance concentration is similar. However, in an area with a larger turbulent flow, the greater the concentration.

AKNOWLEDGEMENT

This study is financed using Dana Selain APBN FSM UNDIP T.A. 2021.

REFERENCES

[1] N. Samec and L. Skerget, "Numerical Modelling of Pollutant Dispersion in Water Systems," *Proc. Int. Conf. Dev. Appl. Comput. Tech. to Environ. Stud.*, vol. 25, pp. 45–54, 1998.

[2] A. D. Polyanin, "Functional Separable Solutions of Nonlinear Convection–Diffusion Equations with Variable Coefficients," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 73, pp. 379–390, 2019, doi: 10.1016/j.cnsns.2019.02.022.

[3] V. F. Morales-Delgado, J. F. Gómez-Aguilar, and M. A. Taneco-Hernández, "Analytical Solution of the Time Fractional Diffusion Equation and Fractional Convection-Diffusion Equation," *Rev. Mex. Fis.*, vol. 65, no. 1, pp. 82–88, 2019, doi: 10.31349/revmexfis.65.82.

[4] F. Wang, C. M. Fan, C. Zhang, and J. Lin, "A Localized Space-Time Method of Fundamental Solutions for Diffusion and Convection-Diffusion Problems," *Adv. Appl. Math. Mech.*, vol. 12, no. 4, pp. 940–958, 2020, doi: 10.4208/aamm.OA-2019-0269.

[5] X. Yue, F. Wang, Q. Hua, and X. Y. Qiu, "A Novel Space–Time Meshless Method for Nonhomogeneous Convection–Diffusion Equations with Variable Coefficients," *Appl. Math. Lett.*, vol. 92, pp. 144–150, 2019, doi: 10.1016/j.aml.2019.01.018.

[6] M. Zhang, W. Zhao, and P. Lin, "Lattice Boltzmann Method for General Convection-Diffusion Equations: MRT Model and Boundary Schemes," *J. Comput. Phys.*, vol. 389, pp. 147–163, 2019, doi: 10.1016/j.jcp.2019.03.045.

[7] J. H. He, "A Simple Approach to One-Dimensional Convection-Diffusion Equation and Its Fractional Modification for E Reaction Arising in Rotating Disk Electrodes," *J. Electroanal. Chem.*, vol. 854, pp. 147–163, 2019, doi: 10.1016/j.jelechem.2019.113565.

[8] I. Boztosun and A. Charafi, "An Analysis of the Linear Advection-Diffusion Equation Using Mesh-Free and Mesh-Dependent Methods," *Eng. Anal. Bound. Elem.*, vol. 26, no. 10, pp. 889–895, 2002, doi: 10.1016/S0955-7997(02)00053-X.

[9] D. L. Clements and M. Lobo, "A Bem for Time Dependent Infiltration from an Irrigation Channel," *Eng. Anal. Bound. Elem.*, vol. 34, no. 12, pp. 1100–1104, 2010, doi: 10.1016/j.enganabound.2010.06.013.

[10] I. Solekhudin and K. C. Ang, "A Laplace Transform Drbem with a Predictor-Corrector Scheme for Time-Dependent Infiltration from Periodic Channels with Root-Water Uptake," *Eng. Anal. Bound. Elem.*, vol. 50, pp. 141–147, 2015, doi: 10.1016/j.enganabound.2014.08.011.

[11] I. Solekhudin, "A Numerical Method for Time-Dependent Infiltration from Periodic Trapezoidal Channels with Different Types of Root-Water Uptake," *IAENG Int. J. Appl. Math.*, vol. 48, no. 1, pp. 84–89, 2018.

[12] Munadi, I. Solekhudin, Sumardi, and A. Zulijanto, "A Numerical Study of Steady Infiltration from a Single Irrigation Channel with an Impermeable Soil Layer," *Eng. Lett.*, vol. 28, no. 3, pp. 1–8, 2020.

[13] B. I. Ang and Y. W.T., "A Dual Reciprocity Element Approach for Axisymmetric Nonlinear Time-Dependent Heat Conduction in a Nonhomogeneous Solid," *Eng. Anal. Bound. Elem.*, vol. 34, no. 8, pp. 697–706, 2010.

[14] N. Y. Ashar, "*An Application of Dual Reciprocity Boundary Element Method on Calculation of Pollutant Concentration Range in Twinned Path with Laminar Water Flow*," Universitas Gadjah Mada, 2020.

[15] N. Y. Ashar and I. Solekhudin, "A Numerical Study of Steady Pollutant Spread in Water from a Point Source," *Eng. Lett.*, vol. 29, no. 3, pp. 840–848, 2021.