BAREKENG: Journal of Mathematics and Its Applications
March 2023 Volume 17 Issue 1 Page 0427–0438
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol17iss1pp0427-0438

GENERALIZED ORLICZ SEQUENCE SPACES

Cece Kustiawan[1*], Al Azhary Masta[2], Dasep[3], Encum Sumiaty[4], Siti Fatimah[5], Sofihara Al Hazmy[6]

[1,2,3,4,5]*Mathematics Study Program, Universitas Pendidikan Indonesia*
Jl. Dr. Setiabudi, Bandung, 40154, Indonesia
[6]*Faculty of Mathematics and Military Natural Sciences, Universitas Pertahanan Republik Indonesia,*
Bogor,16810, Indonesia

*Corresponding author's e-mail: * **cecekustiawan@upi.edu***

ABSTRACT

Article History:

Received: 10th November 2022
Revised: 29th January 2023
Accepted: 11th February 2023

Keywords:

***S**-Convex Function;*
***S**-Young Function;*
Sequence Orlicz Spaces

*Orlicz spaces were first introduced by Z. W. Birnbaum and W. Orlicz as an extension of Labesgue space in 1931. There are two types of Orlicz spaces, namely continuous Orlicz spaces and Orlicz sequence spaces. Some properties that apply to continuous Orlicz spaces are known, as are Orlicz sequence spaces. This study aims to construct new Orlicz sequence spaces by replacing a function in the Orlicz sequence spaces with a wider function. In addition, this study also aims to show that the properties of the Orlicz sequence spaces still apply to the new Orlicz sequence spaces under different conditions. The method in this research uses definitions and properties that apply to the Orlicz sequence spaces in the previous study and uses the **s**-Young function in these new Orlicz sequence spaces. Furthermore, the study results show that the new Orlicz sequence spaces are an extension of the Orlicz sequence spaces in the previous study. And with the characteristics of the s-Young function, it shows that the properties of the Orlicz sequence spaces still apply.*

How to cite this article:

C. Kustiawan, A. A. Masta, Dasep, E. Sumiaty, S. Fatimah and S. A. Hazmy., "GENERALIZED ORLICZ SEQUENCE SPACES," *BAREKENG: J. Math. & App.*, vol. 17, iss. 1, pp. 0427-0438, March 2023.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*
Research Article • Open Access

1. INTRODUCTION

In 1931, Z.W. Birnbaum and W. Orlicz first introduced the Orlicz spaces as a generalization of the Lebesgue spaces. There are two types of Orlicz spaces that have been discussed by many researchers: continuous Orlicz spaces and Orlicz sequence spaces. Research on continuous Orlicz spaces has been carried out by Maligranda, L. (1989), Masta, A. A. (2016), etc. [1]-[9]. While the Orlicz sequence spaces have been studied by Maligranda, L., and Mastylo, M. (2000), and Awad A. Bakery and Rafaf R. (2020) [10]-[19].

Maligranda and Mastylo in 2000 defined that [8], if $\phi : [0,\infty) \to [0,\infty)$ the Orlicz function is a continuous convex function where the value is 0 only at point 0, then the Orlicz sequence space ℓ_ϕ is

$$\ell_\phi = \left\{ \xi = \left(\xi_j\right) : \exists\, \lambda > 0 \ni \ \rho_\phi(\lambda\xi) < \infty \right\}$$

where $\rho_\phi(\lambda\xi) = \sum_{j=1}^{\infty} \phi\left(\left|\xi_j\right|\right)$ for any real sequence $\xi = \left\{\xi_j\right\}$, equipped with the Luxemburg norm that is

$$\|\xi\|_\phi = \inf\left\{\lambda > 0 : \ \rho_\phi(\xi/\lambda) \le 1\right\}.$$

A similar definition is also used by Savas, E. (2004) in his article entitled "Some Sequence Spaces Defined by Orlicz Functions" [12] and by Khusnussaadah, N. (2019) in his article entitled "Completeness of Sequence Spaces Generated by an Orlicz Function" [13]. Both use the same function, namely the Orlicz function, which is a continuous convex function. The definition of the Orlicz function is similar to the definition of the Young function on a continuous Orlicz spaces [2], [3], [12]. Meanwhile, Prayoga, P. S. (2020), in his article titled "Sifat Inklusi dan Perumuman Ketaksamaan Hölder pada Ruang Barisan Orlicz" uses the Young function in defining the Orlicz sequence spaces [14].

In this article, the authors are interested in extending the function used in the Orlicz sequence spaces by using a function that is a continuous s-convex function where the value is 0 only at point 0. In this article, we call such a function s-Young function. This new definition of space is called the generalized Orlicz sequence spaces. In this generalized Orlicz sequence space, the author will show that some properties of the Orlicz sequence spaces still apply to the generalized Orlicz sequence spaces by giving different conditions.

Before discussing the generalized Orlicz sequence space further, first recall the definitions of the Young function and the s-Young function. A function $\phi : [0,\infty) \to [0,\infty)$ is a Young function if ϕ is convex function, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$ and ϕ is continuous [2], [14]. Furthermore, before defining the s-Young function, the definition of the s-convex function is introduced. Kuei Lin Tseng (2007) defines the s-convex function, namely if $0 < s \le 1$ a function $\phi : [0,\infty) \to \mathbb{R}$ is called s-convex if for every $x, y \in [0,\infty)$ and $\alpha, \beta \ge 0$ with $\alpha^s + \beta^s = 1$ satisfies $\phi(\alpha x + \beta y) \le \alpha^s \phi(x) + \beta^s \phi(y)$ [20]. Rian Dermawan (2020) defines the generalization of s-Young function, namely the function $\phi : [0,\infty) \to [0,\infty)$ is the generalization of the s-Young function if ϕ is s-convex, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$, and ϕ is continuous [21].

The next topic for discussion in this article is the method used to obtain the results of the research conducted by the author. After that, the author explains the results of the research that he obtained from his research on the generalized Orlicz sequence spaces and the properties that apply to the sequence spaces.

2. RESEARCH METHODS

In this section, the author will explain the method used by the author in his research on generalized Orlicz sequence spaces. By using the definitions, lemmas, and properties of the Young function and its generalization, as well as the definition of the Orlicz sequence spaces that has been obtained by previous researchers, the author will construct a new sequence space by replacing the Young function on the Orlicz sequence spaces with a generalized version of the Young function, which is the s-Young function. Before explaining further about the generalized Orlicz sequence spaces, the following will explain the definitions, lemmas, and properties that will help the author construct a generalized Orlicz sequence spaces.

Definition 1. Let $I \subseteq \mathbb{R}$ is a interval. A function $\phi : I \to \mathbb{R}$ is said to be convex on I if for any $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ and any point $x, y \in I$ we have

$$\phi(\alpha x + \beta y) \le \alpha\phi(x) + \alpha, \beta\phi(y).$$

Definition 2. If $0 < s \leq 1$ a function $\phi : [0, \infty) \to [0, \infty)$ is called s-convex if for every $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$ satisfies

$$\phi(\alpha x + \beta y) \leq \alpha^s \phi(x) + \beta^s \phi(y).$$

Lemma 3. *If function $\phi : [0, \infty) \to [0, \infty)$ is a convex then ϕ is s-convex function.*

Definition 4. The function $\phi : [0, \infty) \to [0, \infty)$ is a Young function if ϕ is a convex, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$, and ϕ is continuous.

Definition 5. The function $\phi : [0, \infty) \to [0, \infty)$ is an s-Young function if ϕ is an s-convex, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$, and ϕ is continuous.

According to **Lemma 3**, the s-Young function in **Definition 5** has a broader function than the definition of the Young function in **Definition 4**. Therefore, we call the s-Young function is a generalized Young function. This function will be used by the author to define the Orlicz sequence spaces in the next section. Before that, the author also explains in advance the property of the s-Young function, which is as follows:

Lemma 6. *Let $\phi : [0, \infty) \to [0, \infty)$ be an s-Young function, then*

a) $\phi(\beta t) \leq \beta^s \phi(t)$ *for every* $t \geq 0$ *and* $0 \leq \beta \leq 1$ *with* $0 < s \leq 1$.
b) $\rho(t) = \frac{\phi(t)}{t^s}$ *increasing for any* $t > 0$ *with* $0 < s \leq 1$.
c) $\phi(t)$ *increasing for any* $t \geq 0$.

Definition 7. Let ϕ be a Young function, the Orlicz sequence spaces $\ell_\phi(\mathbb{Z})$ is all sequence $X = (x_k)$ of real numbers such that $\sum_{k=1}^{\infty} \phi\left(\frac{|x_k|}{a}\right) < \infty$ there is $a > 0$ and equipped with norm $\|X\|_{\ell_\phi(\mathbb{Z})} = \inf\left\{b > 0 : \sum_{k=1}^{\infty} \phi\left(\frac{|x_k|}{b}\right) \leq 1\right\} < \infty$.

Lemma 8. *If $\|X\|_{\ell_\phi} \neq 0, \forall X = (x_k) \in \ell_\phi$, then $\sum_{k=1}^{\infty} \phi\left(\frac{|x_k|}{\|X\|_{\ell_\phi}}\right) \leq 1$. Futhermore $\|X\|_{\ell_\phi} \leq 1$ if and only if $\sum_{k=1}^{\infty} \phi\left(\frac{|x_k|}{\|X\|_{\ell_\phi}}\right) \leq 1$.*

Lemma 9. *Let $\phi : [0, \infty) \to [0, \infty)$ is a Young function, then $\sum_{k=1}^{\infty} \phi\left(\frac{|x_k|}{\varepsilon}\right) \leq 1$ for any $\varepsilon > 0$ if and only if $\|X\|_{\ell_\phi} = 0$ for every $X = (x_k) \in \ell_\phi$.*

Lemma 10. *Let $\phi : [0, \infty) \to [0, \infty)$ is a Young function and $X = (x_k) \in \ell_\phi$, then $\sum_{k=1}^{\infty} \phi(\alpha|x_k|) = 0, \forall \alpha > 0$ if and only if $\|X\|_{\ell_\phi} = 0$.*

Definition 11. The function $\|\cdot\| : X \to [0, \infty)$ is said to be quasi-norm on X if for every $x, y \in X$ and $\alpha \in \mathbb{R}$, satisfies

a) $\|x\| \geq 0$.
b) $\|x\| = 0$ if and only if $x = 0$.
c) $\|\alpha x\| = |\alpha|\|x\|$.
d) There exist $C \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|)$.

3. RESULTS AND DISCUSSION

Based on the definition of the Orlicz sequence spaces ℓ_ϕ in Definition 7 in section 2, by replacing the Orlicz function or Young function with a s-Young function, a new definition of the Orlicz sequence spaces is obtained. Because the s-Young function is a generalization of the Young function, the Orlicz sequence spaces with the s-Young function are clearly an extension from ℓ_ϕ. Thus, the definition of the Orlicz sequence spaces that the author constructs in this article is called the generalized Orlicz sequence spaces denoted by $\ell_{\phi_s}(\mathbb{R})$, where ϕ_s is a s-Young function. The following is the definition and properties of the sequence spaces $\ell_{\phi_s}(\mathbb{R})$ constructed by the author:

Definition 1. Let ϕ_s be an s-Young function. Orlicz sequence spaces $\ell_{\phi_s}(\mathbb{R})$ is all sequences $X=(x_k)$ of real numbers such that $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{a}\right)<\infty$ there is $a>0$ and equipped with a function $\|X\|_{\ell_{\phi_s}(\mathbb{R})}=\inf\left\{b>0:\ \sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{b}\right)\le 1\right\}$.

Before discussing the properties of $\ell_{\phi_s}(\mathbb{R})$, it will be shown that the function $\|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}$ is well defined on the Orlicz sequence spaces $\ell_{\phi_s}(\mathbb{R})$, that is, for every $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$, then $\|X\|_{\ell_{\phi_s}(\mathbb{R})}=\inf\left\{b>0:\ \sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{b}\right)\le 1\right\}$ exists.

In this article, we suppose that

$$A=\left\{b>0:\ \sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{b}\right)\le 1\right\} \tag{1}$$

it will be shown that $A\neq\emptyset$.

Take any $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$, based on **Definition 1** there is $a>0$ so that $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{a}\right)<\infty$. Suppose that $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{a}\right)=C$, clearly $C\ge 0$. To show $A\neq\emptyset$, divide into two cases for the value of C.

Case 1 If $C\le 1$, then

$$\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{a}\right)=C\le 1$$

Its means $a\in A$, so that $A\neq\emptyset$.

Case 2 If $C>1$, for $0<s\le 1$ clearly $C^{1/s}>1$ or $\frac{1}{C^{1/s}}<1$.

Based on **Lemma 6(a)** in section 2, we have

$$\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{aC^{1/s}}\right)\le\left(\frac{1}{C^{1/s}}\right)^s\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{a}\right)=\frac{1}{C}\cdot C=1$$

Its means $aC^{1/s}\in A$, so that $A\neq\emptyset$.

From these two cases, $A\neq\emptyset$ is obtained so that $\inf A=\|X\|_{\ell_{\phi_s}(\mathbb{R})}$ exists for every $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$, thus the function $\|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}$ is well defined in the Orlicz sequence spaces $\ell_{\phi_s}(\mathbb{R})$. □

The function $\|\cdot\|_{\ell_\phi}$ in **Definitions 7** in section 2 defines a norm function on the Orlicz sequence space ℓ_ϕ, but the function $\|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}$ in the generalized Orlicz sequence spaces is define a quasi-norm function, the following is an explanation in **Theorem 2**.

Theorem 2. *The function* $\|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}:\ell_{\phi_s}(\mathbb{R})\to[0,\infty)$ *is a quasi-norm.*

Proof. Take any $X=(x_k),Y=(y_k)\in\ell_{\phi_s}(\mathbb{R})$ and $\alpha\in\mathbb{R}$. Suppose that $A=\left\{b>0:\ \sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{b}\right)\le 1\right\}$.

1. Will be shown $\|X\|_{\ell_{\phi_s}(\mathbb{R})}\ge 0$.
   Because $\|X\|_{\ell_{\phi_s}(\mathbb{R})}=\inf A$ its clearly $\|X\|_{\ell_{\phi_s}(\mathbb{R})}\ge 0,\forall\ X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$.
2. Will be shown $\|X\|_{\ell_{\phi_s}(\mathbb{R})}=0$ if and only if $X=(x_k)=0$.
   ($\Rightarrow$) Assume that $\|X\|_{\ell_{\phi_s}(\mathbb{R})}=0$. For any $0<\varepsilon<1$ and $\alpha>0$ with $0<s\le 1$, based on **Lemma 6(a)** in section 2, we have

$$\phi_s(\alpha|x_k|)=\phi_s\left(\varepsilon\left(\frac{\alpha|x_k|}{\varepsilon}\right)\right)\le\varepsilon^s\phi_s\left(\frac{\alpha|x_k|}{\varepsilon}\right)$$

$$\Leftrightarrow \sum_{k=1}^{\infty} \phi_s(\alpha|x_k|) \le \varepsilon^s \sum_{k=1}^{\infty} \phi_s\left(\frac{\alpha|x_k|}{\varepsilon}\right) \le \varepsilon^s$$

Since it applies to any $0 < \varepsilon < 1$, it can be concluded that $\sum_{k=1}^{\infty} \phi_s(\alpha|x_k|) = 0$ for any $\alpha > 0$. Therefore, it must be $\phi_s(\alpha|x_k|) = 0$, consequently $x_k = 0, \forall\, k \in \mathbb{N}$, so $X = (x_k) = 0$.

($\Leftarrow$) Assume that $X = (x_k) = 0$. Take any $\varepsilon > 0$, we have

$$\sum_{k=1}^{\infty} \phi_s\left(\frac{|x_k|}{\varepsilon}\right) = \sum_{k=1}^{\infty} \phi_s\left(\frac{0}{\varepsilon}\right) = \sum_{k=1}^{\infty} \phi_s(0) = 0 < 1$$

Its means $\varepsilon \in A$, consequently $\|X\|_{\ell_{\phi_s}(\mathbb{R})} \le \varepsilon$. Hence $\varepsilon > 0$ is arbitrary, we conclude that $\|X\|_{\ell_{\phi_s}(\mathbb{R})} = 0$.

3. Will be shown $\|\alpha X\|_{\ell_{\phi_s}(\mathbb{R})} = |\alpha|\|X\|_{\ell_{\phi_s}(\mathbb{R})}$.

   When $\alpha = 0$ obviously applies it. Now we consider when $\alpha \neq 0$, note that :

$$\begin{aligned}
\|\alpha X\|_{\ell_{\phi_s}(\mathbb{R})} &= \inf\left\{b > 0 : \sum_{k=1}^{\infty} \phi_s\left(\frac{|\alpha x_k|}{b}\right) \le 1\right\} \\
&= \inf\left\{b > 0 : \sum_{k=1}^{\infty} \phi_s\left(\frac{|\alpha||x_k|}{b}\right) \le 1\right\} \\
&= \inf\left\{b > 0 : \sum_{k=1}^{\infty} \phi_s\left(\frac{|x_k|}{\frac{b}{|\alpha|}}\right) \le 1\right\} \\
&= \inf\left\{|\alpha|c > 0 : \sum_{k=1}^{\infty} \phi_s\left(\frac{|x_k|}{c}\right) \le 1\right\} \quad \text{, with } c = \frac{b}{|\alpha|} \\
&= |\alpha| \cdot \inf\left\{c > 0 : \sum_{k=1}^{\infty} \phi_s\left(\frac{|x_k|}{c}\right) \le 1\right\} \\
&= |\alpha|\|X\|_{\ell_{\phi_s}(\mathbb{R})}
\end{aligned}$$

   Thus it is proved that $\|\alpha X\|_{\ell_{\phi_s}(\mathbb{R})} = |\alpha|\|X\|_{\ell_{\phi_s}(\mathbb{R})}$.

4. Will be shown that there is $C \ge 1$ such that $\|X + Y\|_{\ell_{\phi_s}(\mathbb{R})} \le C^{1/s}\left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})}\right)$ with $0 < s \le 1$.

   When $X = Y = 0$ obviously applies it. Now we consider when $X \neq 0$ or $Y \neq 0$.

   If $X \neq 0, Y = 0$ or vice versa, by choosing $C = 1$, we have

$$\|X + Y\|_{\ell_{\phi_s}(\mathbb{R})} = \|X\|_{\ell_{\phi_s}(\mathbb{R})} = 1 \cdot \|X\|_{\ell_{\phi_s}(\mathbb{R})} = 1 \cdot \left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})}\right)$$

   So,when $X \neq 0, Y = 0$ or vice versa, obviously applies it.

   If $X \neq 0$ and $Y \neq 0$, suppose that $C = \left(\frac{\|X\|_{\ell_{\phi_s}(\mathbb{R})}}{\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})}}\right)^s + \left(\frac{\|Y\|_{\ell_{\phi_s}(\mathbb{R})}}{\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})}}\right)^s$ and $B = \left\{b > 0 : \sum_{k=1}^{\infty} \phi_s\left(\frac{|x_k + y_k|}{b}\right) \le 1\right\}$ with $0 < s \le 1$. Will be shown that $C^{\frac{1}{s}}\left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})}\right) \in B$. Note that, based on **Lemma 6** in section 2, we have

$$\sum_{k=1}^{\infty} \phi_s\left(\frac{|x_k + y_k|}{C^{\frac{1}{s}}\left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})}\right)}\right) \le \sum_{k=1}^{\infty} \phi_s\left(\frac{|x_k| + |y_k|}{C^{\frac{1}{s}}\left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})}\right)}\right)$$

$$= \sum_{k=1}^{\infty} \phi_s\left(\frac{\|X\|_{\ell_{\phi_s}(\mathbb{R})}}{C^{\frac{1}{s}}\left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})}\right)} \cdot \frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})}} + \frac{\|Y\|_{\ell_{\phi_s}(\mathbb{R})}}{C^{\frac{1}{s}}\left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})}\right)} \cdot \frac{|y_k|}{\|Y\|_{\ell_{\phi_s}(\mathbb{R})}}\right)$$

$$\leq \left(\frac{\|X\|_{\ell_{\phi_s}(\mathbb{R})}}{C^{\frac{1}{s}} \left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})} \right)} \right)^s \sum_{k=1}^{\infty} \phi_s \left(\frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})}} \right)$$
$$+ \left(\frac{\|Y\|_{\ell_{\phi_s}(\mathbb{R})}}{C^{\frac{1}{s}} \left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})} \right)} \right)^s \sum_{k=1}^{\infty} \phi_s \left(\frac{|y_k|}{\|Y\|_{\ell_{\phi_s}(\mathbb{R})}} \right)$$
$$\leq \left(\frac{\|X\|_{\ell_{\phi_s}(\mathbb{R})}}{C^{\frac{1}{s}} \left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})} \right)} \right)^s + \left(\frac{\|Y\|_{\ell_{\phi_s}(\mathbb{R})}}{C^{\frac{1}{s}} \left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})} \right)} \right)^s$$
$$= \frac{1}{C} \left[\left(\frac{\|X\|_{\ell_{\phi_s}(\mathbb{R})}}{\left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})} \right)} \right)^s + \left(\frac{\|Y\|_{\ell_{\phi_s}(\mathbb{R})}}{\left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})} \right)} \right)^s \right]$$
$$= \frac{1}{C} \cdot C = 1$$

Thus it is proved that $C^{1/s} \left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})} \right) \in B$. Since $\inf B = \|X+Y\|_{\ell_{\phi_s}(\mathbb{R})}$, we conclude that

$$\|X+Y\|_{\ell_{\phi_s}(\mathbb{R})} \leq C^{1/s} \left(\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \|Y\|_{\ell_{\phi_s}(\mathbb{R})} \right)$$

Since applies 1, 2, 3 and 4, based on **Definition 11** in section 2, we can be concluded that $\|\cdot\|_{\ell_{\phi_s}(\mathbb{R})} : \ell_{\phi_s}(\mathbb{R}) \to [0, \infty)$ is a quasi-norm.

□

3.1. Properties Applicable to Generalized Orlicz Sequence Spaces

In this section, it will be shown that the properties that apply to the Orlicz sequence spaces $\ell_\phi(\mathbb{R})$ also apply to the Orlicz sequence spaces $\ell_{\phi_s}(\mathbb{R})$ by using the properties or characteristics of the s-Young function. Recall that the s-Young function is a monotonically increasing function and the fact that if $\phi_s : [0,\infty) \to [0,\infty)$ is a s-Young function, then $\phi_s(\alpha t) \leq \alpha^s \phi_s(t)$ applies for every $t \geq 0$ and $0 \leq \alpha \leq 1$ with $0 < s \leq 1$. Using this fact, the following lemmas will be shown:

Lemma 3. *If* $\|X\|_{\ell_{\phi_s}(\mathbb{R})} \neq 0, \forall\, X = (x_k) \in \ell_{\phi_s}(\mathbb{R})$, *then* $\sum_{k=1}^{\infty} \phi_s \left(\frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})}} \right) \leq 1$.

Proof. Take any $\varepsilon > 0$ and $X = (x_k) \in \ell_{\phi_s}(\mathbb{R})$ such that $\|X\|_{\ell_{\phi_s}(\mathbb{R})} \neq 0$. Since $\|X\|_{\ell_{\phi_s}(\mathbb{R})} = \inf A$ then $\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \varepsilon$ is not the lower bound of A (see **Equation (1)**) consequently, there is $b_1 \in A$ such that

$$\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \varepsilon \geq b_1 \quad \Leftrightarrow \frac{1}{\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \varepsilon} \leq \frac{1}{b_1}$$

Since $|x_k| \geq 0, \forall\, k \in \mathbb{N}$ then $\frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \varepsilon} \leq \frac{|x_k|}{b_1}$. Based on **Lemma 6(c)** in section 2 and $b_1 \in A$, we have

$$\phi_s \left(\frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \varepsilon} \right) \leq \phi_s \left(\frac{|x_k|}{b_1} \right)$$

$$\Leftrightarrow \sum_{k=1}^{\infty} \phi_s \left(\frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})} + \varepsilon} \right) \leq \sum_{k=1}^{\infty} \phi_s \left(\frac{|x_k|}{b_1} \right) \leq 1$$

Since $\varepsilon > 0$ is arbitrary we can be concluded that

$$\sum_{k=1}^{\infty} \phi_s \left(\frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})}} \right) \leq 1 \tag{2}$$

Lemma 4. *If $\phi_s:[0,\infty)\to[0,\infty)$ is an s-Young function and $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$, then $\sum_{k=1}^{\infty}\phi_s(|x_k|)\le 1$ if and only if $\|X\|_{\ell_{\phi_s}(\mathbb{R})}\le 1$.*

Proof. Let $\phi_s:[0,\infty)\to[0,\infty)$ is an s-Young function and take any $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$.

(⇒) Assume $\sum_{k=1}^{\infty}\phi_s(|x_k|)\le 1$ for every $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$. Note that

$$\sum_{k=1}^{\infty}\phi_s(|x_k|)=\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{1}\right)\le 1$$

From **Equation (1)** we get $1\in A$. Because $\|X\|_{\ell_{\phi_s}(\mathbb{R})}=\inf A$ then obviously $\|X\|_{\ell_{\phi_s}(\mathbb{R})}\le 1$ for every $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$.

(⇐) Assume $\|X\|_{\ell_{\phi_s}(\mathbb{R})}\le 1$ for every $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$, then we have $|x_k|\le\frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})}}$.

Since ϕ_s is increasing, then

$$\phi_s(|x_k|)\le\phi_s\left(\frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})}}\right) \tag{3}$$

Based on **Lemma 3** we have

$$\sum_{k=1}^{\infty}\phi_s(|x_k|)\le\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})}}\right)\le 1 \tag{4}$$

From **Equation (4)** it is proved that $\sum_{k=1}^{\infty}\phi_s(|x_k|)\le 1$.

Corollary 5. *Let $\phi_s:[0,\infty)\to[0,\infty)$ be an s-Young function and $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$, then the following statetments are equivalent:*

a) $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\alpha}\right)\le\frac{1}{\alpha^s}$ for any $\alpha\ge 1$ with $0<s\le 1$
b) $\|X\|_{\ell_{\phi_s}(\mathbb{R})}\le 1$.

Proof. (a ⇒ b) Assume $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\alpha}\right)\le\frac{1}{\alpha^s}$ for every $\alpha\ge 1$ with $0<s\le 1$, will be shown $\|X\|_{\ell_{\phi_s}(\mathbb{R})}\le 1$ for any $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$.

Take any $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$. For $\alpha\ge 1$ and $0<s\le 1$ we have $0<\frac{1}{\alpha}\le\frac{1}{\alpha^s}\le 1$, so that from the hypothesis obtained

$$\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\alpha}\right)\le\frac{1}{\alpha^s}\le 1$$

Choose $\alpha=1$, obtained $\sum_{k=1}^{\infty}\phi_s(|x_k|)\le 1$. From **Equation (1)**, its means $1\in A$. Since $\|X\|_{\ell_{\phi_s}(\mathbb{R})}=\inf A$, consequently $\|X\|_{\ell_{\phi_s}(\mathbb{R})}\le 1$ for every $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$.

(b ⇒ a). Assume $\|X\|_{\ell_{\phi_s}(\mathbb{R})}\le 1$ for every $X=(x_k)\in\ell_{\phi_s}(\mathbb{R})$, we will be shown $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\alpha}\right)\le\frac{1}{\alpha^s}$.

From the hypothesis we know $\|X\|_{\ell_{\phi_s}(\mathbb{R})}\le 1$ or $\frac{1}{\|X\|_{\ell_{\phi_s}(\mathbb{R})}}\ge 1$. Based on **Lemma 6(a)**, for every $\alpha\ge 1$ with $0<s\le 1$, we have

$$\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\alpha}\right)\le\frac{1}{\alpha^s}\sum_{k=1}^{\infty}\phi_s(|x_k|) \tag{5}$$

From **Equation (3)** and **Lemma 15** we obtained

$$\frac{1}{\alpha^s}\sum_{k=1}^{\infty}\phi_s(|x_k|) \le \frac{1}{\alpha^s}\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\|X\|_{\ell_{\phi_s}(\mathbb{R})}}\right) \le \frac{1}{\alpha^s}\cdot 1 = \frac{1}{\alpha^s} \qquad (6)$$

From **Equation (5)** and **Equation** (6), it is proved that $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\alpha}\right) \le \frac{1}{\alpha^s}$ for every $\alpha \ge 1$.

Lemma 6. *Let $\phi_s : [0,\infty) \to [0,\infty)$ be an s-Young function and $X = (x_k) \in \ell_{\phi_s}(\mathbb{R})$, then the following statetments are equivalent:*

a) $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\varepsilon}\right) \le 1$ *for every* $\varepsilon > 0$.
b) $\|X\|_{\ell_{\phi_s}(\mathbb{R})} = 0$.

Proof. (a ⇒ b) assume (a) is holds. Take any $X = (x_k) \in \ell_{\phi_s}(\mathbb{R})$. Because $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\varepsilon}\right) \le 1$ applies for every $\varepsilon > 0$, from **Equation (1)** we obtained $\varepsilon \in A$. Based on **Definition 13** we have $0 \le \|X\|_{\ell_{\phi_s}(\mathbb{R})} \le \varepsilon$, consequently $\|X\|_{\ell_{\phi_s}(\mathbb{R})} = 0$ for every $X = (x_k) \in \ell_{\phi_s}(\mathbb{R})$.

(b ⇒ a) assume (b) is holds. By hypothesis, $\|X\|_{\ell_{\phi_s}(\mathbb{R})} = 0$ for every $X = (x_k) \in \ell_{\phi_s}(\mathbb{R})$, we will be shown that $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\varepsilon}\right) \le 1$ for every $\varepsilon > 0$.

By contradiction, suppose there is $\varepsilon_0 > 0$ such that $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\varepsilon_0}\right) > 1$ and consequently $\varepsilon_0 \notin A$. Then take any $b \in A$ (see **Equation (1)**), its obviously $\inf A = \|X\|_{\ell_{\phi_s}(\mathbb{R})} \le b$. In this case, the relationship between ε_0 and b has two cases, that is

Case 1 If $\varepsilon_0 > b$ or $\frac{1}{\varepsilon_0} < \frac{1}{b}$, so that for every $X = (x_k) \in \ell_{\phi_s}(\mathbb{R})$ we obtained $\frac{|x_k|}{\varepsilon_0} < \frac{|x_k|}{b}$.

Since ϕ_s is increasing and $b \in A$, we have

$$\phi_s\left(\frac{|x_k|}{\varepsilon_0}\right) < \phi_s\left(\frac{|x_k|}{b}\right)$$

$$\Leftrightarrow \sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\varepsilon_0}\right) < \sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{b}\right) \le 1$$

This contradicts the statement $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\varepsilon_0}\right) > 1$, so the false assumption must not exist if $\varepsilon_0 > 0$, such that $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\varepsilon_0}\right) > 1$.

Case 2 If $\varepsilon_0 < b$. Since $b \in A$ is arbitrary then ε_0 is lower bound of A, consequently $\|X\|_{\ell_{\phi_s}(\mathbb{R})} = \inf A \ge \varepsilon_0 > 0$. This contradicts the hypothesis that $\|X\|_{\ell_{\phi_s}(\mathbb{R})} = 0$, so the false assumption must not exist if $\varepsilon_0 > 0$, such that $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\varepsilon_0}\right) > 1$.

From cases 1 and 2, it can be concluded that $\sum_{k=1}^{\infty}\phi_s\left(\frac{|x_k|}{\varepsilon}\right) \le 1$ for every $\varepsilon > 0$.

Lemma 7. *Let $\phi_s : [0,\infty) \to [0,\infty)$ be an s-Young function and $X = (x_k) \in \ell_{\phi_s}(\mathbb{R})$, then the following statetments are equivalent:*

a) $\sum_{k=1}^{\infty}\phi_s(\alpha|x_k|) = 0$ *for every* $\alpha > 0$
b) $\|X\|_{\ell_{\phi_s}(\mathbb{R})} = 0$.

Proof. (a ⇒ b) assume (a) is holds. By hypothesis, $\sum_{k=1}^{\infty}\phi_s(\alpha|x_k|) = 0$ for every $\alpha > 0$, such that

$$\sum_{k=1}^{\infty} \phi_s \left(\frac{|x_k|}{\frac{1}{\alpha}} \right) = 0 < 1$$

Its means $\frac{1}{\alpha} \in A$ (see **Equation (1)**), consequently $0 \leq \|X\|_{\ell_{\phi_s}(\mathbb{R})} \leq \frac{1}{\alpha}$. Since $\alpha > 0$ is arbitrary, we can concluded that $\|X\|_{\ell_{\phi_s}(\mathbb{R})} = 0$.

(b ⇒ a) assume (b) is holds. For any $0 < \varepsilon < 1$ and $\alpha > 0$ with $0 < s \leq 1$, by **Lemma 6(a)** in section 2 we have

$$\phi_s(\alpha|x_k|) = \phi_s\left(\varepsilon\left(\frac{\alpha|x_k|}{\varepsilon}\right)\right) \leq \varepsilon^s \phi_s\left(\frac{\alpha|x_k|}{\varepsilon}\right) \quad , \forall\, k \in \mathbb{N}$$

$$\Leftrightarrow \sum_{k=1}^{\infty} \phi_s(\alpha|x_k|) \leq \varepsilon^s \sum_{k=1}^{\infty} \phi_s\left(\frac{\alpha|x_k|}{\varepsilon}\right) \leq \varepsilon^s$$

Since $0 < \varepsilon < 1$ is arbitrary, we can concluded that $\sum_{k=1}^{\infty} \phi_s(\alpha|x_k|) = 0$ for every $\alpha > 0$.

3.2. Completeness

In the following section, the author will show that the space $\left(\ell_{\phi_s}(\mathbb{R}), \|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}\right)$ is a Quasi-Banach space by showing that the generalized Orlicz sequence spaces $\ell_{\phi_s}(\mathbb{R})$ is complete, that is, any Cauchy sequence in $\ell_{\phi_s}(\mathbb{R})$ converges to a point in $\ell_{\phi_s}(\mathbb{R})$. The following explanation is presented in **Theorem 20** below.

Theorem 8 *The spaces* $\left(\ell_{\phi_s}(\mathbb{R}), \|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}\right)$ *is a Quasi-Banach space.*

Proof. In **Theorem 2**, it has been shown that $\|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}$ defines a quasi-norm on the sequence space $\ell_{\phi_s}(\mathbb{R})$. We will then show that $\left(\ell_{\phi_s}(\mathbb{R}), \|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}\right)$ is a quasi-Banach space. Take any Cauchy sequence $X = (x_n)$ in $\ell_{\phi_s}(\mathbb{R})$, meaning that for every $\varepsilon > 0$, there are $n_0 \in \mathbb{N}$ such that for every $n, m \geq n_0$ holds

$$\|x_m - x_n\|_{\ell_{\phi_s}(\mathbb{R})} < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary then there are $n_k \in \mathbb{N}$ and $T > 0$ that holds $\|x_m - x_n\|_{\ell_{\phi_s}(\mathbb{R})} < \frac{1}{T2^k}$, $\forall\, m, n \geq n_k$. Furthermore, it can be obtained from sub-sequences $\left(x_{n_k}\right)$ such that for $n \geq n_k$ and $T > 0$ it satisfies

$$\left\|x_{n_{k+1}} - x_{n_k}\right\|_{\ell_{\phi_s}(\mathbb{R})} < \frac{1}{T2^k} \tag{7}$$

Now define the sequence of functions (g_m), that is

$$g_m(x) = \sum_{k=1}^{m} \left|x_{n_{k+1}} - x_{n_k}\right| < \infty \tag{8}$$

It will be shown first that (g_m) in $\ell_{\phi_s}(\mathbb{R})$. From the following description, by choosing $T = 2^{m/s}$ and from **Equation (7)** we have

$$\|g_m(x)\|_{\ell_{\phi_s}(\mathbb{R})} = \left\|\sum_{k=1}^{m} \left|x_{n_{k+1}} - x_{n_k}\right|\right\|_{\ell_{\phi_s}(\mathbb{R})}$$

$$= \left\|\left|x_{n_2} - x_{n_1}\right| + |x_3 - x_2| + \cdots + \left|x_{n_{m+1}} - x_{n_m}\right|\right\|_{\ell_{\phi_s}(\mathbb{R})}$$

$$< 2^{\frac{m}{s}} \sum_{k=1}^{m} \left\|x_{n_{k+1}} - x_{n_k}\right\|_{\ell_{\phi_s}(\mathbb{R})} < 2^{\frac{m}{s}} \sum_{k=1}^{m} \frac{1}{2^{\frac{m}{s}} 2^k} = \sum_{k=1}^{m} \frac{1}{2^k}$$

When $m \to \infty$, then $\sum_{k=1}^{\infty} \frac{1}{2^k}$ is a geometric series with a ratio of $\frac{1}{2}$, so that $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{1-\frac{1}{2}} - 1 = 1$. It means $\|g_m(x)\|_{\ell_{\phi_s}(\mathbb{R})} < \sum_{k=1}^{m} \frac{1}{2^k} \le 1$. Based on **Lemma 16**, we obtained $\sum_{k=1}^{\infty} \phi_s(|g_m(x)|) \le 1$, so that $(g_m) \in \ell_{\phi_s}(\mathbb{R})$. Next, suppose

$$\lim_{m\to\infty} \big(g_m(x)\big) = g(x) = \sum_{k=1}^{\infty} |x_{n_{k+1}} - x_{n_k}| \tag{9}$$

Because $g_m(x) \to g(x)$ in $\mathbb{R}$ and ϕ_s is continuous function, then $\phi_s\big(g_m(x)\big) \to \phi_s\big(g(x)\big)$ in $\mathbb{R}$, such that

$$\sum_{k=1}^{\infty} \phi_s\big(g(x)\big) = \lim_{m\to\infty} \sum_{k=1}^{\infty} \phi_s\big(g_m(x)\big) \le 1$$

It means $g(x) \in \ell_{\phi_s}(\mathbb{R})$.

Now, consider that

$$x_{n_{m+1}} = \sum_{k=1}^{m} (x_{n_{k+1}} - x_{n_k}) + x_{n_1}$$

$$\Leftrightarrow \lim_{m\to\infty} x_{n_{m+1}} = \sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k}) + x_{n_1} = g(x) + x_{n_1}$$

Suppose $g(x) + x_{n_1} = x$, it is obviously $x \in \ell_{\phi_s}(\mathbb{R})$. Because $\big(x_{n_{m+1}}\big)$ is sub-sequences of $X = (x_n)$ that converges to x, and $X = (x_n)$ is a Cauchy sequence, it is can concluded that $X = (x_n)$ converges to x. Consequently for $0 < \varepsilon < \min\{1, \sum_{k=1}^{\infty} \phi_s(1)\}$, there is $K \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon^{1+\frac{1}{s}}}{\left(\sum_{k=1}^{\infty} \phi_s(1)\right)^{\frac{1}{s}}} \quad , \forall\, n \ge K \tag{10}$$

Suppose $B = \left\{ b > 0 : \sum_{n=1}^{\infty} \phi_s\left(\frac{|x_n + x|}{b}\right) \le 1 \right\}$. Also note that for every $n \ge K$, that holds

$$\sum_{n=1}^{\infty} \phi_s\left(\frac{|x_n - x|}{\varepsilon}\right) < \sum_{n=1}^{\infty} \phi_s\left(\frac{\frac{\varepsilon^{1+\frac{1}{s}}}{\left(\sum_{k=1}^{\infty}\phi_s(1)\right)^{\frac{1}{s}}}}{\varepsilon}\right)$$

$$= \sum_{n=1}^{\infty} \phi_s\left(\left(\frac{\varepsilon}{\sum_{k=1}^{\infty}\phi_s(1)}\right)^{\frac{1}{s}}\right)$$

$$\le \left(\frac{\varepsilon}{\sum_{k=1}^{\infty}\phi_s(1)}\right) \sum_{n=1}^{\infty} \phi_s(1)$$

$$= \varepsilon < 1$$

It means we obtained $\varepsilon \in B$, consequently $\inf B = \|x_n - x\|_{\ell_{\phi_s}(\mathbb{R})} \le \varepsilon$, $\forall\, n \ge K$, we can write $x_n \to x$.

Thus, every Cauchy sequence in $\ell_{\phi_s}(\mathbb{R})$ converges to a point in $\ell_{\phi_s}(\mathbb{R})$, therefore the spaces $\ell_{\phi_s}(\mathbb{R})$ is complete, and it is proven that the space $\left(\ell_{\phi_s}(\mathbb{R}), \|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}\right)$ is a Quasi-Banach space.

4. CONCLUSIONS

Based on the research results obtained by the author regarding the Orlicz sequence spaces $\ell_{\phi_s}(\mathbb{R})$ and the characteristics of the sequence in that spaces, the author concludes that the generalized Orlicz sequence spaces $\ell_{\phi_s}(\mathbb{R})$ is an extension of the Orlicz sequence spaces ℓ_{ϕ}. The generalized Orlicz sequence spaces $\ell_{\phi_s}(\mathbb{R})$ is equipped with the $\|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}$ function, which defines a quasi-norm function on $\ell_{\phi_s}(\mathbb{R})$. In addition, the lemmas or properties of the Orlicz sequence spaces also apply to the generalized Orlicz sequence spaces under several different conditions. And finally, the author concludes that the generalized Orlicz sequence space is a complete space, and as a result, the space $\left(\ell_{\phi_s}(\mathbb{R}), \|\cdot\|_{\ell_{\phi_s}(\mathbb{R})}\right)$ is a Quasi-Banach space.

AKNOWLEDGEMENT

This research is supported by Hibah Penguatan Kompetensi Universitas Pendidikan Indonesia 2022.

REFERENCES

[1] Maligranda, L., “Orlicz Spaces and Interpolation,” *Thesis*, *Departemento de Matematica*, Universidade Estadual de Campinas, 1989.

[2] Masta, A. A., Gunawan, H. and Setya-Budhi, W., “Inclusion Property of Orlicz and Weak Orlicz Spaces,” J. Math. Fund. Sci, 48(3), 193–203, 2016

[3] Masta, A. A., Gunawan, H. and Wono Setya-Budhi, “On Inclusion Properties of Two Versions of Orlicz-Morrey Spaces*,” Mediteranean Journal of Mathematics*, 14-228, 2017.

[4] Gunawan, H., Kikianty, E., & Schwanke, C., “Discrete Morrey Spaces and Their Inclusion Properties*,’ Mathematische Nachrichten*, 291(8-9), 1-14, 2017.

[5] Orlicz, W., “ Linear Functional Analysis (Series in Real Analysis Volume 4),“ Singapore: World Scientific, 1992.

[6] Rao, M., & Ren, Z., “Theory of Orlicz spaces“. New York: Marcel Dekker, Inc, 1991.

[7] Masta, A. A., “ Sifat Inklusi pada Ruang Orlicz-Morrey“. Bandung: Institut Teknologi Bandung, 2018.

[8] Taqiyuddin, M., & Rosjanuardi, R., “Inclusion Properties of Weighted Weak Orlicz Spaces,“ *Journal of Engineering Science and Technology*, *16*(5), 3974-3986, 2021.

[9] Fatimah, S., Masta, A., Ifronika, I., Wafiqoh, R., & Agustine, “Generalized Hölder’s inequality in Orlicz sequence spaces,“ In *Proceedings of the 7th Mathematics, Science, and Computer Science Education International Seminar, MSCEIS 2019, 12 October 2019, Bandung, West Java, Indonesia*. 2020.

[10] Maligranda, L. & Mastylo, M., “Inclusion Mappings Between Orlicz Sequence Spaces,” *Journal of Functional Analysis,”* 176, 264-279, 2000.

[11] Awad A Bakery & Afaf R., “Some Properties of Pre-quasi Norm On Orlicz Sequence Space*,” Journal of Innequalities and Applications*, 55, 2020.

[12] Savas, E. and savas, R.,”Some Sequence Spacess Defined by Orlicz Functions*,” Archivum Mathematicum*, 40, 33-40, 2004.

[13] Khusnussaadah, N. & Supama., “Completeness of Sequence Spaces Generated by an Orlicz Function*,” Journal of Sciences and Data Analysis*, 19, 1-14, 2019.

[14] Prayoga, P. S., Al Azhary Masta, and Siti fatimah, “Sifat Inklusi dan Perumuman Ketaksamaan Hölder Pada Ruang Barisan Orlicz*,” Jurnal EurekaMatika*,8, No. 2, 2020.

[15] Abbas, N. M. “Some Properties On Orlicz Sequence Spaces“. *Journal of Babylon University/Pure and Applied Sciences, 21*, 2340-2345, 2013.

[16] Kamthan, P. K., & Gupta, M. “Sequence Spaces and Series“. New York: Marcel Dekker, Inc, 1981.

[17] Prayoga, Pradipta; Fatimah, Siti; Masta, Al., “Several Properties of Discrete Orlicz Spaces,“ In: *Proceedings of the 7th Mathematics, Science, and Computer Science Education International Seminar, MSCEIS 2019, 12 October 2019, Bandung, West Java, Indonesia*. 2020.

[18] Fatimah, S., et al, "Sufficient and necessary conditions for generalized Hölder’s inequality in p-summable sequence spaces," *Journal of Physics: Conference Series*. Vol. 1280. No. 2. IOP Publishing, 2019.

[19] Al Hazmy, S., Kustiawan, C., & Rukmana, I., "Discrete Orlicz-Morrey Spaces And Their Inclusion Properties," *Journal of Engineering Science and Technology*, *16*(3), 2018-2027, 2021.

[20] Tseng, K., Hwang, S. & Dragomir, S., “On Some New Inequalities Of Hermite-Hadamard-Fejér Type Involving Convex Functions*,” Demonstratio Mathematica*, *40*(1), 51-64, 2007.

[21] Dermawan, R., Fatimah, S., Hazmy, S. A., Masta, A. A., & Kustiawan, C., “*Generalized of Young’s Function”* . AIP Proceedings, 2022.