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ON THE THIRD ORDER SOLUTION OF KdV EQUATION BY USING HOMOTOPY PERTURBATION METHOD

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ABSTRACT

In this research, we discussed the solution of the KdV equation using the Homotopy Perturbation method. The KdV equation that describes the water wave equation is solved by using the mixing method between Homotopy and Perturbation methods. Homotopy was built with embedding parameter $p \in [0,1]$, which undergoes a deformation process from linear problems to nonlinear problems, and the assumed solution of the KdV equation is expressed in the form of a power series p up to the third order. The result shows that in each order solution, we obtained resonance term. For handling the condition, we used the Lindsteadt-Poincare method. The wave number k_2 and dispersion relation ω can be obtained in the second-order solution as the effect of using the Lindsteadt-Poincare method.

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1. INTRODUCTION

Mathematical models can be used to describe problems that occur in nature. The problems that occur in nature can usually be modeled in the form of a nonlinear differential equation which is difficult to find an exact solution. A popular method for solving nonlinear differential equations is the perturbation method[1], but according to He [2], the perturbation method has many drawbacks, namely it relies on small parameters, and the determination of small parameters must be precise otherwise, it will result in large errors. Thus, it is necessary to develop a method to solve nonlinear problems that are difficult to solve exactly and do not depend on small parameters.

The homotopy perturbation method is one of the developments of the perturbation method, which was first proposed by J. He in 1999. The application of this method has been carried out on the Duffing equation with an error value of less than 5.8%(see [3]). The homotopy perturbation method was also applied to the pendulum equation and the error value was not more than 1.5% [4]. Both studies are mathematical models in the form of nonlinear ordinary differential equations. Problems in nature are also often modeled with nonlinear partial differential equations. This method has also been applied to nonlinear partial differential equations and the results show that the homotopy perturbation method is very effective and simple [5]. Mohyud-Din and Noor [6] also apply the homotopy perturbation method to nonlinear differential equations and the results obtained are simpler than the adomian decomposition method [7]

In this paper, we use the homotopy perturbation method to solve the KdV equation which is the uni directional water wave equation. The KdV equation has been used in many studies in the context of wave problems in fluid dynamics and its derivation has been carried out for various conditions and approaches [8].

2. RESEARCH METHODS

The KdV equation that we use in this paper is given as (Debnath, [9])

$$\eta_t + c\eta_x + \frac{3}{2}\frac{c}{h}\eta\eta_x + \frac{ch^2}{6}\eta_{xxx} = 0, \tag{1}$$

where $\boldsymbol{c} = \sqrt{\boldsymbol{gh}}$ and g is gravity acceleration, h represents water depth and the total depth $\boldsymbol{H} = \boldsymbol{h} + \boldsymbol{\eta}$. While $\boldsymbol{\eta}$ is elevation of wave. The first two terms $(\boldsymbol{\eta_t} + \boldsymbol{c\eta_x})$ describe the evolution of waves. The term with the coefficient $\left(\frac{\boldsymbol{3c}}{\boldsymbol{2h}}\right)$represents the slope of the nonlinear wave, and the term with the coefficient $\left(\frac{\boldsymbol{ch^2}}{\boldsymbol{6}}\right)$ describes the dispersion. Thus, the Korteweg-de Vries equation is a balance between time evolution, nonlinearity, and dispersion [10].

Initially, we make the KdV equation in the form of wave propagates so we take $\boldsymbol{\eta} = \boldsymbol{\eta}(\boldsymbol{X})$ with $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{Ut}$. Then we apply the the homotopy perturbation method the KdV equation to get the solution and we also make simulation for various parameter of wave in the solution.

The basic idea of Homotopy Perturbation Method for solving nonlinear differential equations is given as follows, consider the following differential equations

$$A(u) - f(r) = 0, \quad r \, \epsilon \, \Omega \tag{2}$$

with boundary condition

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0, \quad r \in \Gamma \tag{3}$$

where $\boldsymbol{A}$ is the general differential operator, $\boldsymbol{f(r)}$ is known as the known function, $\boldsymbol{u}$ is the function to be determined, $\boldsymbol{B}$ is the boundary operator, $\boldsymbol{\Gamma}$ is the boundary of the domain (Ω), and $\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial t}}$ denotes the differential along the normal to $\boldsymbol{\Gamma}$.

In general, operator $\boldsymbol{A}$ is divided into two parts $\boldsymbol{L}$ and $\boldsymbol{N}$, where $\boldsymbol{L}$ is linear operator while $\boldsymbol{N}$ is nonlinear operator. Therefore, Equation (2) can be written as follows:

$$L(x) + N(x) - f(r) = 0, \quad r\epsilon\Omega \tag{4}$$

In the case of the nonlinear **Equation (2)**, it does not include small parameters, we can construct the homotopy equation as follows.

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{5}$$

where

$$v(r,p): \Omega \times [0,1] \to R \tag{6}$$

In **Equation (5)**, $\boldsymbol{p\epsilon[0,1]}$ is the embedding parameter and $\boldsymbol{u_0}$ is the first approximation that satisfies the boundary conditions.

From **Equation (5)** we have

$$H(v,0) = L(v) - L(u_0) = 0, \tag{7}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{8}$$

The process of moving $\boldsymbol{p}$ from zero to one represents moving of $\boldsymbol{v(r,p)}$ from $\boldsymbol{u_0}$ to $\boldsymbol{u_r}$. In topology, this is called deformation, and $\boldsymbol{L(v) - L(u_0)}, \boldsymbol{A(v) - f(r)}$ are homotopies **[11]**.

We introduce embedding parameters $\boldsymbol{p}$ in a more natural way, which is not affected by artificial factors. Furthermore, we can consider $\boldsymbol{p}$ as a small parameter for $\boldsymbol{0 \le p \le 1}$. Therefore, it is reasonable to assume that solution of **Equation (5)** can be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{9}$$

Therefore, the approximate solution of **Equation (2)** can be obtained as

$$\boldsymbol{\lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots} \tag{10}$$

3. RESULTS AND DISCUSSION

Consider the KdV **Equation (1)**, take $\eta = \eta(X)$ with $X = x - Ut$. We obtain

$$(c-U)\eta' + \frac{3c}{2h}\eta\eta' + \frac{ch^2}{6}\eta''' = 0 \tag{11}$$

where $\eta' =$ U is the speed of the wave and (X) represents the shape of the wave. **Equation (2)** is integrated once with respect to X, we get

$$(c-U)\eta + \frac{3c}{4h}\eta^2 + \frac{ch^2}{6}\eta'' = G, \tag{12}$$

where G is the constant of integration. Dividing **Equation (11)** by $\frac{ch^2}{6}$, we obtain

$$\eta'' + \frac{6(c-U)}{ch^2}\eta + \frac{9}{2h^3}\eta^2 = K, \qquad c,h \neq 0, K = \frac{6G}{ch^2} \tag{13}$$

with initial values $\eta(0) = A$ and $\eta'(0) = 0$.

Furthermore, by using the homotopy method, **Equation (13)** is built into a homotopy equation as follows:

$$H(\eta,p) = (1-p)\left[\eta'' + \frac{6(c-U)}{ch^2}\eta\right] + p\left[\eta'' + \frac{6(c-U)}{ch^2}\eta + \frac{9}{2h^3}\eta^2\right] = K. \tag{14}$$

It is clear that when $p = 0$, **Equation (14)** becomes a linear equation. When $p = 1$, **Equation (14)** becomes **Equation (13)**. Thus, the process of moving p from zero to one is a linear oscillator process to the original nonlinear oscillator. By assuming the solution of **Equation (14)** in the terms of a power series p:

$$\eta = \eta_0 + p\eta_1 + p^2\eta_2 + p^3\eta_3 + \cdots. \tag{15}$$

and substituting **Equation (15)** into **Equation (14)** and takes the monochromatic mode as its input. In the first order term there is a secular term which causes the amplitude to increase over time. to overcome the secular terms, we use Linstedt-Poincaire method by assuming the coefficients of linear terms as a power series in p (see **[12]**).Then, we get

$$\frac{6(c-U)}{ch^2} = \omega^2 + p\omega_1 + p^2\omega_2 + p^3\omega_3 + \cdots. \tag{16}$$

and also expanding K as power series in p

$$K = pk_1 + p^2k_2 + p^3k_3 + \ldots \tag{17}$$

Substituting **Equation (15)**, **Equation (16)**, and **Equation (17)** into **Equation (14)**, we obtain the linear differential equation as follows:

$$p^0: \eta_0'' + \omega^2\eta_0 = 0 \tag{18}$$

$$p^1: \eta_1'' + \omega^2\eta_1 + \omega_1\eta_0 + \frac{9}{2h^3}\eta_0{}^2 = k_1 \tag{19}$$

$$p^2: \eta_2'' + \omega^2\eta_2 + \omega_1\eta_1 + \omega_2\eta_0 + \frac{9}{2h^3}2\eta_0\eta_1 = k_2 \tag{20}$$

$$p^3: \eta_3'' + \omega^2\eta_3 + \omega_1\eta_2 + \omega_2\eta_1 + \omega_3\eta_0 + \frac{9}{2h^3}(2\eta_0\eta_2 + \eta_1{}^2) = k_3. \tag{21}$$

Then we obtain the zero order solution of **Equation (18)**

$$\eta_0 = A\cos\omega X. \tag{22}$$

By substituting **Equation (22)** in to the **Equation (19)**, we obtain

$$\eta_1'' + \omega^2\eta_1 = -\omega_1 A\cos\omega X - \frac{9}{4h^3}A^2\cos 2\omega X - \frac{9}{4h^3}A^2 + k_1. \tag{23}$$

In the fact that we get *secular term* in **Equation (23)**. For handling this *secular term*, we use Linstedt-Poincaire method by assuming the coefficients of linear terms as a power series in p, so we get

$$\omega_1 = 0 \tag{24}$$

and

$$k_1 = \frac{9}{4h^3}A^2, \tag{25}$$

Therefore, **Equation (23)** will be

$$\eta_1'' + \omega^2\eta_1 = -\frac{9}{4h^3}A^2\cos 2\omega X. \tag{26}$$

Then the solution of **Equation (26)** is

$$\eta_1 = \frac{3A^2}{4\omega^2h^3}\cos 2\omega X. \tag{27}$$

We substitute **Equation (24)** and **Equation (27)** into **Equation (20)** then we obtain second order equation as follows.

$$\eta_2'' + \omega^2\eta_2 + A\cos\omega X\left(\omega_2 + \frac{27A^2}{8\omega^2h^6}\right) + \frac{27A^3}{8\omega^2h^6}\cos 3\omega X - k_2 = 0. \tag{28}$$

In the same way for handling secular term in **Equation (28)**, we use *Linstedt-Poincare method* then we obtain

$$\omega_2 = -\frac{27A^2}{8\omega^2h^6} \tag{29}$$

and

$$k_2 = 0, \tag{30}$$

The **Equation (20)** will be

$$\eta_2'' + \omega^2\eta_2 = -\frac{27A^3}{8\omega^2h^6}\cos 3\omega X. \tag{31}$$

The solution of the second order **Equation (31)** is obtained as follows.

$$\eta_2(X) = \frac{27A^3}{64\omega^4h^6}\cos 3\omega X. \tag{32}$$

In the third order equation we have,

$$\eta_3'' + \omega^2\eta_3 = -\omega_3 A\cos\omega t - \frac{81A^4}{64\omega^4 h^9} + k_3 + \left(\frac{81A^4}{128\omega^4 h^9}\right)\cos 2\omega X - \left(\frac{405A^4}{128\omega^4 h^9}\right)\cos 4\omega X. \quad (33)$$

By using *Linstedt-Poincare* method [13], we can drop the secular term and we obtain

$$\omega_3 = 0 \quad (34)$$

and we obtain also

$$k_3 = \frac{81A^4}{64\omega^4 h^9}. \quad (35)$$

Therefore Equation (33) can be rewritten as follows.

$$\eta_3'' + \omega^2\eta_3 = \left(\frac{81A^4}{128\omega^4 h^9}\right)\cos 2\omega X - \left(\frac{405A^4}{128\omega^4 h^9}\right)\cos 4\omega X. \quad (36)$$

The third order solution of Equation (36) is obtained as

$$\eta_3 = \frac{27A^4}{128\omega^6 h^9}\cos 4\omega X. \quad (37)$$

By using homotopy perturbation method we can rewrite Equation (16) and Equation (17) as follows.

$$\frac{6(c-U)}{ch^2} = \omega^2 - \frac{27A^2}{8\omega^2 h^6} \quad (38)$$

$$K = \frac{9A^2}{4h^3} + \frac{81A^4}{64\omega^4 h^9}. \quad (39)$$

By using Equation (29), we obtain ω as follows.

$$\omega = \frac{1}{2h^2 c}\sqrt{3hc\left(4ch - 4hU + \sqrt{16c^2h^2 - 32ch^2U + 16h^2U^2 + 6c^2A^2}\right)}. \quad (40)$$

The Equation (40) is called the dispersion relation of the wave (see [14],[15]). The solution of KdV equation up to third order solution is given by

$$\eta = \eta_0 + \eta_1 + \eta_2 + \eta_3$$

$$\eta = A\cos\omega X + \frac{3A^2}{4\omega^2 h^3}\cos 2\omega X + \frac{27A^3}{64\omega^4 h^6}\cos 3\omega X + \frac{27A^4}{128\omega^6 h^9}\cos 4\omega X. \quad (41)$$

The total solution and third order solution of Equation (1) by using the homotopy perturbation method are described in Figure 1. In the solution, we take A = 0,04 m, water depth h = 5 m, gravity acceleration g = 9,81 m/s^2. Velocity of wave U=*1* m/s. We observed each order solution at time $t = 10$. While the wave profiles or the total solution based on the solution for the KdV equation at different times are given in Figure 2.

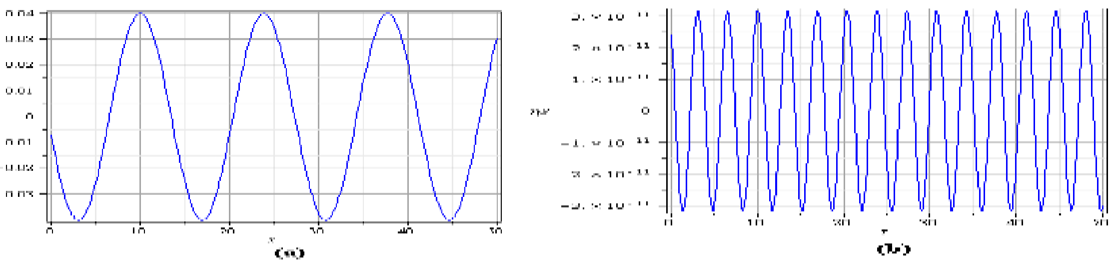

Figure 1. (a)The total solution and (b) the third order solution of the KdV equation at t=10.

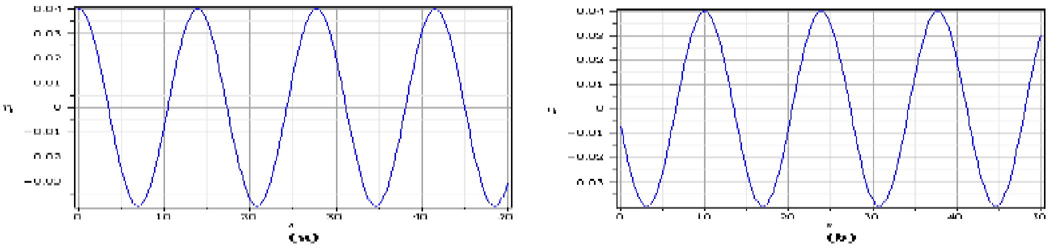

Figure 2. The wave profile of the KdV equation at (a) t=0, (b) t=10,

4. CONCLUSIONS

In this paper, we studied the solution of the KdV equation up to the third order by using homotopy perturbation method. We found that in each order solution, there exist secular term. For handling this term, we used *Linstedt-Poincare* method, such that we obtained frequency and wave number in each order of equations. Finally, we obtained the total solution which is sum of the zero order solution, first order solution, the second order solution and the third order solution and also we found the dispersion relation of the wave.

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