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CENTRALITY MEASURES ON BRINKMANN GRAPH: BEFORE AND AFTER NODE DELETION

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Abstract. *Brinkmann graph is a 4-regular graph with 21 nodes and 42 edges discovered by Gunnar Brinkmann in 1992. To our knowledge, the research specifically on Brinkmann graph is still hard to find. Therefore, this research was carried out to analyze the Brinkmann graph in term of its centrality. The centrality measures used are degree, betweenness, and closeness centrality. In this paper, we presented the centrality measures not only on the Brinkmann graph but also on the Brinkmann graph after node deletion to see how the impact of node deletion to the centrality of graph. Before deletion, the results showed that according to the betweenness centrality, there exist 7 nodes who act as mediators or bridges in the Brinkmann graph. Therefore, when a node among these nodes has deleted, it affected not only any other mediator nodes and the furthest nodes from the deleted node but also the nodes that are adjacent to the deleted nodes.*

Keywords: *degree, betweenness, closeness, centrality, Brinkmann, node deletion..*

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1. INTRODUCTION

Brinkmann graph is a 4-regular graph discovered by Gunnar Brinkmann in 1992. It has 21 nodes and 42 edges, girth 5, radius 3, diameter 3, chromatic index 5 and chromatic number 4 [1]. The following figure is how Brinkmann Graph looks like.

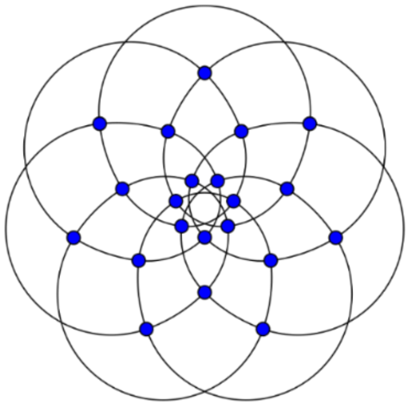


Figure 1. Brinkmann Graph [2]

Many researchers have done studies on 4-regular graphs such as [3], [4], [5], [6] and so on, but to our knowledge, the research specifically on Brinkmann graph is still hard to find.

As a regular graph, every node in the Brinkmann graph obviously has the same degree. In terms of degree centrality, the node with higher degree means the more central it is [7]. In the case of Brinkmann graph, it can be interpreted that each node plays the same role in the graph since all nodes have degree 4. If we talk about degree centrality, we can say that degree centrality is one of the well-known centrality measures used in various studies of network [8]. Centrality measure is useful to find the center of graph namely the node that has an important role in the graph [9]. The study on centrality measures in social networks has been presented by Das K, *et al* in [10]. In their survey paper, they provided various research works on measures of centrality in social network and some of its applications in drug, biology, transportation, traffic, classroom, research, and security. In this paper, we presented the other measures namely the betweenness and closeness not only on the Brinkmann graph but also on the Brinkmann graph after node deletion because we were not only curious about the centrality measures other than degree centrality on the Brinkmann graph but also how the impact of node deletion to the centrality of graph.

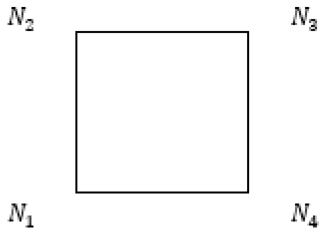
2. RESEARCH METHODS

2.1. Methods

This research was carried out through several steps, namely: graph conversion, processing, results, and analysis.

a. Graph Conversion

The Brinkmann Graph shown in Figure 1 was first converted into an adjacency matrix. The adjacency matrix is a matrix whose rows and columns are indexed by the node set of the graph and entries are 1 for adjacent nodes and 0 otherwise [11]. Here is the example of the adjacency matrix.



(a)

$$\begin{array}{c c} & \begin{array}{cccc} N_1 & N_2 & N_3 & N_4 \end{array} \\ \begin{array}{c} N_1 \\ N_2 \\ N_3 \\ N_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b)

Figure 2. (a) Graph Cycle C_4 (b) The Corresponding Adjacency Matrix

Next, the matrix was input into software RStudio as raw data and then changed back to a graph so that R could plot it as a graph.

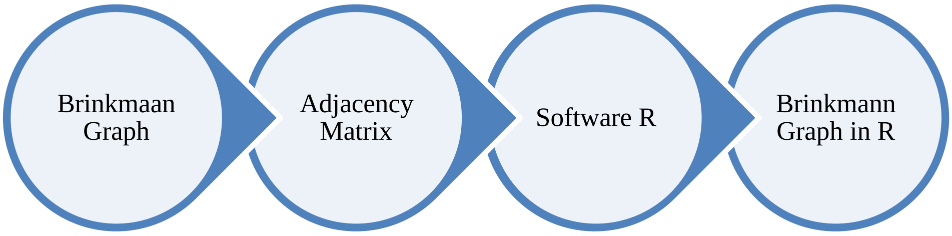


Figure 3. Graph Conversion

b. Processing

In this step, the graph in RStudio was measured its centralities. After that, some nodes were deleted from the graph, and then the centrality of the graph after deletion was remeasured.

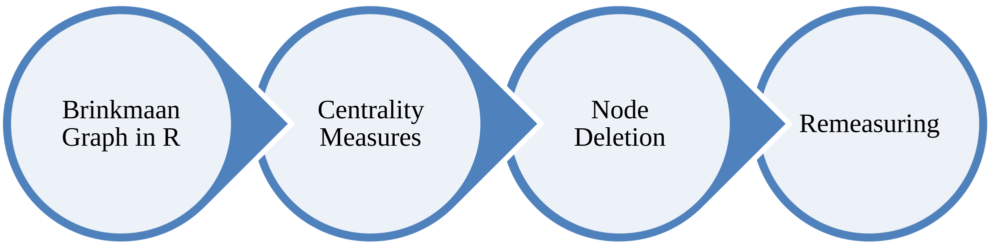


Figure 4. Processing

The centrality measures used in this paper are degree, betweenness, and closeness centrality that can be explained as follows:

1) Degree Centrality (DC)
Degree is the sum of edges connecting a node and other nodes [12], notated as $\deg(i)$. The degree centrality of node i ($DC(i)$) can be measured using equation below:

$$DC(i) = \deg(i) \quad (1)$$

The higher its value then the more connection it has.

2) Betweenness Centrality (BC)
Betweenness centrality is a measure based on the shortest path through an entire graph [13]. This centrality measure can define the node who plays as a mediation in the graph [14]. If the number of

shortest paths from node j to k is denoted as σ_{jk} and the number of those paths through node i as $\sigma_{jk}(i)$ then the betweenness centrality can be measured by equation 2.

$$BC(i) = \sum_{i \neq j \neq k} \frac{\sigma_{jk}(i)}{\sigma_{jk}} \quad (2)$$

The higher its value then the more important the node is.

3) Closeness Centrality (CC)

Closeness centrality indicates the average number of edges that must be traversed from a node to all other nodes [15]. Here is the equation to measure closeness centrality where d_{ij} is the shortest path between node i and all other nodes notated $j \in N$.

$$CC(i) = \frac{1}{\sum_{j \in N} d_{ij}} \quad (3)$$

The higher its value then the closer the node is to other nodes.

c. Result and Analysis

This last section will contain the discussion about the results obtained namely degree, betweenness, and closeness centrality before and after node deletion.

3. RESULTS AND DISCUSSION

3.1. Adjacency Matrix of Brinkmann Graph

The nodes of Brinkmann graph in Figure 1 were first labeled to avoid misunderstanding in further discussion. The graph with labeled nodes can be seen in Figure 5

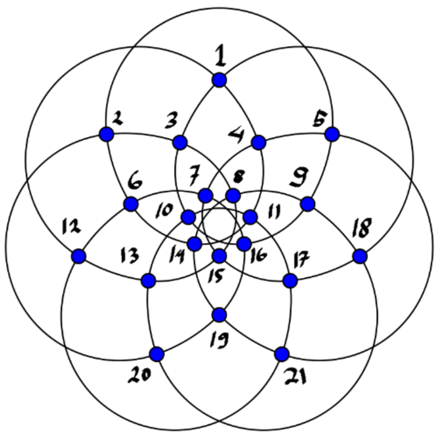


Figure 5. Labeled Brinkmann Graph

This graph was then represented as an adjacency matrix with 21 rows and columns as the number of nodes in Brinkmann graph is 21. The associated adjacency matrix is shown in Figure 6.

N_i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	0	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0
2	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
3	1	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
4	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
5	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
6	0	1	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0
7	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
8	0	0	1	0	0	0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0
9	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	1	0	0	0
10	0	0	1	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0
11	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0
12	1	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
13	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1	0
14	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0
15	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0	1	0	0	0	0
16	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	1	0	0
17	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0	0	1
18	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	1
20	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	0
21	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0

Figure 6. The Adjacency Matrix of Brinkmann Graph

This adjacency matrix was input in software RStudio and returned into a graph in RStudio as raw data to be processed further. Here are the script and the results:

```
> M = matrix(c(0,0,1,1,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0, 0,0,1,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,
1,0, 1,1,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0, 1,0,0,0,1,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0, 0,1,0,
1,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1, 0,1,0,0,0,0,1,0,0,0,0,1,0,1,0,0,0,0,0,0,0, 0,0,0,1,0,1,0,0,
0,0,0,0,0,0,1,1,0,0,0,0,0, 0,0,1,0,0,0,0,0,1,0,0,0,0,1,1,0,0,0,0,0,0, 0,0,0,0,1,0,0,1,0,0,0,0,0,
0,0,1,0,1,0,0,0, 0,0,1,0,0,0,0,0,0,0,1,0,1,0,0,1,0,0,0,0,0, 0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,1,0,
0,0,0, 1,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1, 0,0,0,0,0,0,0,0,0,1,0,1,0,0,1,0,0,0,0,1,0, 0,0,
0,0,0,1,0,1,0,0,1,0,0,0,0,0,0,0,1,0,0, 0,0,0,0,0,0,1,1,0,0,0,0,1,0,0,0,1,0,0,0,0, 0,0,0,0,0,0,1,
0,1,1,0,0,0,0,0,0,0,0,1,0,0, 0,0,0,0,0,0,0,0,0,0,1,0,0,0,1,0,0,1,0,0,1, 1,0,0,0,0,0,0,0,1,0,0,0,
0,0,0,0,1,0,0,1,0, 0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,1,1, 0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,
1,1,0,0, 0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,1,0,1,0,0),nrow=21,ncol=21,dimnames=list(c
("1","2","3","4","5","6","7","8","9","10","11","12","13","14","15","16","17","18","19","20","2
1"),c("1","2","3","4","5","6","7","8","9","10","11","12","13","14","15","16","17","18","19","2
0","21")))
> BG=graph_from_adjacency_matrix(M, mode="undirected", weighted = NULL)
> plot(BG)
>
>
>
>
>
>
>
>
>
>
>
>
```

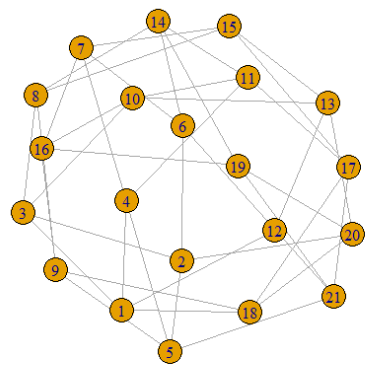


Figure 7. Brinkmann Graph in RStudio

3.2. Centrality Measures Before Node Deletion

As mentioned above, the centrality measures on Brinkmann graph in this article are degree, betweenness and closeness centrality. First, the figure below shows the distance of all pair of nodes on the Brinkmann graph that has been done using software RStudio:

```
> distances(BG)
   1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
1  0 2 1 1 2 2 2 2 2  2  2  1  2  3  3  3  2  1  3  2  2
2  2 0 1 2 1 1 2 2 2  2  3  2  2  2  3  3  3  2  2  1  2
3  1 1 0 2 2 2 3 1 2  1  2  2  2  2  2  2  3  2  3  2  3
4  1 2 2 0 1 2 1 3 2  2  1  2  3  2  2  2  2  2  3  3  2
5  2 1 2 1 0 2 2 2 1  3  2  2  3  3  3  2  2  2  2  2  1
6  2 1 2 2 2 0 1 2 3  3  2  1  2  1  2  2  3  3  2  2  2
7  2 2 3 1 2 1 0 2 2  2  2  2  2  2  1  1  2  3  2  3  3
8  2 2 1 3 2 2 2 0 1  2  2  3  2  1  1  2  2  2  2  3  3
9  2 2 2 2 1 3 2 1 0  2  3  3  3  2  2  1  2  1  2  2  2
10 2 2 1 2 3 3 2 2 2  0  1  2  1  2  2  1  2  3  2  2  3
11 2 3 2 1 2 2 2 2 3  1  0  3  2  1  2  2  1  2  2  3  2
12 1 2 2 2 2 1 2 3 3  2  3  0  1  2  2  3  2  2  2  2  1
13 2 2 2 3 3 2 2 2 3  1  2  1  0  3  1  2  2  2  2  1  2
14 3 2 2 2 3 1 2 1 2  2  1  2  3  0  2  2  2  3  1  2  2
15 3 3 2 2 3 2 1 1 2  2  2  2  1  2  0  2  1  2  3  2  2
16 3 3 2 2 2 2 1 2 1  1  2  3  2  2  2  0  3  2  1  2  2
17 2 3 3 2 2 3 2 2 2  2  1  2  2  2  1  3  0  1  2  2  1
18 1 2 2 2 2 3 3 2 1  3  2  2  2  3  2  2  1  0  2  1  2
19 3 2 3 3 2 2 2 2 2  2  2  2  2  1  3  1  2  2  0  1  1
20 2 1 2 3 2 2 3 3 2  2  3  2  1  2  2  2  2  1  1  0  2
21 2 2 3 2 1 2 3 3 2  3  2  1  2  2  2  2  1  2  1  2  0
```

Figure 8. Distance of Each Pair of Nodes

Further, here are the results of centrality measures with the highlight ones are the highest value:

Table 1. Centrality measures on Brinkmann graph before node deletion

Node (i)	$DC(i)$	$BC(i)$	$CC(i)$
1	4	10.333	0.25
2	4	10.333	0.25
3	4	9.5	0.25
4	4	9.5	0.25
5	4	10.333	0.25
6	4	9.5	0.25
7	4	10.167	0.25
8	4	10.167	0.25
9	4	9.5	0.25
10	4	10.167	0.25
11	4	10.167	0.25
12	4	10.333	0.25
13	4	9.5	0.25
14	4	10.167	0.25
15	4	10.167	0.25
16	4	10.167	0.25
17	4	9.5	0.25
18	4	10.333	0.25
19	4	9.5	0.25
20	4	10.333	0.25
21	4	10.333	0.25

In Table 1, for degree and closeness centrality, the score of each node is the same. It means that each node has equivalent connections and closeness to all other nodes. However, in term of betweenness centrality, it can be seen that the values vary. If we rank the results, then we have node 1, 2, 5, 12, 18, 20 and 21 as first rank with 10.333 that is the highest BC value. It means that these nodes are mostly passed through in the shortest path between two nodes.

3.3. Centrality Measures After Node Deletion

In this part, we will present the centrality measures after deleting a node from Brinkmann Graph. In the previous section, we have found that there exist the influential nodes in the graph according to BC measure, namely node 1, 2, 5, 12, 18, 20 and 21. Since their BC values are the same that is 10.333, we can choose any node among these ones to be deleted from the graph, for example node 1. Certainly, the distance in Figure 8 change as well and can be seen in Figure 9 as follows:

```
> distances(DN)
   2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
2  0 1 2 1 1 2 2 2  2  3  2  2  2  3  3  3  2  2  1  2
3  1 0 3 2 2 3 1 2  1  2  3  2  2  2  2  3  3  3  2  3
4  2 3 0 1 2 1 3 2  2  1  3  3  2  2  2  2  3  3  3  2
5  1 2 1 0 2 2 2 1  3  2  2  3  3  3  2  2  2  2  2  1
6  1 2 2 2 0 1 2 3  3  2  1  2  1  2  2  3  3  2  2  2
7  2 3 1 2 1 0 2 2  2  2  2  2  2  1  1  2  3  2  3  3
8  2 1 3 2 2 2 0 1  2  2  3  2  1  1  2  2  2  2  3  3
9  2 2 2 1 3 2 1 0  2  3  3  3  2  2  1  2  1  2  2  2
10 2 1 2 3 3 2 2 2  0  1  2  1  2  2  1  2  3  2  2  3
11 3 2 1 2 2 2 2 3  1  0  3  2  1  2  2  1  2  2  3  2
12 2 3 3 2 1 2 3 3  2  3  0  1  2  2  3  2  3  2  2  1
13 2 2 3 3 2 2 2 3  1  2  1  0  3  1  2  2  2  2  1  2
14 2 2 2 3 1 2 1 2  2  1  2  3  0  2  2  2  3  1  2  2
15 3 2 2 3 2 1 1 2  2  2  2  1  2  0  2  1  2  3  2  2
16 3 2 2 2 2 1 2 1  1  2  3  2  2  2  0  3  2  1  2  2
17 3 3 2 2 3 2 2 2  2  1  2  2  2  1  3  0  1  2  2  1
18 2 3 3 2 3 3 2 1  3  2  3  2  3  2  2  1  0  2  1  2
19 2 3 3 2 2 2 2 2  2  2  2  2  1  3  1  2  2  0  1  1
20 1 2 3 2 2 3 3 2  2  3  2  1  2  2  2  2  1  1  0  2
21 2 3 2 1 2 3 3 2  3  2  1  2  2  2  2  1  2  1  2  0
```

Figure 9. Distance of Each Pair of Nodes after Deletion

Here is the remeasuring of the centrality measures after node deletion.

Table 2. Centrality measures on Brinkmann graph after node deletion

Node (i)	$DC(i)$	$BC(i)$	$CC(i)$
1	-	-	-
2	4	13.417	0,026
3	3	4.75	0,024
4	3	4.75	0,024
5	4	13.417	0,026
6	4	10.5	0,026
7	4	10.333	0,026
8	4	10.333	0,026
9	4	10.5	0,026
10	4	11.333	0,026
11	4	11.333	0,026
12	3	4.917	0,024
13	4	10.583	0,026
14	4	10.5	0,027
15	4	11	0,027
16	4	10.5	0,027
17	4	10.583	0,026
18	3	4.917	0,024
19	4	9.5	0,027
20	4	11.417	0,026
21	4	11.417	0,026

In Table 2, when node 1 is deleted from the graph, it is obvious that the degree of nodes connected to node 1, namely node 3, 4, 12, 18, will decrease. Consequently, the DC value of these nodes becomes smaller than other nodes. This is also the cause of the BC and CC values of nodes 3, 4, 12, and 18 falling off since these nodes are adjacent to node 1 which eventually affect the shortest path in the graph. For example, the shortest path from node 10 to 18 in Figure 5 is 10-3-1-18. If node 1 is deleted then surely the shortest path from node 10 to 18 will change and exclude node 3 as part of the new shortest path since no edge linking node 3 to 18 directly.

In term of the BC values, we can see in Table 2 that the value of node 2 and 5 increase and they remain the most important nodes in the graph. In Table 1, they were originally part of nodes with the highest BC value. However, when node 1 is deleted, the values of node 2 and 5 go up. It happens because the shortest paths between two nodes through node 1 has gone and diverted mostly to either node 2 or 5. Furthermore, the BC value of node 19 does not change. It means the deletion of node 1 gives no impact to the shortest paths between two nodes that go through node 19. This is reasonable as the shortest distance between node 1 and 19 are 3, equal to diameter of Brinkmann graph, which means there is no shortest path through both node 1 and 19.

Moreover, in term of closeness to all nodes, each node was previously at the same level. However, the deletion node 1 brings effect to the CC value of all nodes even though not quite significant. Node 14, 15, 16, and 19 seem to have the highest value in closeness centrality measure. The reason is among other nodes in Brinkmann graph, node 1 is the furthest node from node 14, 15, 16, and 19. Therefore, if node 1 is deleted then the closeness value of node 14, 15, 16 and 19 rise slightly.

4. CONCLUSIONS

This article aims to show the centrality measures namely degree, betweenness and closeness centrality on Brinkmann graph before and after node deletion. Before deletion, all nodes are at the same level according to the degree and closeness centrality value. However, in betweenness centrality, there are 7 important nodes that act as bridges in the graph. Next, after one of these bridges deleted, the centrality value of nodes linked to the deleted node has decreased while the further nodes from the deleted node have either remained the same or slightly increased.

REFERENCES

[1] W. D. Joyner and C. G. Melles, *Adventures in Graph Theory*. Springer International Publishing, 2017. [Online]. Available: https://books.google.co.id/books?id=NdFEDwAAQBAJ

[2] W. Commons, “File:Brinkmann graph LS.svg — Wikimedia Commons, the free media repository.” 2020. [Online]. Available: https://commons.wikimedia.org/w/index.php?title=File:Brinkmann_graph_LS.svg&oldid=453381461

[3] J. Liang, D. Lou, Z. Qin, and Q. Yu, “A polynomial algorithm determining cyclic vertex connectivity of 4-regular graphs,” *J Comb Optim*, vol. 38, no. 2, pp. 589–607, 2019.

[4] N. Ramya and M. Muthukumar, “On coloring of 4-regular graphs,” *International Journal of Pure and Applied Mathematics*, vol. 116, 2020.

[5] G. Lancia, E. Pippia, and F. Rinaldi, “Proving hamiltonian properties in connected 4-regular graphs: an ILP-based approach,” *arXiv preprint arXiv:2112.05087*, 2021.

[6] W. Liu, Q. Cui, and Y. Wang, “On 4-connected 4-regular graphs without even cycle decompositions,” *Discrete Mathematics*, vol. 345, no. 10, p. 113012, 2022, doi: https://doi.org/10.1016/j.disc.2022.113012.

[7] S. P. Tahalea, E. Salouw, and A. W. A. Wibowo, “TOURISM ON INSTAGRAM: A SOCIAL NETWORK ANALYSIS,” *BAREKENG*, vol. 16, no. 1, pp. 199–206, Mar. 2022.

[8] J. Zhang and Y. Luo, “Degree centrality, betweenness centrality, and closeness centrality in social network,” in *2017 2nd international conference on modelling, simulation and applied mathematics (MSAM2017)*, 2017, pp. 300–303.

[9] C. Sciarra, G. Chiarotti, F. Laio, and L. Ridolfi, “A change of perspective in network centrality,” *Scientific Reports*, vol. 8, no. 1, p. 15269, 2018, doi: 10.1038/s41598-018-33336-8.

[10] K. Das, S. Samanta, and M. Pal, “Study on centrality measures in social networks: a survey,” *Social Network Analysis and Mining*, vol. 8, no. 1, p. 13, 2018, doi: 10.1007/s13278-018-0493-2.

[11] R. B. Bapat, *Graphs and Matrices*. Springer London, 2014. [Online]. Available: https://books.google.co.id/books?id=LWCSBAAAQBAJ

[12] Y. Yustiawan, W. Maharani, and A. A. Gozali, “Degree Centrality for Social Network with Opsahl Method,” *Procedia Computer Science*, vol. 59, pp. 419–426, 2015, doi: 10.1016/j.procs.2015.07.559.

[13] J. Matta, G. Ercal, and K. Sinha, “Comparing the speed and accuracy of approaches to betweenness centrality approximation,” *Computational Social Networks*, vol. 6, no. 1, p. 2, 2019, doi: 10.1186/s40649-019-0062-5.

[14] I. Tsalouchidou, R. Baeza-Yates, F. Bonchi, K. Liao, and T. Sellis, “Temporal betweenness centrality in dynamic graphs,” *International Journal of Data Science and Analytics*, vol. 9, no. 3, pp. 257–272, 2020, doi: 10.1007/s41060-019-00189-x.

[15] R. Goldstein and M. S. Vitevitch, “The influence of closeness centrality on lexical processing,” *Front Psychol*, vol. 8, p. 1683, 2017.