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RADIO LABELING OF BANANA GRAPHS

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ABSTRACT

Let G=(V, E) be a graph. An L(3,2,1) labeling of G is a function $f: V \to \mathbb{N} \cup \{0\}$ such that for every $u, v \in V, |f(u) - f(v)| \geq 3$ if $d(u,v) = 1$, $|f(u) - f(v)| \geq 2$ if $d(u,v) = 2$, and $|f(u) - f(v)| \geq 1$ if $d(u,v) = 3$. Let $k \in \mathbb{N}$, a $k - L(3,2,1)$ labeling is a labeling $L(3,2,1)$ where all labels are not greater than k. An $L(3,2,1)$ number of G, denoted by $\lambda_{3,2,1}(G)$, is the smallest non-negative integer k such that G has a $k - L(3,2,1)$ labeling. In this paper, we determine $\lambda_{3,2,1}$ of banana graphs.

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1. INTRODUCTION

In real-world circumstances, graph theory has numerous applications. Find an assignment that satisfies various constraints and minimizes the value of the given objective function given a collection of radio transmitters to be assigned operating frequencies. The first frequency assignment issues arose as a result of the discovery that transmitters assigned to the same or closely related frequencies could interfere with one another **[1]**. To minimize interference, assigning different transmitters to different noninterfering frequencies, or coming as close to this as possible within the constraints, was a simple solution. Another issue is determining how to set a different frequency and minimize the frequency range used to reduce costs.

Roberts developed a new technique for frequency assignment in 1991, classifying stations as closed or very closed **[2]**. If the distance between two stations is one and two, they are said to be 'very close' and 'closed,' respectively. $L(2,1)$- labeling has been extensively researched in the past,**[3]**–**[6]**. Calamoneri et al. studied $L-h,\ k$ labeling of co-comparability, interval, and circular-arc graphs in 2009 **[7]**. They demonstrated that for co-comparability graphs, $\lambda_{h,k}\,(G) \leq max(h,2k)2\Delta + k$ and IG $\lambda_{h,k}(G) \leq max(h,2k)\Delta$. They also demonstrated that $\lambda_{h,k}(G) \leq max(h,2k)2\Delta + hw\$$ for CAG. Paulet also demonstrated that $\lambda_{2,1}(G) \leq max\{4\Delta - 2{,}5\Delta - 8\}$ and $\lambda_{0,1}(G) = \Delta - 1$ for permutation graphs **[8]**.

In 1992, Griggs and Yeh **[9]** modeled the problem in constructing a graph by assuming the transmitter were a set of vertices, very close transmitters were assumed to be neighboring vertices, and sufficiently close transmitters were assumed to be two vertices apart on the graph. Defined labeling $L(2,1)$ as the answer to the problem of setting the frequency. In 1992, they studied the labeling number $L(2,1)$, denoted by $L(2,1)$, in several classes of graphs between other path graphs, cycle graphs, cube graphs, and tree graphs. In addition, they studied the lower and upper bounds for $L(2,1)$ on any graphs with maximum degree Δ. In 2004, Liu and Sahou defined $L(3,2,1)$ labeling. They get results about $\lambda_{3,2,1}$ in several classes of graphs, with determining boundaries for $\lambda_{3,2,1}$ on irregular graphs, Halin Graphs, and Planar Graphs with a maximum degree Δ. Then in 2005, Clipperton cs **[10]** studied $\lambda_{3,2,1}$ on classes of graphs like path graphs, cycle graphs, n-ary tree graph and regular caterpillar graphs.

In 2011, Chia studied some general concepts regarding labeling $L(3,2,1)$ and gave an upper bound of $\lambda_{3,2,1}$ for any graph with a maximum degree Δ. Furthermore, they also described the labeling $L(3,2,1)$ on ary tree, rooted tree, and Cartesian product of path graphs and cycle graph. In **[10]**, Jean displayed that $\lambda_{d,2,1}(K_n) = d(n-1) + 1$ and $\lambda_{d,2,1}(K_{m,n}) = d + 2(m+n) - 3$. $L(3,2,1)$-labeling have been studied in the present year **[10]**. Amanathulla and Madhumangal in 2017 **[11]**, **[12]**, defined labeling $L(3,2,1)$ interval graphs issues. Furthermore, they also did research in 2021 to upgrade their previous studies with $L(3,2,1)$-labeling issues with trapezoids **[12]**.

Kusbudiono and his friends, in 2022 **[13]**, studied about $L(2,1)$ labeling of lollipop and pendulum graphs. The $L(2,1)$ creating the labeling for the lollipops $L\_\{m,n\}$ with $m \geq\ 3$ and n positive integers is discussed in this paper. The goal of this research is to find the number with the smallest possible span that comes from the labeling $L(2,1)$ on the lollipop graph $L_{m,n}$ and they may symbolize $\lambda_{2,1} - L_{m,n}$. Moreover, find the span value with the smallest range using the $L(2,1)$ labeling on the pendulum graph. Simulation software for $L(2,1)$ labeling of lollipop graphs is also produced by it with large m and n values. They found that the smallest possible length of a span for a lollipop graph is $\lambda_{2,1}\ - L_{m,n} = 2m - 2$ and length of a pendulum graphs, P_n^k with $k \geq\ 4$ and $n \geq$ is $k + 1$.

2. RESEARCH METHODS

The research method used is to determine the value of $\lambda_{3,2,1}$ for banana graphs. The value is obtained by specifying the lower limit for $\lambda_{3,2,1}$ and determine its upper limit. Determination of the lower limit of $\lambda_{3,2,1}$ the graph studied begins by using the lower limit of $\lambda_{3,2,1}$ by Chia et al. **[14]** or using contradiction techniques. While the upper limit is $\lambda_{3,2,1}$. determined by the labeling construction method $L(3,2,1)$.

Two vertices x and y on the graph G is known as a neighbor if xy is the edge on G. The degree of a vertex v, $d(v)$ is many edges adjacent to v. Complement of graph G, is notated $\bar{G}$ is a graph where $V(\overline{G}) = V(G)$ and $E(\bar{G}) = [V(G)]^2 \backslash E(G)$. Star graph S_n is the adjacent graph with order $n+1$, where a degree of a vertex is n that is named the center vertex. The following is definitions of radio labeling.

Definition 1. Let $G = (V, E)$ be a graph and f be a mapping \$$f : V \to \mathbb{N}$. f is an $L(3,2,1)$ - labeling of G if, for all $x, y \in V$,

$$|f(x) - f(y)| \geq \begin{cases} 3, & if\ d(x,y) = 1; \\ 2, & if\ d(x,y) = 2; \\ 1, & if\ d(x,y) = 3 \end{cases}$$

Definition 2. Let $k \in \mathbb{N} \cup \{0\}$. $k - L(3,2,1)$ labeling is labeling $L(3,2,1)$ with every label used not greater than k. $L(3,2,1)$ number on G graph, denoted as $\lambda_{3,2,1}(G)$, is the smallest number k so that G graph has the labeling $k - L(3,2,1)$.

3. RESULTS AND DISCUSSION

3.1. Bounds of $\lambda_{3,2,1}$ Banana Graphs

The first step we should take is to determine the smallest number of Banana graphs that were obtained using the determined lowest bound and upper bound for $\lambda_{3,2,1}$. Determination is lower bound by using Chia [15] or the contradiction method. While the upper bound of $\lambda_{3,2,1}$ gets by construction method of $L(3,2,1)$ labeling. Before we discuss deep to determine the lowest bound and upper bound of $\lambda_{3,2,1}$ Banana graph, take a look at the subsequent corollary and lemma holds.

Lemma 1. *If G' subgraph of G, then $\lambda_{3,2,1}(G') \leq \lambda_{3,2,1}(G)$.*

Proof. By contradiction, suppose $\lambda_{3,2,1}(G') > \lambda_{3,2,1}(G)$. Let $\lambda_{3,2,1}(G') = k_1$ and $\lambda_{3,2,1}(G) = k_2$, meaning that k_1 is the smallest number, so graph G has the labeling $k_1 - L(3,2,1)$. Let f $k_1 - L(3,2,1)$ labeling on a graph G. $G' \subset G, V(G') \subset V(G), E(G') \subset E(G)$ and f labeling $k_1 - L(3,2,1)$ on graph G. Thus, for every vertices $u, v \in V(G') \subset V(G)$, $|f(u) - f(v)| \geq 3$ for every vertices $u, v \in V(G') \subset V(G)$ with $d(u,v) = 1$, $|f(u) - f(v)| \geq 2$ for every vertices $u, v \in V(G') \subset V(G)$ with $d(u,v) = 2$ and $|f(u) - f(v)| \geq 1$ for every vertices $u, v \in V(G') \subset V(G)$ with $d(u,v) = 3$. This means that there is $k_1 - L(3,2,1)$ on graph G'. $\lambda_{3,2,1}(G) = k_2$ and $k_2 > k_1$, meaning that there is a number smaller than k_2 (named k_1) so that graph G' has labeling $k_1 - L(3,2,1)$. This contradicts the minimal $\lambda_{3,2,1}(G')$. Therefore $\lambda_{3,2,1}(G') \leq \lambda_{3,2,1}(G)$.

Lemma 2. *For star graph $S_n = K_{1,n} = \{v\} + \overline{K_n}$ then $\lambda_{3,2,1}(S_n) = 2n+1$. Further, if f is $2n+1 - L(3,2,1)$, labeling on S_n graph, then $f(v) \in \{0, 2n+1\}$.*

The followings are corollaries from Lemma 2.

Corollary 1. For any graph G with $\Delta(G) = \Delta > 0$, then $\lambda_{3,2,1}(G) \geq 2\Delta + 1$. If $\lambda_{3,2,1}(G) = 2\Delta + 1$ and f is any labeling $2\Delta + 1 - L(3,2,1)$, then for every $v \in V(G)$ where $d(v) = \Delta$ such that $f(v) \in \{0, 2\Delta + 1\}$.

Proof. Based on **Lemma 1**, $\lambda_{3,2,1}(G) \geq \lambda_{3,2,1}(S_\Delta) = 2\Delta + 1$. If equality occurs, then by using **Lemma 2**, we get for any f is $2\Delta + 1$ −labelling, applies $f(v) \in \{0, 2\Delta + 1\}$ for every $v \in V(G)$ with $d(v) = \Delta$.

Corollary 2. Let G be a graph with $\lambda_{3,2,1}(G) = 2\Delta + 1$ and f is $2\Delta + 1 - L(3,2,1)$ on graph G. If for any vertices $v \in V(G)$ with $d(v) = \Delta$ and $f(v) = 0$ or $f(v) - 2\Delta + 1$, then $f(N(v)) = \{f(u) | u, v \in V(G)\} = \{2i + 1 | 1 \leq i \leq \Delta\}$ or $f(N(v)) = \{2(i-1\} | 1 \leq i \leq \Delta\})$.

To demonstrate that there is $2\Delta + 2 - L(3,2,1)$ labeling, consider the following lemma and corollary.

Corollary 3. Let G be a graph with $\Delta \geq 1$. If there is $v_1, v_2 \in V(G)$ with $d(v_1, v_2) = 2$ and $d(v_1) = d(v_2) = \Delta$, then $\lambda_{3,2,1}(G) \geq 2\Delta + 2$.

Proof. By contradiction, suppose $\lambda_{3,2,1}(G) \leq 2\Delta + 1$. Cause corollary 2.1, then $\lambda_{3,2,1}(G) = 2\Delta + 1$ and for any f $2\Delta + 1 - L(3,2,1)$ labeling on graph G, $f(v) \in \{0, 2\Delta + 1\}$ for every $v \in V(G)$ with $d(v) = \Delta$. Let f $2\Delta + 1 - L(3,2,1)$ labeling on graph G. Hence there are $v_1, v_2 \in V(G)$ with $d(v_1, v_2) = 2$ and $d(v_1) = d(v_2) = \Delta$. Thus, $f(v_1) \neq f(v_2)$ then $\{f(v_1), f(v_2)\} = \{0, 2\Delta + 1\}$. Without compromising generally, let $f(v_1) = 0$ and $f(v_2) = 2\Delta + 1$. Based on corollary 2.2, we get $f(N(v_1)) = \{f(u) | uv_1 \in E(G)\} = \{2i + 1 | \, 1 \leq i \leq \Delta\}$. Since u is a neighbor of v_1, thus $f(u) = 2\Delta + 1$, then $\{f(u) - f(v_2)| = 0$ with $d(u, v_2) \leq 3$. Contradiction.

Meanwhile, the Banana graph $B_{m,n}$ is a graph obtained from m of a copy of the graph S_n and a vertex v where $v \notin V(S_n)$, by connecting a leaf of each graph S_n with a vertex v.

Theorem 1. *Let m and n be positive integers with $m \geq 3$ and $n \geq 1$. If $B_{m,n}$ is a banana graph, then*

$$\lambda_{3,2,1}(B_{m,n}) = \begin{cases} 2\Delta + 1, for\ m \neq n; \\ 2\Delta + 2, for\ m = n. \end{cases}$$

Proof. Suppose $V(B_{m,n}) = \{v_0, \dots, v_m\} \cup \{v_1^0, \dots, v_1^{n-1}\} \cup \dots \cup \{v_m^0, \dots, v_m^{n-1}\}$. The determination of the lower bound for $\lambda_{3,2,1}(B_{m,n})$ uses direct evidence by taking advantage of the consequences present in the previous. To prove it is divided into two cases.

1) Case $m \neq n$

   In this case, the distance of each vertex of Δ is not equal to two. Based on corollary **1,** obtained $\lambda_{3,2,1}(B_{m,n}) \geq 2\Delta + 1\ for\ m \neq n$.

2) Case $m = n$

   Note that if $m = n$, then Δ is located at the vertex v_i^0 for $i \in \{1, 2, \dots, m\}$ and in v_0. Since $d(v_i^0) = \Delta = d(v_i)$ and $d(v_i^0, v_i) = 2$, then based on the corollary **1**, $\lambda_{3,2,1}(B_{m,n}) \geq 2\Delta + 2$ for $m = n$.

Furthermore, the determination of the upper bound for $\lambda_{3,2,1}(B_{m,n})$ is divided into two cases.

1) Case $m \neq n$
   This section is subdivided into two subsections. Namely, $m > n$ and $m < n$, because the functions of each case are different.

   a. For $m > n$
      Claim that if $m > n$, then $\lambda_{3,2,1}(B_{m,n}) = 2\Delta + 1$. To indicate that $\lambda_{3,2,1}(B_{m,n}) = 2\Delta + 1$ define the labeling function $L(3,2,1)$ on $(B_{m,n})$ with $m > n$ as follows:

$$f(v_i) = \begin{cases} 0, & if\ \ i = 0; \\ 2i + 1, & if\ \ i \in [1, m] \end{cases}$$

$$f(v_i^0) = \begin{cases} 2m, & if\ \ i = 1; \\ 2(i - 1), & if\ \ i \in [2, m] \end{cases}$$

$$f(v_1^j) = \begin{cases} 1, & if\ \ j = 1; \\ 2j + 1, & if\ \ j \in [2, n - 1] \end{cases}$$

$$f(v_i^j) = \begin{cases} 2j - 1, & if\ 1 \leq j \leq min(i - 2, n - 1), i \in [2, m]; \\ 2j + 5, & if\ min(i - 1, n) \leq j \leq n - 1, i \in [2, m] \end{cases}$$

      Furthermore, with construction that f is $L(3,2,1)$ labeling for $m > n$. Take any vertex $u, v \in V(B_{m,n})$ with $d(u, v) = 1$, $d(u, v) = 2$ and $d(u, v) = 3$. Then we get f represents the labeling of $2\Delta + 1$ - $L(3,2,1)$ on $B_{m,n}$ for $m > n$.

   b. For $m < n$
      Claim that if $m < n$ then $\lambda_{3,2,1}(B_{m,n}) = 2\Delta + 1$. To indicate that $\lambda_{3,2,1}(B_{m,n}) = 2\Delta + 1$ define the labeling function $L(3,2,1)$ on $(B_{m,n})$ with $m < n$ as follows:

$$f(v_i) = \begin{cases} 2n - 1, & if\ \ i = 0; \\ 2\,i - 2, & if\ \ i \in [1, m] \end{cases}$$

$$f(v_i^0) = 2n + 1, \quad if\ i \in [1, m]$$

$$f(v_1^j) = \begin{cases} 2j-2, & if\ \ 1 \le j \le min(i-1, n-1), i \in [1,m] \\ 2j, & if\ min(i,n) \le j \le n-1, i \in [1,m] \end{cases}$$

Furthermore, with construction that f is $L(3,2,1)$ labeling for $m < n$. Take any vertex $u, v \in V(B_{m,n})$ with $d(u,v) = 1$, $d(u,v) = 2$ and $d(u,v) = 3$. Then we get f represents the labeling of $2\Delta + 1$ - $L(3,2,1)$ on $B_{m,n}$ for $m < n$.

2) Case $m = n$
Claim that if $m = n$, then $\lambda_{3,2,1}(B_{m,n}) = 2\Delta + 2$. To indicate that $\lambda_{3,2,1}(B_{m,n}) = 2\Delta + 2$ define the labeling function $L(3,2,1)$ on $(B_{m,n})$ with $m = n$ as follows:

$$f(v_i) = \begin{cases} 3, & if\ \ i = 0; \\ 0, & if\ \ i = 1 \\ 2i+2, & if\ i \in [2,m] \end{cases}$$

$$f(v_i^0) = \begin{cases} 5, & if\ \ i = 1; \\ 1, & if\ \ i \in [2,m] \end{cases}$$

$$f(v_1^j) = \begin{cases} 2, & if\ \ j = 1; \\ 2j+4, & if\ \ j \in [2, n-1] \end{cases}$$

$$f(v_i^j) = \begin{cases} 2j-2, & if\ \ 1 \le j \le min(i-1, n-1), i \in [2,m] \\ 2j+4, & if\ min(i,n) \le j \le n-1, i \in [2,m] \end{cases}$$

Furthermore, with the construction that f is $L(3,2,1)$ labeling for $m = n$. Take any vertex $u, v \in V(B_{m,n})$ with $d(u,v) = 1$, $d(u,v) = 2$ and $d(u,v) = 3$. Then we get f represents the labeling of $2\Delta + 2$ - $L(3,2,1)$ on $B_{m,n}$ for $m = n$.

4. CONCLUSIONS

A graph obtained from m of a copy of the graph S_n and a vertex v where $v \notin V(S_n)$ by connecting a leaf of each graph S_n with a vertex of v called the banana graph $B_{m,n}$. The numbers $L(3,2,1)$ of the banana graph $B_{m,n}$ with $m \ge 3$ and $n \ge 1$, are as follows:

$$\lambda_{3,2,1}(B_{m,n}) = \begin{cases} 2\Delta + 1, for\ m \ne n; \\ 2\Delta + 2, for\ m = n. \end{cases}$$

REFERENCES

[1] W. K. Hale, “Frequency assignment: Theory and applications,” *Proc. IEEE*, vol. 68, no. 12, pp. 1497–1514, 1980.
[2] F. S. Roberts, “T-colorings of graphs: recent results and open problems,” *Discrete Math.*, vol. 93, no. 2–3, pp. 229–245, 1991.
[3] A. A. Bertossi and C. M. Pinotti, “Approximate L (δ1, δ2,…, δt)-coloring of trees and interval graphs,” *Networks An Int. J.*, vol. 49, no. 3, pp. 204–216, 2007.
[4] D. Indriati and T. S. M. N. Herlinawati, “L (d, 2, 1)-labeling of star and sun graphs,” *Math. Theory Model.*, vol. 4, no. 11, 2012.
[5] N. Khan, M. Pal, and A. Pal, “L (0, 1)-labeling of cactus graphs,” 2012.
[6] B. M. Kim, W. Hwang, and B. C. Song, “$L(3,2,1)$-Labeling for the Product of a Complete Graph and a Cycle,” *Taiwan. J. Math.*, vol. 19, no. 3, May 2015, doi: 10.11650/tjm.19.2015.4632.
[7] T. Calamoneri, “The L(h, k)-Labelling Problem: An Updated Survey and Annotated Bibliography,” *Comput. J.*, vol. 54, no. 8, pp. 1344–1371, Aug. 2011, doi: 10.1093/comjnl/bxr037.
[8] S. Ghosh, P. Podge, N. C. Debnath, and A. Pal, “Efficient Algorithm for L (3, 2, 1)-Labeling of Cartesian Product Between Some Graphs,” in *Proceedings of 32nd International Conference on*, 2019, vol. 63, pp. 111–120.
[9] J. A. Gallian, “A dynamic survey of graph labeling,” *Electron. J. Comb.*, vol. 1, no. DynamicSurveys, p. DS6, 2018.
[10] J. Clipperton, “L (d, 2, 1)-labeling of simple graphs,” *Rose-Hulman Undergrad. Math. J.*, vol. 9, no. 2, p. 2, 2008.
[11] S. Amanathulla and M. Pal, “L (3, 2, 1)-and L (4, 3, 2, 1)-labeling problems on interval graphs,” *AKCE Int. J. Graphs Comb.*, vol. 14, no. 3, pp. 205–215, 2017.
[12] S. Amanathulla, B. Bera, and M. Pal, “L(2,1,1)-Labeling of Circular-Arc Graphs,” *J. Sci. Res.*, vol. 13, no. 2, pp. 537–544, May 2021, doi: 10.3329/JSR.v13i2.50483.
[13] Kusbudiono, I. A. Umam, I. Halikin, and M. Fatekurohman, “L (2,1) Labeling of Lollipop and Pendulum Graphs,” 2022, doi:

10.2991/case.k.220202.010.

[14] M.-L. Chia, D. Kuo, H. Liao, C.-H. Yang, and R. K. Yeh, “$ L (3, 2, 1) $-Labeling Of Graphs,” *Taiwan. J. Math.*, vol. 15, no. 6, pp. 2439–2457, 2011.

[15] M. Chia, D. Kuo, H. Liao, C. Yang, and R. K. Yeh, “L (3 , 2 , 1)-labeling of graphs,” vol. 15, no. 6, pp. 2439–2457, 2011.