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BIPARTITE GRAPH ASSOCIATED WITH ELEMENTS AND COSETS OF SUBRINGS OF FINITE RINGS

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ABSTRACT

Let R be a finite ring. The bipartite graph associated to elements and cosets of subrings of R is a simple undirected graph $\Gamma(R)$ with vertex set $R \cup \boldsymbol{S}_R$, where $\boldsymbol{S}_R$ is the set of all subrings of R, and two vertices $r \in R$ and $S \in \boldsymbol{S}_R$ are adjacent if and only if $rS = Sr$. In this study, we investigate some basic properties of the graph $\Gamma(R)$. In particular, we investigate some properties of $\Gamma(M_2(\mathbb{Z}_n))$, where $M_2(\mathbb{Z}_n)$ is the ring of matrices over $\mathbb{Z}_n$. Also, we study the diameter of the bipartite graph associated to the quaternion ring.

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1. INTRODUCTION

The study of algebraic structures and graph theory has been considerable attention over the past several years. There are many important notions of such interplay, for instance, see [1]–[6].

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For any edge uv, where u and v are in $V(\Gamma)$, we call u and v as the end points of uv. We say that Γ is connected if there is a path between every pair of vertices of Γ. The length of the smallest cycle contained in a graph Γ is called the girth of Γ. The distance between u and v in graph Γ is the length of the shortest path between u and v and denoted $d(u,v)$. The length of the longest path between two distinct vertices of connected graph Γ is called the diameter of Γ and it is denoted by $\operatorname{diam}(\Gamma)$. A bipartite graph is a graph whose vertices can be partitioned into two disjoint sets A and B such that every edge connects a vertex in A and a vertex in B. In the case that every vertex in A is adjacent to every vertex in B, we call the graph as a complete bipartite graph. A cycle that meets every vertex in a graph exactly once is called a Hamiltonian cycle and a graph that includes a Hamiltonian cycle is called a Hamiltonian graph. A planar graph is a graph that can be drawn in the plane without crossings but possibly at the end points. By a vertex cover, we mean a set of some vertices of a graph that contains at least one of the end points of every edge in the graph. Moreover, a vertex cover having the smallest possible number of vertices for a given graph is known as a minimum vertex cover of Γ, denoted $\beta(\Gamma)$. A matching M of a graph Γ is a set of edges of Γ having no common end points. A matching of Γ is said to be maximum if Γ has no matching M' with $|M'| > |M|$.

Throughout this article, R denotes any finite ring. For any subring S of R and for any $r \in R$, the sets $rS = \{rs | s \in S\}$ and $Sr = \{sr | s \in S\}$ are called as a left coset and a right coset, respectively. The set of all subrings of R is denoted by $\boldsymbol{S}_R$. For further definitions and theorems of graph theory, group theory, and matrices over commutative theory, we refer to [7]–[14], respectively.

This article concerns on the bipartite graph associated with elements and cosets of subrings of R that is motivated by [5]. In Section 2, we introduce the bipartite graph associated with elements and cosets of subrings of R and we investigate some basic properties of the graph including connectivity, diameter, girth, and planarity. We also study the hamiltonicity property of this graph. Moreover, we present some relations between graph theory and matrices over $\mathbb{Z}_n$ through this graph and give some conjectures about minimum vertex cover and maximum matching. We close the result and discussion section by giving the diameter of the bipartite graph associated to quaternion ring for some cases with the definition of quaternion ring refers to [15].

2. RESULTS AND DISCUSSION

In this section, firstly, we give definition of a bipartite graph associated to elements and cosets of subrings of finite rings and some basic properties of the graph.

Definition 1. Let R be a finite ring. The bipartite graph associated to element and cosets of subrings of R is a simple undirected graph $\Gamma(R)$ with vertex set $V(\Gamma(R)) = R \cup \boldsymbol{S}_R$ where $\boldsymbol{S}_R$ is the set of all subrings of R and two vertices $r \in R$ and $S \in \boldsymbol{S}_R$ are adjacent if and only if $rS = Sr$.

Theorem 1. *Let R be a ring. Then $\Gamma(R)$ has no isolated vertex.*

Proof. If R is a zero ring, i.e., $R = \{0\}$, then trivially 0 is adjacent to R and hence, $\Gamma(R)$ is a complete graph with two vertices. Assume that $R \neq \{0\}$. Then, for every vertex $r \in R$ and $S \in \boldsymbol{S}_R$, r is adjacent to $\{0\} \in \boldsymbol{S}_R$, since $r\{0\} = \{0\} = \{0\}r$, and S is adjacent to 0, since $0S = \{0\} = S0$. Thus, the degree of r and S are at least 1 which implies $\Gamma(R)$ has no isolated vertex.

Theorem 2. *For any ring R, the graph $\Gamma(R)$ is connected with diam$\big(\Gamma(R)\big)$ at most 3.*

Proof. We are going to prove that for every two arbitrary vertices, there exists a path of length at most 3 between them. We consider the following cases.

(i) For every $r_1, r_2, \in R$, it is obvious that r_1 and r_2 have common neighbours $\{0\}$ in $\boldsymbol{S}_R$. Hence, we have a path $r_1 - \{0\} - r_2$ of length 2.

(ii) Similar to case (i), two arbitrary vertices S_1 and S_2 in $\boldsymbol{S}_R$ have common neighbors 0 in R and we have path $S_1 - 0 - S_2$ of length 2.

(iii) For every vertex $r \in R$ and $S \in \boldsymbol{S}_R$, by the above two cases, we have a path $r - \{0\} - 0 - S_2$ of length 3.

Hence, $\Gamma(R)$ is connected and $\text{diam}\big(\Gamma(R)\big) \leq 3$.

Theorem 3. *If R is a nonzero commutative ring, then* $diam\big(\Gamma(R)\big) = 2$.

Proof. Let $u, v \in V\big(\Gamma(R)\big)$. If u and v are in the same partition, then $d(u,v) = 2$. Without losing the generality of the proof, let $u \in R$ and $v \in \boldsymbol{S}_R$. Since R is a commutative ring, then we have $uv = vu$ for any $u \in R, v \in \boldsymbol{S}_R$. Thus, $d(u,v) = 1$, implying $\text{diam}\big(\Gamma(R)\big) = 2$.

We have a conjecture for the converse of **Theorem 3** as given as follows.

Conjecture 1. *If $diam\big(\Gamma(R)\big) = 2$ then R is not necessarily a commutative ring.*

Theorem 4. *The diameter of $\Gamma(R)$ is 1 if and only if $R = \{0\}$.*

Proof. If $R = \{0\}$ then it is clear that $\text{diam}\big(\Gamma(R)\big) = 1$. For otherwise, suppose $R \neq \{0\}$ and $\text{diam}\big(\Gamma(R)\big) = 1$. Since $R \neq \{0\}$ then there exists $a \neq 0$ such that $a \in R$. It means that $d(0,a) = 2$, which contradicts $\text{diam}\big(\Gamma(R)\big) = 1$.

Recall that for any ring R, the set $Z(R) = \{x \in R | (\forall y \in R) xy = yx\}$ is called the center of R. In the following theorem give a sufficient condition for the graph $\Gamma(R)$ has girth 4, related to the center of R.

Theorem 5. *For any ring R, if $Z(R) \neq \{0\}$ and $|\boldsymbol{S}_R| \geq 2$, then the girth of $\Gamma(R)$ is equal to 4.*

Proof. First of all, note that if for any bipartite graph has no odd cycle, then the girth of $\Gamma(R)$ cannot be 3. On the other hand, since $Z(R) \neq \{0\}$ and $|\boldsymbol{S}_R| \geq 2$, then there exists $0 \neq x \in Z(R)$ and $H_1, H_2 \in \boldsymbol{S}_R$ such that $0 - H_1 - x - H_2 - 0$. Therefore, we have a cycle of length 4. Thus, we conclude that the girth of Γ(R) is equal to 4.

The following theorem gives a necessary condition for the graph $\Gamma(R)$ to be Hamiltonian.

Theorem 6. *Let $\Gamma(R)$ be a Hamiltonian graph. Then $|R| = |\boldsymbol{S}_R|$.*

Proof. Let $\Gamma(R)$ be a hamiltonian graph. Then, we have a cycle that meets all vertices of $\Gamma(R)$. Let the cycle begin from a vertex $x_1 \in R$. Then there exists $y_1 \in \boldsymbol{S}_R$ such that $x_1 - y_1$, and also we will have $x_2 \in R$ such that $x_1 - y_1 - x_2$. If we do this until $|R|$ steps, then we have $x_{|R|} \in R$ and $y_{|\boldsymbol{S}_R|} \in \boldsymbol{S}_R$ such that $x_1 - y_1 - x_2 - \cdots - x_{|R|} - y_{|\boldsymbol{S}_R|} - x_1$. Since $\Gamma(R)$ is Hamiltonian, then the cycle must meet all vertices in R and $\boldsymbol{S}_{\boldsymbol{R}}$. Therefore, we conclude that $|R| = |\boldsymbol{S}_R|$.

In the following theorem we give a property of $\Gamma(R)$ for a particular R.

Theorem 7. *Let $\mathbb{Z}_p$ be the ring of integers modulo p, where p is a prime number. Then $\Gamma(\mathbb{Z}_p)$ is planar.*

Proof. From the definition of the bipartite graph $\Gamma(\mathbb{Z}_p)$ and the structure of $\mathbb{Z}_p$, we can see that the vertex set of $\Gamma\big(\mathbb{Z}_p\big)$ consists of p elements of $\mathbb{Z}_p$ and precisely two subrings (the two trivial subrings) $\{0\}$ and $\mathbb{Z}_p$ in $\boldsymbol{S}_{\mathbb{Z}_p}$. Thus, $\Gamma\big(\mathbb{Z}_p\big)$ it is complete bipartite graph which is planar as shown in **Figure 1**.

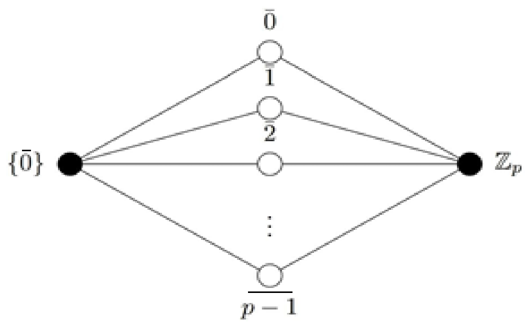

Figure 1. The graph $\boldsymbol{\Gamma}\big(\mathbb{Z}_{\boldsymbol{p}}\big)$

Let R be a commutative ring with unity and let $M_n(R)$ be the ring of square matrices of size $n \times n$ over R. From the theory of matrices over rings, we have following theorem.

Theorem 8. [9] *Let J be an ideal of $M_n(R)$. Then, $J = M_n(I)$ for a unique ideal $I \subseteq R$.*

As an example, $(\mathbb{Z}2, +, \cdot)$ is a commutative ring with unity. Furthermore, $\mathbb{Z}2$ is a field. Therefore, the only ideals of $\mathbb{Z}2$ are $\{0\}$ and $\mathbb{Z}2$. By Theorem 8, the only ideals of $Mn(\mathbb{Z}2)$ are $\{0_{n \times n}\}$ and $Mn(\mathbb{Z}2)$.

Theorem 9. [9] *Let $A = \{I \subseteq R | I \text{ ideal}\}$ and $B = \{J \subseteq M_n(R) | J \text{ ideal}\}$. Then, there exists a bijective function from A to B, mapping each I in A to $M_n(I)$ in B.*

Let $\mathbb{Z}_n$ be the ring of integers modulo n. Then $M_n(\mathbb{Z}_n)$ is a ring.

Furthermore, in [10] Grigore Calugăreanu has determined the subrings generated by some units matrices. Let R be a commutative ring with unity. Let $N = \{1, 2, \dots, n\}$ and let $\rho \subseteq N \times N$. Then $M_\rho \subseteq M_n(R)$ is defined as the set of matrices generated by $\{E_{ij} : (i,j) \in \rho\}$ where E_{ij} is a unit matrix. In the following lemma it is given the necessary and sufficient condition for M_ρ to be subring.

Lemma 1. [9] *M_ρ is a subring if only if ρ transitive ($i.e\ \rho \circ \rho \subseteq \rho$).*

Example 1. Given $M_{\rho_1} = \left\{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} | a, b, c \in \mathbb{Z}_2\right\}$ and $M_{\rho_2} = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z}_2\right\}$. Then M_{ρ_1} is not a subring in $M_2(\mathbb{Z}2)$ but M_{ρ_2} is a subring in $M_2(\mathbb{Z}2)$. We know that

$$\begin{aligned} M_{\rho_1} &= \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle \\ &= \langle E_{11}, E_{12}, E_{21} \rangle. \end{aligned}$$

It means that $\rho_1 = \{(1,1), (1,2), (2,1)\}$, on the other hand

$$\rho_1 \circ \rho_1 = \{(1,1), (1,2), (2,1), (2,2)\} \nsubseteq \rho_1.$$

Then, according to **Lemma 1**, M_{ρ_1} is not a subring in $M_2(\mathbb{Z}2)$. Furthermore, we have

$$\begin{aligned} M_{\rho_2} &= \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle \\ &= \langle E_{11}, E_{12}, E_{22} \rangle. \end{aligned}$$

Thus, $\rho_2 = \{(1,1), (1,2), (2,2)\}$, and

$$\rho_2 \circ \rho_2 = \{(1,1), (1,2), (2,2)\} = \rho_2,$$

which implies M_{ρ_2} is a subring.

Theorem 10. *The diameter of $\Gamma\left(M_2(\mathbb{Z}_n)\right)$ is 3, for any natural number n >1.*

Proof. By **Theorem 2**, we have $\operatorname{diam}\left(\Gamma(R)\right) \leq 3$ for every ring R. Then we only need to find two vertices v_1, v_2 such that $d(v_1, v_2) = 3$. Let $v_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and $v_2 = M_2(\mathbb{Z}_n)$ then we have

$$v_1 \cdot v_2 = v_1 \cdot M_2(\mathbb{Z}_n) = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in \mathbb{Z}_n\right\}$$

$$v_2 \cdot v_1 = M_2(\mathbb{Z}_n) \cdot v_1 = \left\{\begin{pmatrix} a & a \\ b & b \end{pmatrix} | a, b \in \mathbb{Z}_n\right\}.$$

We know that $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in v_2 v_1$ but $A \notin v_1 v_2$. It means that $v_1 v_2 \neq v_2 v_1$. On the other hand, there exists $v_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, v_4 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ satisfies $v_1 - v_2 - v_3 - v_4$. We conclude that $d(v_1, v_2) = 3$ and $\operatorname{diam} \Gamma\left(M_2(\mathbb{Z}_n)\right) = 3$.

The minimum cover of the graph $M_2(\mathbb{Z}_n)$ is conjectured as the following.

Conjecture 2. *The minimum vertex cover of $M_2(\mathbb{Z}_n)$ is $\beta\left(M_2(\mathbb{Z}_n)\right) = \left|S_{(M_2(\mathbb{Z}_n))}\right|$.*

Theorem 11. (*Kőnig's Theorem*) *Let $G(V, E)$ be a bipartite graph. The size of a maximum matching in G equals the size of a minimum vertex cover of G.*

According to **Conjecture 2** and Kőnig's Theorem then the size of maximum matching can be conjectured as given below.

Conjecture 3. *The size of maximum matching in $M2(\mathbb{Z}n)$ is $\left|S_{(M_2(\mathbb{Z}_n))}\right|$.*

Now, we will discuss the bipartite graph associated to quaternion ring. Prior, we give the definition of the quaternion ring.

Definition 2. **[15]** Let $\mathbb{C}$ and $\mathbb{R}$ denote the fields of the complex and real numbers, respectively. Let $\mathbb{Q}$ be a four dimensional vector space over $\mathbb{R}$ with an ordered basis, denoted by e, i, j and k. A real quaternion, simply called quaternion is a vector

$$x = x_0e + x_1i + x_2j + x_3k \in \mathbb{Q}$$

with real coefficients x_0, x_1, x_2, x_3.

Real numbers and complex numbers can be thought of as quaternions in the natural way. Thus $x_0e + x_1i + x_2j + x_3k$ can simply be written as $x_0 + x_1i + x_2j + x_3k$. Moreover, Adkin in **[11]** defined $\mathbb{H} = \mathbb{Q}(-1, -1; \mathbb{R})$ as the quaternion ring with addition and a multiplication on $\mathbb{H}$ are given in the following definition.

Definition 3. Let $p, q \in \mathbb{H}$ with $p = x_0 + x_1i + x_2j + x_3k$, $q = y_0 + y_1i + y_2j + y_3k$ and $x_m, y_m \in \mathbb{R}$ for $m = 1,2,3$. The addition of p and q on $\mathbb{H}$ is given by

$$\begin{aligned} p + q &= (x_0 + x_1i + x_2j + x_3k) + (y_0 + y_1i + y_2j + y_3k) \\ &= (x_0 + y_0) + (x_1 + y_1)i + (x_2 + y_2)j + (x_3 + y_3)k \end{aligned}$$

and multiplication of p and q is given by

$$\begin{aligned} p \cdot q = &\ (x_0 + x_1i + x_2j + x_3k)(y_0 + y_1i + y_2j + y_3k) \\ = &\ (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) + (x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)i + \\ &\ (x_0y_2 + x_2y_0 + x_3y_1 - x_1y_3)j + (x_0y_3 + x_3y_0 + x_1y_2 - x_2y_1)k. \end{aligned}$$

Beside the addition and the multiplication above, the product of any two quaternions e, i, j, k is defined due to the requirement that $e = 1 + 0_i + 0_j + 0_k$ and

$$i^2 = j^2 = k^2 = -1,$$

and

$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

Proposition 1. *If we define set $\mathbb{H}(\mathbb{Z}_n) = \mathbb{Q}(-1, -1; \mathbb{Z}_n)$ with addition and multiplication as given in Definition 3, then $\mathbb{H}(\mathbb{Z}_n)$ is a ring. (We called the ring as the quaternion ring by $\mathbb{Z}_n$.)*

Theorem 12. *If $n > 1$ and n is odd, then the diameter of the graph associated to quaternion ring $\mathbb{H}(\mathbb{Z}_n)$ is 3.*

Proof. Let $\mathbb{H}(\mathbb{Z}_n)$ be the quaternion ring by $\mathbb{Z}_n$ with n odd and let $\Gamma(\mathbb{H}(\mathbb{Z}_n))$ be the associated graph of $\mathbb{H}(\mathbb{Z}_n)$. Take $x = (j + 2k) \in \mathbb{H}(\mathbb{Z}_n)$ and $y = \mathbb{H}(\mathbb{Z}_n) \in \boldsymbol{S}_{\mathbb{H}(\mathbb{Z}_n)}$ as vertices of $\Gamma(\mathbb{H}(\mathbb{Z}_n))$. Then we have

$$\begin{aligned} x\mathbb{H}(\mathbb{Z}_n) = &\ (j + 2k).\{a_0 + a_1i + a_2j + a_3k | a_0, a_1, a_2, a_3 \in \mathbb{Z}_n\} \\ = &\ \{(-a_2 - 2a_3) + (a_3 - 2a_2)i + (a_0 + 2a_1)j + (2a_0 - a_1)k | a_0, a_1, a_2, a_3 \in \mathbb{Z}_n\} \\ = &\ \{\big((n-1)a_2 + (n-2)a_3\big) + (a_3 + (n-2)a_2)i + (a_0 + 2a_1) \\ &\ +(2a_0 + (n-1)a_1)k |\ a_0, a_1, a_2, a_3 \in \mathbb{Z}_n\} \end{aligned}$$

and

$$\begin{aligned} \mathbb{H}(\mathbb{Z}_n)x = &\ \{a_0 + a_1i + a_2j + a_3k | a_0, a_1, a_2, a_3 \in \mathbb{Z}_n\} \cdot (j + 2k) \\ = &\ \{(-a_2 - 2a_3) + (2a_2 - a_3)i + (a_0 - 2a_1)j + (2a_0 + a_1)k | a_0, a_1, a_2, a_3 \in \mathbb{Z}_n\} \\ = &\ \{\big((n-1)a_2 + (n-2)a_3\big) + (2a_2 + (n-1)a_3)i + (a_0 + (n-2)a_1)j \\ &\ +(2a_0 + a_1)k | a_0, a_1, a_2, a_3 \in \mathbb{Z}_n\}. \end{aligned}$$

Since n is odd, there is $m \in \mathbb{N}$ such that $n = 2m + 1$, so that we have

$$x\mathbb{H}(\mathbb{Z}_n) = \quad (2m)a_2 + (2m-1)a_3\big) + (a_3 + (2m-1)a_2)i + (a_0 + 2a_1)j$$
$$(2a_0 + (2m)a_1)k | a_0, a_1, a_2, a_3 \in \mathbb{Z}_n\}$$

and

$$\mathbb{H}(\mathbb{Z}_n)x = \quad \{\big((2m)a_2 + (2m-1)a_3\big) + (2a_2 + (2m)a_3)i + (a_0 + (2m-1)a_1)j$$
$$+(2a_0 + a_1)k | a_0, a_1, a_2, a_3 \in \mathbb{Z}_n\}.$$

It is easy to see that $(2m-1)j + k \in \mathbb{H}(\mathbb{Z}_n)x$ as we can take $a_0 = a_2 = a_3 = 0$ and $a_1 = 1$ so that we have $\big((2m)a_2 + (2m-1)a_3\big) + (2a_2 + (2m)a_3)i + (a_0 + (2m-1)a_1)j + (2a_0 + a_1)k = 0 + 0i + (2m-1)j + k = (2m-1)j + k$. On the other hand, $(2m-1)j + k \notin x\mathbb{H}(\mathbb{Z}_n)$ as the element of the coordinate bases k always be an even number. Thus, x and y are not adjacent, but x is adjacent to $\{0\}$ and y is adjacent to 0 so that we have path

$$x = \{0\} - 0 - y = \mathbb{H}(\mathbb{Z}_n).$$

We conclude that $\operatorname{diam}\Big(\Gamma\big(\mathbb{H}(\mathbb{Z}_n)\big)\Big) = 3$ for $n > 1$ odd.

3. CONCLUSIONS

According to the description above we know that the bipartite graph associated to elements and cosets of subrings of any ring R has no isolated vertex and has a diameter 2 if R is nonzero and commutative. But however, the complete information on the structure of the graphs in general is not yet obtained. This could be interesting for future works.

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