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RAINBOW VERTEX-CONNECTION NUMBER ON COMB PRODUCT OPERATION OF CYCLE GRAPH ($\mathbf{C_4}$) AND COMPLETE BIPARTITE GRAPH ($\mathbf{K_{3,n}}$)

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ABSTRACT

Rainbow vertex-connection number is the minimum colors assignment to the vertices of the graph, such that each vertex is connected by a path whose edges have distinct colors and is denoted by $rvc(G)$. The rainbow vertex connection number can be applied to graphs resulting from operations. One of the methods to create a new graph is to perform operations between two graphs. Thus, this research uses comb product operation to determine rainbow-vertex connection number resulting from comb product operation of cycle graph and complete bipartite graph $(C_4 \rhd K_{3,n})$ & $(K_{3,n} \rhd C_4)$. The research finding obtains the theorem of rainbow vertex-connection number at the graph of $(C_4 \rhd K_{3,n})$ is 5 for $2 \leq n \leq 7$ while the theorem of rainbow vertex-connection number at the graph of $(K_{3,n} \rhd C_4) = 3n-1$ for $n = 2 \ \wedge n+3$ for $3 \leq n \leq 7$.

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1. INTRODUCTION

One of the subjects in the field of combinatorial mathematics is graph theory, which has been widely used to solve a problem. The use of graphs is to present discrete objects and the relationships between them [1]. Graph theory gets a lot of attention because the models used can be applied in everyday life. Among them such as scheduling transportation departures, scheduling courses, determining the shortest trajectory, and many more. Some theories support this, graphs theory solving and analysis of automated electric networks [2], and applications of graphs theory for solving electrical circuits [3].

Graph labeling is one of the topics in graph theory. Graph labeling is presented by vertex and edge along with a set of natural number members called labels. There are three types of labeling on a graph (G), namely point labeling, side labeling, and total labeling [4].

Graph coloring is a special case in graph labeling. Giving color to the vertex or edge of a graph is the definition of graph coloring. Three types of coloring are obtained in graphs, namely vertex coloring, edge coloring, and field coloring. Vertex coloring gives color to a vertex, where each of the two adjacent vertex acquires a different color. Edge coloring gives color to the edge, where each of the two adjacent edges obtains a different color. Field coloring gives color to the field, where each of the two adjacent fields obtains a different color [5][6]. The rainbow connection is a development of graph coloring. A rainbow connection is the coloring of the edge of the graph, where each vertex on the graph is connected by a trajectory that has a different edge color. Rainbow connection number of a graph (G), symbolized by $rc(G)$ is a number that states the number of minimum coloring in a graph (G) [7] [8].

Krivelevich and Yuster began introducing the rainbow vertex connection in 2009, that was a development of the rainbow connection concept [9]. Rainbow vertex connection is the coloring of the vertex of a graph, so that each vertex on the graph is connected by a trajectory that has an interior vertex with different colors. Rainbow vertex connection number symbolized by $rvc(G)$ is a number that expresses the number of minimum coloring in a graph (G). Numbers connected to rainbow vertex can also be applied to graphs that are the result of operations. One method for obtaining a new graph form is to perform an operation between two graphs. There are various types of operations in the graph, such as join $(\oplus)$, combined $(\cup)$, cartesian $(\times)$, corona $(\odot)$, and comb $(\rhd)$ operations.

In recent studies, we have studied and developed rainbow-connected numbers as a result of surgery. The rainbow connection numbers of the sum and multiplication operations of graphs [10] [11]. The determination of rainbow connection numbers in the results of cartesian product operations against circle graphs and bipartite graphs complete with track graphs [12]. The total rainbow connected numbers resulting from comb operations on graph cycles and graph paths [13]. Rainbow connection number of comb product of graphs. This research give sharp lower and upper bounds for the rainbow connection number of comb product between two connected graphs [14]. The numbers connected to strong rainbow vertex on the graph resulting from edge comb operations [15].

Based on the background description above, this study discusses the number of connected rainbow vertex by using vertex comb operation, where the graph to be operated is the cycle graph (C_4) and the complete bipartite graph $(K_{3,n})$.

2. RESEARCH METHODS

This research uses literature study research methods (library research). In this study, a study of books, textbooks, journals, and scientific articles on the number of rainbow vertex connected to the results of comb $\boldsymbol{rvc(G)}$ operations with the aim of obtaining information and methods used in the completion of this study.

3. RESULTS AND DISCUSSION

In this section, we determine rainbow-vertex connection number resulting from comb product operation of cycle graph and complete bipartite graph $(\boldsymbol{C_4 \rhd K_{3,n}})$ & $(\boldsymbol{K_{3,n} \rhd C_4})$.

3.1. Comb Cycle Graph and Complete Bipartite Graph $(C_4 \rhd K_{3,n})$

In this section we discuss Comb Cycle Graphs and Complete Bipartite Graphs $(C_4 \rhd K_{3,n})$.

Definition 1: Suppose $n \geq 2$. The C_4 graph is a cycle graph with 4 vertices. The $K_{3,n}$ graph is a complete bipartite graph with a size of $3, n$. Thus, the comb operation for the Cycle graph and the Complete Bipartite graph is denoted with $(C_4 \rhd K_{3,n})$. Suppose $(C_4 \rhd K_{3,n})$ is the graph G, then graph G is formed by the set of vertex and edge defined as follows.

$$V(G) = \{i \in [1,4]\} \cup \{i \in [1,4], j \in [1,n]\} \cup \{u_{i,k} \mid i \in [1,4], k \in [1,2]\}$$

$$\begin{aligned} E(G) = &\{i \in [1,4], u_4 + 1 = u_1\} \cup \{i = 1, j \in [1,n]\} \cup \\ &\{i = 2, j \in [1,n]\} \cup \{i = 3, j \in [1,n]\} \cup \\ &\{i = 4, j \in [1,n]\} \cup \{u_{i,k} v_{i,j}, u_{i,k+1} v_{i,j} \mid i \in [1,4], k = 1, j \in [1,n]\} \end{aligned}$$

Graph image of comb operation $C_4 \rhd K_{3,2}$ is shown in Figure 1.

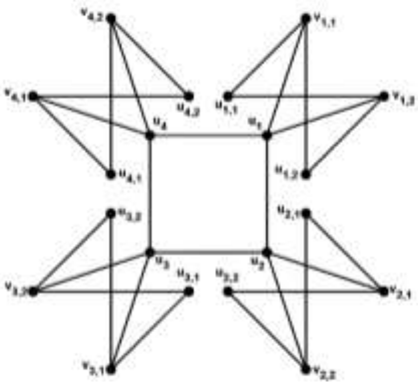

Figure 1. Graph of comb operation $C_4 \rhd K_{3,2}$

3.2. Rainbow Vertex-Connection Number on Comb Product Operation of Cycle Graph and Complete Bipartite Graph $(C_4 \rhd K_{3,n})$

In this section, we describe Rainbow Vertex-Connection Number on Comb Product Operation of Cycle Graph and Complete Bipartite.

Theorem 1: *Suppose C_4 is a cycle graph with four vertices and $K_{3,n}$ is a bipartite graph complete with $2 \leq n \leq 7$. If $G \cong (C_4 \rhd K_{3,n})$, then*

$$rvc(G) = 5$$

Proof. Known $rvc(G) \geq diam - 1$, so to prove Theorem 1 is enough to show that $rvc(G) \leq diam - 1$. For this reason, the coloring of $c: V(G) \to \{1,2,3,4,5\}$ is defined as follows.

$$\begin{aligned} c(u_i) &= c(u_{i,k}, u_{i,k+1}) = i \bmod 5, && \text{for } i \in [1,4] \wedge k = 1 \\ c(v_{2,1}) &= c(v_{i,j}) = 1, && \text{for } i = 3 \wedge j \in [2,n] \\ c(v_{4,1}) &= c(v_{i,j}) = 2i, && \text{for } i = 1 \wedge j \in [2,n] \\ c(v_{3,1}) &= c(v_{i,j}) = 5 \bmod 2i - 2, && \text{for } i = 4 \wedge j \in [2,n] \\ c(v_{i,j}) &= 4i, && \text{for } i = 1 \wedge j = 1 \\ c(v_{i,j}) &= i + 1 \bmod 2i, && \text{for } i = 2 \wedge j \in [2,n] \end{aligned}$$

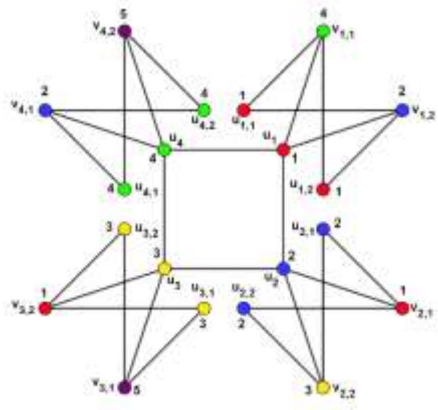

Figure 2. Rainbow vertex-connection number of graph $C_4 \rhd K_{3,2}$

It will then be shown that for each x and y in $V(C_4 \rhd K_{3,n})$ there is an $x - y$ rainbow vertex-connection.

Table 1. Rainbow vertex trajectory $(C_4 \rhd K_{3,n})$

Case	x	y	Condition	Rainbow Trajectory
1	u_i	u_k	$i = 1, k = 3$	u_i, u_2, u_k
			$i = 2, k = 4$	u_i, u_3, u_k
2	u_i	$u_{k,j}$	$i \in [1,4], k \in [1,4], j \in [1,2]$	$u_i, v_{i,1}, u_{k,j}$
3	u_i	$u_{k,l}$	$i = 1, k = \{2,4\}, l \in [1,2]$	$u_i, u_k, v_{k,1}, u_{k,l}$
			$i = 2, k = \{1,3\}, l \in [1,2]$	$u_i, u_k, v_{k,1}, u_{k,l}$
			$i = 3, k = \{2,4\}, l \in [1,2]$	$u_i, u_k, v_{k,1}, u_{k,l}$
			$i = 4, k = \{1,3\}, l \in [1,2]$	$u_i, u_k, v_{k,1}, u_{k,l}$
4	u_i	$u_{k,m}$	$i \in [1,2], k = i + 2, m \in [1,2]$	$u_i, u_{i+1}, u_{i+2}, u_{k,1}, u_{k,m}$
			$i = \{3,4\}, k = i - 2, m \in [1,2]$	$u_i, u_{i-1}, u_{i-2}, v_{k,1}, u_{k,m}$
5	$u_{i,j}$	$u_{k,l}$	$i = \{1,2,3,4\}, j = 1,$ $k = \{1,2,3,4\}, l = 2$	$u_{i,j}, v_{i,j}, u_{k,l}$
6	$u_{i,j}$	$u_{k,m}$	$i = 1, j \in [1,2], k = 2, m \in [1,2]$	$u_{i,j}, v_{i,1}, u_i, u_k, v_{k,2}, u_{k,m}$
			$i = 1, j \in [1,2], k = 4, m \in [1,2]$	$u_{i,j}, v_{i,1}, u_i, u_k, v_{k,2}, u_{k,m}$
			$i = 2, j \in [1,2], k = 3, m \in [1,2]$	$u_{i,j}, v_{i,1}, u_i, u_k, v_{k,2}, u_{k,m}$
			$i = 3, j \in [1,2], k = 4, m \in [1,2]$	$u_{i,j}, v_{i,1}, u_i, u_k, v_{k,2}, u_{k,m}$
7	$u_{i,j}$	$u_{k,o}$	$i = 1, j \in [1,2], k = 3, o \in [1,2]$	$u_{i,j}, v_{i,1}, u_i, u_{i+1}, u_k, v_{k,i}, u_{k,o}$
			$i = 2, j \in [1,2], k = 4, o \in [1,2]$	$u_{i,j}, v_{i,1}, u_i, u_{i+1}, u_k, v_{k,i}, u_{k,o}$
8	u_i	$v_{k,j}$	$i = 1, k = 2, j \in [1, n]$	$u_i, u_k, v_{k,j}$
			$i = 1, k = 4, j \in [1, n]$	$u_i, u_k, v_{k,j}$
			$i = 2, k = \{1,3\}, j \in [1, n]$	$u_i, u_k, v_{k,j}$
			$i = 3, k = \{2,4\}, j \in [1, n]$	$u_i, u_k, v_{k,j}$
			$i = 4, k = \{1,3\}, j \in [1, n]$	$u_i, u_k, v_{k,j}$
9	u_i	$v_{k,l}$	$i = 1, k = 3, l \in [1, n]$	$u_i, u_{i+1}, u_k, v_{k,l}$
			$i = 2, k = 4, l \in [1, n]$	$u_i, u_3, u_k, v_{k,l}$
			$i = 3, k = 1, l \in [1, n]$	$u_i, u_2, u_k, v_{k,l}$
			$i = 4, k = 2, l \in [1, n]$	$u_i, u_1, u_k, v_{k,l}$

Case	x	y	Condition	Rainbow Trajectory
10	$u_{i,j}$	$v_{k,l}$	$i = 1, j \in [1,2], k = 2, l \in [1, n]$	$u_{i,j}, v_{i,1}, u_i, u_k, v_{k,l}$
			$i = 1, j \in [1,2], k = 4, l \in [1, n]$	$u_{i,j}, v_{i,2}, u_i, u_k, v_{k,l}$
			$i = 2, j \in [1,2], k = 1, l \in [1, n]$	$u_{i,j}, v_{i,2}, u_i, u_k, v_{k,l}$
			$i = 2, j \in [1,2], k = 3, l \in [1, n]$	$u_{i,j}, v_{i,1}, u_i, u_k, v_{k,l}$
			$i = 3, j \in [1,2], k = 2, l \in [1, n]$	$u_{i,j}, v_{i,1}, u_i, u_k, v_{k,l}$
			$i = 3, j \in [1,2], k = 4, l \in [1, n]$	$u_{i,j}, v_{i,1}, u_i, u_k, v_{k,l}$
			$i = 4, j \in [1,2], k = 1, l \in [1, n]$	$u_{i,j}, v_{i,1}, u_i, u_k, v_{k,l}$
			$i = 4, j \in [1,2], k = 3, l \in [1, n]$	$u_{i,j}, v_{i,1}, u_i, u_k, v_{k,l}$
11	$u_{i,j}$	$v_{k,m}$	$i = 1, j \in [1,2], k = 3, m \in [1, n]$	$u_{i,j}, v_{i,1}, u_{i+1}, u_k, v_{k,m}$
			$i = 2, j \in [1,2], k = 4, m \in [1, n]$	$u_{i,j}, v_{i,1}, u_{i+1}, u_k, v_{k,m}$
			$i = 3, j \in [1,2], k = 1, m \in [1, n]$	$u_{i,j}, v_{i,1}, u_{i+1}, u_k, v_{k,m}$
			$i = 4, j \in [1,2], k = 2, m \in [1, n]$	$u_{i,j}, v_{i,2}, u_i, u_k, v_{k,m}$
12	$v_{i,j}$	$v_{k,l}$	For $n = 2$	
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 2$	$v_{i,j}, u_i, v_{k,l}$
			For $n = 3$	
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 2$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 3$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{1,2,3,4\}, j = 2, k = \{1,2,3,4\}, l = 3$	$v_{i,j}, u_i, v_{k,l}$
			For $n = 4$	
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 2$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 3$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{1,2,3,4\}, j = 2, k = \{1,2,3,4\}, l = 3$	$v_{i,j}, u_i, v_{k,l}$
			$i = 1, j = \{1,2,3\}, k = 1, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			$i = 2, j = \{1,2,3\}, k = 2, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			$i = 3, j = \{1,2,3\}, k = 3, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			$i = 4, j = \{1,2,3\}, k = 4, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			For $n = 5$	
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 2$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 3$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{1,2,3,4\}, j = 2, k = \{1,2,3,4\}, l = 3$	$v_{i,j}, u_i, v_{k,l}$
			$i = 1, j = \{1,2,3\}, k = 1, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			$i = 2, j = \{1,2,3\}, k = 2, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			$i = 3, j = \{1,2,3\}, k = 3, l = \{4,5\}$	$v_{i,j}, u_i, v_{k,l}$
			$i = 4, j = \{1,2,3\}, k = 4, l = \{4,5\}$	$v_{i,j}, u_i, v_{k,l}$
			$i = 1, j = \{1,2,3,4\}, k = 1, l = 5$	$v_{i,j}, u_i, v_{k,l}$
			$i = 2, j = \{1,2,3,4\}, k = 2, l = 5$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{3,4\}, j = 4, k = \{3,4\}, l = 5$	$v_{i,j}, u_i, v_{k,l}$

Case	x	y	Condition	Rainbow Trajectory
			For $n = 6$	
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 2$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 3$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{1,2,3,4\}, j = 2, k = \{1,2,3,4\}, l = 3$	$v_{i,j}, u_i, v_{k,l}$
			$i = 1, j = \{1,2,3\}, k = 1, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			$i = 2, j = \{1,2,3\}, k = 2, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			$i = 3, j = \{1,2,3\}, k = 3, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			$i = 1, j = \{1,2,3,4\}, k = 1, l = 5$	$v_{i,j}, u_i, v_{k,l}$
			$i = 1, j = \{1,2,3,4,5\}, k = 1, l = 6$	$v_{i,j}, u_i, v_{k,l}$
			$i = 2, j = \{1,2,3,4\}, k = 2, l = 5$	$v_{i,j}, u_i, v_{k,l}$
			$i = 2, j = \{1,2,3,4,5\}, k = 2, l = 6$	$v_{i,j}, u_i, v_{k,l}$
			$i = 3, j = \{1,2,3,4\}, k = 3, l = 5$	$v_{i,j}, u_i, v_{k,l}$
			$i = 3, j = \{1,2,3,4,5\}, k = 3, l = 6$	$v_{i,j}, u_i, v_{k,l}$
			$i = 4, j = \{1,2,3\}, k = 4, l \in [4,6]$	$v_{i,j}, u_i, v_{k,l}$
			$i = 4, j = 4, k = 4, l = \{5,6\}$	$v_{i,j}, u_i, v_{k,l}$
			$i = 4, j = 5, k = 4, l = 6$	$v_{i,j}, u_i, v_{k,l}$
				$v_{i,j}, u_i, v_{k,l}$
			For $n = 7$	
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 2$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{1,2,3,4\}, j = 1, k = \{1,2,3,4\}, l = 3$	$v_{i,j}, u_i, v_{k,l}$
			$i = \{1,2,3,4\}, j = 2, k = \{1,2,3,4\}, l = 3$	$v_{i,j}, u_i, v_{k,l}$
			$i = 1, j = \{1,2,3\}, k = 1, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			$i = 2, j = \{1,2,3\}, k = 2, l = 4$	$v_{i,j}, u_i, v_{k,l}$
			$i = 1, j = \{1,2,3,4\}, k = 1, l = 5$	$v_{i,j}, u_i, v_{k,l}$
			$i = 1, j = \{1,2,3,4,5\}, k = 1, l = 6$	$v_{i,j}, u_i, v_{k,l}$
			$i = 2, j = \{1,2,3,4\}, k = 2, l = 5$	$v_{i,j}, u_i, v_{k,l}$
			$i = 2, j = \{1,2,3,4,5\}, k = 2, l = 6$	$v_{i,j}, u_i, v_{k,l}$
			$i = 3, j = \{1,2,3,4\}, k = 3, l = 5$	$v_{i,j}, u_i, v_{k,l}$
			$i = 3, j = \{1,2,3,4,5\}, k = 3, l = 6$	$v_{i,j}, u_i, v_{k,l}$
			$i = 4, j = \{1,2,3\}, k = 4, l \in [4,7]$	$v_{i,j}, u_i, v_{k,l}$
			$i = 4, j = 4, k = 4, l \in [5,7]$	$v_{i,j}, u_i, v_{k,l}$
			$i = 4, j = 5, k = 4, l \in [6,7]$	$v_{i,j}, u_i, v_{k,l}$
			$i = 4, j = 6, k = 4, l = 7$	$v_{i,j}, u_i, v_{k,l}$
			$i = 1, j \in [1,6], k = 1, l = 7$	$v_{i,j}, u_i, v_{k,l}$
			$i = 2, j \in [1,6], k = 2, l = 7$	$v_{i,j}, u_i, v_{k,l}$
			$i = 3, j \in [1,6], k = 3, l = 7$	$v_{i,j}, u_i, v_{k,l}$
			$i = 3, j = \{1,2,3\},, k = 3, l = 4$	$v_{i,j}, u_i, v_{k,l}$
13	$v_{i,j}$	$v_{k,m}$	$i = 1, j \in [1, n], k = \{2,4\}, m \in [1, n]$	$v_{i,j}, u_i, u_k, v_{k,m}$

Case	x	y	Condition	Rainbow Trajectory
			$i=2,\ j\in[1,n], k=3, m\in[1,n]$	$v_{i,j}, u_i, u_k, v_{k,m}$
			$i=3,\ j\in[1,n], k=4, m\in[1,n]$	$v_{i,j}, u_i, u_k, v_{k,m}$
14	$v_{i,j}$	$v_{k,o}$	$i=1,\ j\in[1,n], k=3, o\in[1,n]$	$v_{i,j}, u_i, u_{i+1}, u_k, v_{k,o}$
			$i=2,\ j\in[1,n], k=4, o\in[1,n]$	$v_{i,j}, u_i, u_{i+1}, u_k, v_{k,o}$

Based on the trajectory of the vertex rainbow in **Table 1,** the theorem $rvc(C_4 \rhd K_{3,n}) = 5$ is proven.

3.3. Comb Complete Bipartite Graph and Cycle Graph ($K_{3,n} \rhd C_4$)

In this section, we describe comb complete bipartite graph and cycle graph and constructing figure Rainbow Vertex-Connection Number of Graph $K_{3,2} \rhd C_4$.

Definition 2: Suppose $n \geq 2$. The $K_{3,n}$ graph is a complete bipartite graph with a size of $3, n$. The C_4 graph is a cycle graph with 4 vertices. Thus, the comb operation for the Complete Bipartite graph and the Cycle graph is denoted with $(K_{3,n} \rhd C_4)$. Suppose $(K_{3,n} \rhd C_4)$ is the graph G, then graph G is formed by the set of vertex and edge defined as follows.

$$V(G) = \{i \in [1, n+3]\} \cup \{i \in [1, n+3], j \in [1,2]\} \cup \{w_i \mid i \in [1, n+3]\}$$
$$E(G) = \{i \in [4, n+3]\} \cup \{i \in [4, n+3], j \in [1,2]\} \cup \{i \in [4, n+3]\} \cup$$
$$\{i \in [1, n+3], j = 1\} \cup \{w_i v_{i,j}, w_i v_{i,j+1} \mid i \in [1, n+3], j = 1\}$$

Graph image of comb operation $K_{3,2} \rhd C_4$ is shown in **Figure 3**.

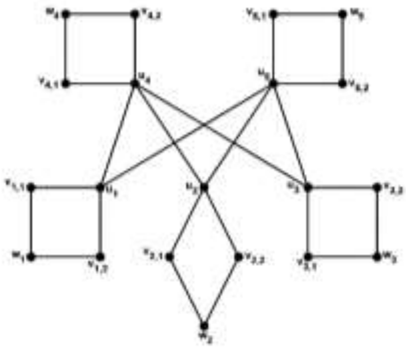

Figure 3. Rainbow Vertex-Connection Number of Graph $K_{3,2} \rhd C_4$

3.4. Rainbow Vertex-Connection Number on Comb Product Operation of Complete Bipartite Graph and Cycle Graph ($K_{3,n} \rhd C_4$)

In this section, we describe Rainbow Vertex-Connection Number on Comb Product Operation of Complete Bipartite Graph and Cycle Graph ($K_{3,n} \rhd C_4$)

Theorem 2. *Suppose $K_{3,n}$ is a complete bipartite graph with $2 \leq n \leq 7$ and C_4 is a cycle graph with four vertices. If $G \cong (C_4 \rhd K_{3,n})$, then*

$$rvc(G) = \begin{cases} 3n-1, & for \quad n=2 \\ n+3, & for \quad 3 \leq n \leq 7 \end{cases}$$

Proof. Known $rvc(G) \geq diam(G) - 1$, so to prove **Theorem 2** is enough to show that $rvc(G) \leq diam(G) - 1$, if $rvc(G) \neq diam(G) - 1$ or $rvc(G) < diam(G) - 1$ it will be proven as the case below:

Case 1: $rvc(G) = 3n - 1$, for $n = 2$

Known $diam(K_{3,n} \rhd C_4) = 6$, then $rvc \geq 5$, for which it is defined coloring $c: V(G) \to \{1,2,3,4,5\}$ as follows:

$$c(v_{i,j}) = \begin{cases} 1, & i \in [1,2n], j = 1, i \quad even \\ 1, & i \in [3,3n], j = 2, i \quad odd \\ 4, & i \wedge j = 1 \\ 5, & i \wedge j = 2 \end{cases}$$

$$c(v_{i,j}) = c(v_{i+1,j+1}) = n, \qquad i = n+1 \wedge j = 1$$

$$c(v_{i,j+1}) = c(v_{i+2n,j}) = n+1, \qquad i \wedge j = 1$$

Rainbow Vertex-Connection Number on Comb Product Operation of Complete Bipartite Graph and Cycle Graph $(K_{3,n} \trianglerighteq C_4)$ can be seen in Figure 4.

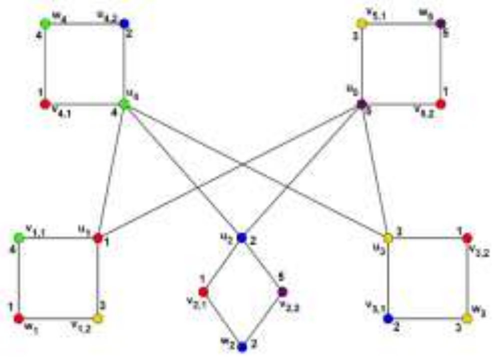

Figure 4. Rainbow Vertex-Connection Number of Graph $K_{3,2} \trianglerighteq C_4$

It will then be shown that for each x and y in $V(K_{3,n} \trianglerighteq C_4)$ there is an $x - y$ rainbow vertex-connection.

Table 2. Rainbow vertex trajectory $(K_{3,2} \trianglerighteq C_4)$

Case	x	y	Condition	Rainbow Trajectory
1	u_i	u_k	$i = 1, k = n$	u_i, u_{i+3}, u_k
			$i \in [1,n], k = n+1$	u_i, u_{i+3}, u_k
			$i = 2n, k = 2n+1$	u_i, u_{i-2}, u_k
2	w_i	w_k	$i = \{1, n+1\}, k = 2n$	$w_i, v_{i,2}, u_i, u_k, v_{k,2}, w_k$
			$i \in [1,n], k = n+3$	$w_i, v_{i,1}, u_i, u_k, v_{k,1}, w_k$
			$i = n, k = 2n$	$w_i, v_{i,2}, u_i, u_k, v_{k,1}, w_k$
			$i = n+1, k = n+3$	$w_i, v_{i,1}, u_i, u_k, v_{k,2}, w_k$
3	w_i	w_l	$i = 1, l = n$	$w_i, v_{i,2}, u_i, u_{i+3}, u_{i+1}, v_{i+1,2}, w_l$
			$i = 1, l = n+1$	$w_i, v_{i,1}, u_i, u_5, u_l, v_{l,1}, w_l$
			$i = 2n, l = 2n+1$	$w_i, v_{i,1}, u_i, u_2, u_l, v_{l,1}, w_l$
			$i = n, l = n+1$	$w_i, v_{i,2}, u_i, u_4, u_l, v_{l,2}, w_l$
4	u_i	w_k	$i \in [1, n+3], k \in [1, n+3]$	$u_i, v_{i,1}, w_k$
5	u_i	w_l	$i \in [1, n+1], l = \{2n, 2n+1\}$	$u_i, u_l, v_{l,1}, w_l$
			$i = \{2n, 2n+1\}, l \in [1, n+1]$	$u_i, u_l, v_{l,2}, w_l$
6	u_i	w_m	$i = 2n, m = 2n+1$	$u_i, u_2, u_m, v_{m,2}, w_m$
			$i = 1, m = \{n, n+1\}$	$u_i, u_4, u_m, v_{m,1}, w_m$
			$i = 2n+1, m = 2n$	$u_i, u_2, u_m, v_{m,1}, w_m$
			$i = n, m = \{1, n+1\}$	$u_i, u_4, u_m, v_{m,2}, w_m$
			$i = n+1, m = \{1, n\}$	$u_i, u_4, u_m, v_{m,2}, w_m$
7	u_i	$v_{k,j}$	$i \in [1, n+1], j \in [1,n], k = \{2n, 2n+1\}$	$u_i, u_k, v_{k,j}$
			$i = \{2n, 2n+1\}, j \in [1,n], k \in [1, n+1]$	$u_i, u_k, v_{k,j}$
8	u_i	$v_{l,j}$	$i = \{1,3\}, j \in [1,n], l = n$	$u_i, u_4, u_l, v_{l,j}$
			$i = \{1,2\}, j \in [1,n], l = n+1$	$u_i, u_4, u_l, v_{l,j}$
			$i = \{2,3\}, j \in [1,n], l = 1$	$u_i, u_4, u_l, v_{l,j}$
			$i = 2n, j \in [1,n], l = 2n+1$	$u_i, u_2, u_l, v_{l,j}$
			$i = 2n+1, j \in [1,n], l = 2n$	$u_i, u_2, u_l, v_{l,j}$

Case	x	y	Condition	Rainbow Trajectory
9	w_i	$v_{k,j}$	$i = 1, j \in [1, n], k = \{2n, 2n + 1\}$	$w_i, v_{i,2}, u_i, u_k, v_{k,j}$
			$i = \{n, n + 1\}, j \in [1, n], k = \{2n, 2n + 1\}$	$w_i, v_{i,2}, u_i, u_k, v_{k,j}$
			$i = 2n, j \in [1, n], k = \{n, n + 1\}$	$w_i, v_{i,1}, u_i, u_k, v_{k,j}$
			$i = 2n + 1, j \in [1, n], k \in [1, n]$	$w_i, v_{i,1}, u_i, u_k, v_{k,j}$
			$i = \{2n, 2n + 1\}, j \in [1, n], k = \{1, n + 1\}$	$w_i, v_{i,2}, u_i, u_k, v_{k,j}$
10	w_i	$v_{l,j}$	$i = \{1, n + 1\}, j \in [1, n], l = n$	$w_i, v_{i,l}, u_i, u_{2l}, u_l, v_{l,j}$
			$i = n, j \in [1, n], l = \{1, n + 1\}$	$w_i, v_{i,2}, u_i, u_4, u_l, v_{l,j}$
			$i = 1, j \in [1, n], l = n + 1$	$w_i, v_{i,1}, u_i, u_5, u_l, v_{l,j}$
			$i = n + 1, j \in [1, n], l = 1$	$w_i, v_{i,l}, u_i, u_4, u_l, v_{l,j}$
			$i = 2n, j \in [1, n], l = 2n + 1$	$w_i, v_{i,1}, u_i, u_2, u_l, v_{l,j}$
			$i = 2n + 1, j \in [1, n], l = 2n$	$w_i, v_{i,1}, u_i, u_2, u_l, v_{l,j}$
11	$v_{i,j}$	$v_{l,k}$	$i \in [1,2n + 1], j = 1,$ $l \in [1,2n + 1], k = 2$	$v_{i,j}, u_i, v_{l,k}$
12	$v_{i,j}$	$v_{l,m}$	$i \in [1, n + 1], j \in [1, n],$ $l = \{2n, 2n + 1\}, m \in [1, n]$	$v_{i,j}, u_i, u_l, v_{l,m}$
13	$v_{i,j}$	$v_{l,o}$	$i = 1, j \in [1, n], l = \{n, n + 1\}, o \in [1, n]$	$v_{i,j}, u_i, u_4, u_l, v_{l,o}$
			$i = n, j \in [1, n], l = n + 1, o \in [1, n]$	$v_{i,j}, u_i, u_4, u_l, v_{l,o}$
			$i = 2n, j \in [1, n], l = 2n + 1, o \in [1, n]$	$v_{i,j}, u_i, u_2, u_l, v_{l,o}$

Based on the trajectory of the vertex rainbow in **Table 2**, the theorem $rvc(K_{3,2} \rhd C_4) = 3n - 1$ proven.

Case 2: $rvc(G) = n + 3$, for $3 \le n \le 7$
For $n = 3$

Known $diam(G) = 6$ for $n = 3$, then $rvc \ge 6$. Will be shown $rvc(G) \ge n + 3 = 6$. Assuming $rvc(G) \le n + 2 = 5$, there is a coloring$-n + 2$ rainbow on graph G with color definition $c': V(G) \to \{1,2, \dots, n + 2\}$. Without reducing generality, suppose coloring is defined as follows

$$u_i = i, i \in [1,6].$$

Noticed the vertex u_6 cannot be colored 1. If u_6 is given color 1, then there will be a trajectory that is not rainbow, namely the $v_{1,1}, u_1, u_6, v_{6,1}$. Furthermore, the u_6 vertex cannot be colored 2. If given color 2, then there will be a track that is not rainbow, namely the track $v_{2,1}, u_2, u_6, v_{6,1}$. Furthermore, the u_6 vertex cannot be colored 3. If given color 3, then there will be a trajectory that is not rainbow, namely the track $v_{3,1}, u_3, u_6, v_{6,1}$. Furthermore, the u_6 vertex cannot be colored 4. If given color 4, then there will be a track that is not rainbow, namely the $v_{4,1}, u_4, u_2, u_6, v_{6,1}$. Furthermore, the u_6 vertex cannot be colored $n + 2 = 5$. If given the color $n + 2 = 5$, then there will be a path that is not rainbow, namely the $v_{5,1}, u_5, u_2, u_6, v_{6,1}$.

Because the graph $(K_{3,3} \rhd C_4)$ for $n = 3$ is not a coloring$-n + 2$ rainbow vertex, so the assumption is wrong. Then it should be $rvc(G) \ge n + 3 = 6$.

$rvc(G) \le n + 3$ will be shown with the color definition $c': V(G) \to \{1,2, \dots, n + 3\}$ as follows.

$$\begin{aligned}
&c(u_i) = c(w_i) = i \bmod 3n - 2, && i \in [1,2n] \\
&c(v_{i,j}) = c(v_{i+1,j+1}) = n - 1, && i = n \wedge j = 1 \\
&c(v_{i,j}) = c(v_{i+4,j-1}) = c(v_{i+5,j}) = n, && i = 1 \wedge j = 2
\end{aligned}$$

$$c(v_{i,j}) = \begin{cases} 1, & i \in [1,2n], j = 1, i \quad even \\ 1, & i \in [3,2n], j = 2, i \quad odd \\ 4, & i \wedge j = 1 \\ 5, & i \wedge j = 2 \end{cases}$$

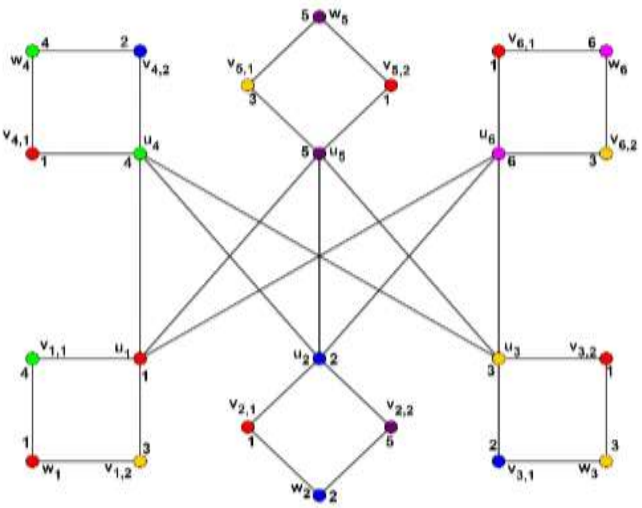

Figure 5. Rainbow vertex-connection number of graph $K_{3,3} \rhd C_4$

It will then be shown that for each x and y in $V(K_{3,3} \rhd C_4)$ there is an $x - y$ rainbow vertex-connection.

Table 3. Rainbow vertex trajectory $(K_{3,3} \rhd C_4)$

Case	x	y	Condition	Rainbow Trajectory
1	u_i	u_k	$i = 1, k = \{n-1, n\}$	u_i, u_{n+1}, u_k
			$i = n+1, k = \{2n-1, 2n\}$	u_i, u_2, u_k
			$i = n-1, k = n$	u_i, u_4, u_k
			$i = 2n-1, k = 2n$	u_i, u_2, u_k
2	w_i	w_k	$i = \{1, n\}, k = n+1$	$w_i, v_{i,2}, u_i, u_k, v_{k,2}, w_k$
			$i = \{1,2\}, k = 2n-1$	$w_i, v_{i,1}, u_i, u_k, v_{k,1}, w_k$
			$i = 2, k = n+1$	$w_i, v_{i,2}, u_i, u_k, v_{k,1}, w_k$
			$i = \{1,2\}, k = 2n$	$w_i, v_{i,1}, u_i, u_k, v_{k,2}, w_k$
			$i = n, k = 2n-1$	$w_i, v_{i,1}, u_i, u_k, v_{k,2}, w_k$
			$i = n, k = 2n$	$w_i, v_{i,1}, u_i, u_k, v_{k,1}, w_k$
3	w_i	w_l	$i = 2n-1, l = 2n$	$w_i, v_{i,1}, u_i, u_2, u_l, v_{l,1}, w_l$
			$i = 1, l = 2$	$w_i, v_{i,2}, u_i, u_{i+3}, u_{i+1}, v_{i+1,l}, w_l$
			$i = 1, l = n$	$w_i, v_{i,1}, u_i, u_5, u_l, v_{l,1}, w_l$
			$i = 2, l = n$	$w_i, v_{i,2}, u_i, u_4, u_l, v_{l,2}, w_l$
			$i = n+1, l = 2n-1$	$w_i, v_{i,1}, u_i, u_2, u_l, v_{l,1}, w_l$
			$i = n+1, l = 2n$	$w_i, v_{i,1}, u_i, u_2, u_l, v_{l,2}, w_l$
4	u_i	w_k	$i \in [1, 2n], k \in [1, 2n]$	$u_i, v_{i,1}, w_k$
5	u_i	w_l	$i \in [1, n], l \in [n+1, 2n]$	$u_i, u_l, v_{l,1}, w_l$
			$i \in [n+1, 2n], l \in [1, n]$	$u_i, u_l, v_{l,2}, w_l$
6	u_i	w_m	$i = 1, m = \{n-1, n\}$	$u_i, u_4, u_m, v_{m,1}, w_m$
			$i = 2, m = \{1, n\}$	$u_i, u_4, u_m, v_{m,2}, w_m$
			$i = n, m = \{1, 2\}$	$u_i, u_4, u_m, v_{m,2}, w_m$
			$i = n+1, m = \{2n-1, 2n\}$	$u_i, u_2, u_m, v_{m,2}, w_m$
			$i = 2n-1, m = \{2n-2, 2n\}$	$u_i, u_2, u_m, v_{m,1}, w_m$
			$i = 2n, m = \{2n-2, 2n-1\}$	$u_i, u_2, u_m, v_{m,1}, w_m$

Case	x	y	Condition	Rainbow Trajectory
7	u_i	$v_{k,j}$	$i \in [1,n], j \in [1,n-1], k = \{n+1,2n\}$	$u_i, u_k, v_{k,j}$
			$i = \{n+1,2n\}, j \in [1,n-1], k \in [1,n]$	$u_i, u_k, v_{k,j}$
8	u_i	$v_{l,j}$	$i = \{1,n\}, j \in [1,n-1], l = n-1$	$u_i, u_4, u_l, v_{l,j}$
			$i = \{1,n-1\}, j \in [1,n-1], l = n$	$u_i, u_4, u_l, v_{l,j}$
			$i = \{n-1,n\}, j \in [1,n-1], l = 1$	$u_i, u_4, u_l, v_{l,j}$
			$i = n+1, j \in [1,n-1], l = \{2n-1,2n\}$	$u_i, u_2, u_l, v_{l,j}$
			$i = 2n-1, j \in [1,n-1], l = \{n+1,2n\}$	$u_i, u_2, u_l, v_{l,j}$
			$i = 2n, j \in [1,n-1], l = \{n+1,n+2\}$	$u_i, u_2, u_l, v_{l,j}$
9	w_i	$v_{k,j}$	$i = 1, j \in [1,n-1], k \in [n+1,2n]$	$w_i, v_{i,2}, u_i, u_k, v_{k,j}$
			$i = \{n-1,n\}, j \in [1,n-1], k \in [n+1,2n]$	$w_i, v_{i,1}, u_i, u_k, v_{k,j}$
			$i = n+1, j \in [1,n-1], k = \{n-1,n\}$	$w_i, v_{i,1}, u_i, u_k, v_{k,j}$
			$i = \{n+1,n+2\}, j \in [1,n-1], k \in [1,n]$	$w_i, v_{i,2}, u_i, u_k, v_{k,j}$
			$i = 2n-1, j \in [1,n], k = \{1,n-1\}$	$w_i, v_{i,1}, u_i, u_k, v_{k,j}$
			$i = 2n, j \in [1,n], k = \{1,n-1\}$	$w_i, v_{i,2}, u_i, u_k, v_{k,j}$
			$i = 2n, j \in [1,n], k = n$	$w_i, v_{i,1}, u_i, u_k, v_{k,j}$
10	w_i	$v_{l,j}$	$i = \{1,n\}, j \in [1,n-1], l = n-1$	$w_i, v_{i,l}, u_i, u_{2l}, u_l, v_{l,j}$
			$i = n-1, j \in [1,n-1], l = \{1,n\}$	$w_i, v_{i,2}, u_i, u_{2i}, u_l, v_{l,j}$
			$i = 1, j \in [1,n-1], l = n$	$w_i, v_{i,1}, u_i, u_{2l-1}, u_l, v_{l,j}$
			$i = n, j \in [1,n-1], l = 1$	$w_i, v_{i,l}, u_i, u_{i+1}, u_l, v_{l,j}$
			$i = n+1, j \in [1,n-1], l = \{2n-1,2n\}$	$w_i, v_{i,1}, u_i, u_2, u_l, v_{l,j}$
			$i = 2n-1, j \in [1,n-1], l = \{n+1,2n\}$	$w_i, v_{i,1}, u_i, u_2, u_l, v_{l,j}$
			$i = 2n, j \in [1,n-1], l = \{n+1,n+2\}$	$w_i, v_{i,1}, u_i, u_2, u_l, v_{l,j}$
11	$v_{i,j}$	$v_{l,k}$	$i \in [1,2n], j = 1, l \in [1,2n], k = 2$	$v_{i,j}, u_i, v_{l,k}$
12	$v_{i,j}$	$v_{l,m}$	$i \in [1,n], j \in [1,n-1],$ $l = \{n+1,2n\}, m \in [1,n-1]$	$v_{i,j}, u_i, u_l, v_{l,o}$
13	$v_{i,j}$	$v_{l,o}$	$i = 1, j \in [1,n-1],$ $l = \{n-1,n\}, o \in [1,n-1]$	$v_{i,j}, u_i, u_4, u_l, v_{l,o}$
			$i = 2, j \in [1,n-1], l = n, o \in [1,n-1]$	$v_{i,j}, u_i, u_4, u_l, v_{l,o}$
			$i = n+1, j \in [1,n-1],$ $l = \{2n-1,2n\}, o \in [1,n-1]$	$v_{i,j}, u_i, u_2, u_l, v_{l,o}$
			$i = n+2, j \in [1,n-1],$ $l = 2n, o \in [1,n-1]$	$v_{i,j}, u_i, u_2, u_l, v_{l,o}$

Based on the trajectory of the vertex rainbow in **Table 3**, the theorem $rvc(K_{3,3} \rhd C_4) = n + 3$, for $3 \leq n \leq 7$ is proven.

Furthermore, for $n = 4, n = 5, n = 6, n = 7$, using the same method as $n = 3$ using proof of contradiction, it can be shown that each graph has $rvc(G) = n + 3$ rainbow coloring.

4. CONCLUSIONS

Based on the results and discussion can be concluded that:

1. To find out the numbers connected rainbows on the graph the operation of the comb graph cycle and complete bipartite graph $(C_4 \rhd K_{3,n})$ can use theorems:
   Suppose C_4 is a cycle graph with four vertices and $K_{3,n}$ is a bipartite graph complete with $2 \leq n \leq 7$. If $G \cong (C_4 \rhd K_{3,n})$, then $rvc(G) = 5$
2. To find out the numbers connected by the rainbow on the graph resulting from the operation of the complete bipartite graph comb and the cycle graph $(K_{3,n} \rhd C_4)$ can use theorem:
   Suppose $K_{3,n}$ is a complete bipartite graph with $2 \leq n \leq 7$ and C_4 is a cycle graph with four vertices. If $G \cong (C_4 \rhd K_{3,n})$, then $rvc(G) = 3n - 1\ for\ n = 2$ and $rvc(G) = n + 3\ for\ 3 \leq n \leq 7$

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